

SHINING STARS'**SAQ STAR QUESTIONS PLUS**SETS

- Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form:

(i) $\{P, R, I, N, C, A, L\}$	(a) $\{x : x \text{ is a positive integer and is a divisor of } 18\}$
(ii) $\{0\}$	(b) $\{x : x \text{ is an integer and } x^2 - 9 = 0\}$
(iii) $\{1, 2, 3, 6, 9, 18\}$	(c) $\{x : x \text{ is an integer and } x + 1 = 1\}$
(iv) $\{3, -3\}$	(d) $\{x : x \text{ is a letter of the word PRINCIPAL}\}$
- Match each of the set on the left in the roster form with the same set on the right described in set-builder form:

(i) $\{1, 2, 3, 6\}$	(a) $\{x : x \text{ is a prime number and a divisor of } 6\}$
(ii) $\{2, 3\}$	(b) $\{x : x \text{ is an odd natural number less than } 10\}$
(iii) $\{M, A, T, H, E, I, C, S\}$	(c) $\{x : x \text{ is natural number and divisor of } 6\}$
(iv) $\{1, 3, 5, 7, 9\}$	(d) $\{x : x \text{ is a letter of the word MATHEMATICS}\}$

RELATIONS AND FUNCTIONS

- Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function from Z to Z defined by $f(x) = ax + b$, for some integers a, b . Determine a, b . Also find $f(x)$.
- Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.	(ii) $A \times C$ is a subset of $B \times D$.
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LINEAR INEQUALITIES

- IQ of a person is given by the formula $IQ = MA/CA \times 100$ where MA is mental age and CA is chronological age. If $80 \leq IQ \leq 140$ for a group of 12 years old children, find the range of their mental age.
- Solve the inequality & show the graph of the solution on number line: $\frac{x}{2} \geq \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$
- Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

PERMUTATIONS AND COMBINATIONS

- Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that (i) all vowels occur together (ii) all vowels do not occur together.
- In how many ways can the letters of the word PERMUTATIONS be arranged if the

(i) Words start with P and end with S,	(ii) Vowels are all together,
(iii) There are always 4 letters between P and S?	

BINOMIAL THEOREM

- If n is a positive integer, then prove that $C_0 + C_1 \cdot \frac{x}{2} + C_2 \cdot \frac{x^2}{3} + \dots + C_n \cdot \frac{x^n}{n+1} = \frac{(1+x)^{n+1} - 1}{(n+1)x}$

STRAIGHT LINES

- By using the concept of equation of a line, prove that $(3, 0), (-2, -2), (8, 2)$ are collinear.
- Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

DERIVATIVES

- Find the derivative of $\sin x + \cos x$ from the first principle
- Find the derivative of $1/x^2$
- Find the derivative of $x + 1/x - 1$

SHINING STARS'

SAQ STAR QUESTIONS PLUS**SETS**

1. Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form:

- | | |
|---------------------------|--|
| (i) {P, R, I, N, C, A, L} | (a) {x : x is a positive integer and is a divisor of 18} |
| (ii) {0} | (b) {x : x is an integer and $x^2 - 9 = 0$ } |
| (iii) {1, 2, 3, 6, 9, 18} | (c) {x : x is an integer and $x + 1 = 1$ } |
| (iv) {3, -3} | (d) {x : x is a letter of the word PRINCIPAL} |

Sol: For the sets given on the right side let us write all the elements.

(a) The positive divisors of 18 are 1, 2, 3, 6, 9, 18 $\therefore$ (a) $\leftrightarrow$ (iii)

(b) $x^2 - 9 = 0 \Rightarrow (x - 3)(x + 3) = 0 \Rightarrow x = 3, -3$. $\therefore$ (b) $\leftrightarrow$ (iv)

(c) $x + 1 = 1 \Rightarrow x = 0$. $\therefore$ (c) $\leftrightarrow$ (ii)

(d) Drop the two repeated letters P & I in PRINCIPAL.

Then, we get PRINCAL $\therefore$ (d) $\leftrightarrow$ (i)

$\therefore$ the matching : (i) $\rightarrow$ (d); (ii) $\rightarrow$ (c); (iii) $\rightarrow$ (a); (iv) $\rightarrow$ (b)

2. Match each of the set on the left in the roster form with the same set on the right described in set-builder form:

- | | |
|--------------------------------|---|
| (i) {1, 2, 3, 6} | (a) {x : x is a prime number and a divisor of 6} |
| (ii) {2, 3} | (b) {x : x is an odd natural number less than 10} |
| (iii) {M, A, T, H, E, I, C, S} | (c) {x : x is natural number and divisor of 6} |
| (iv) {1, 3, 5, 7, 9} | (d) {x : x is a letter of the word MATHEMATICS}. |

Sol: For the sets given on the right side let us write all the elements.

a) Divisors of 6 are 1, 2, 3, 6. Among those primes are 2, 3. $\therefore$ (a) $\leftrightarrow$ (ii)

b) Odd natural numbers less than 10 are 1, 3, 5, 7, 9 $\therefore$ (b) $\leftrightarrow$ (iv)

c) Natural numbers and divisors of 6 are 1, 2, 3, 6 $\therefore$ (c) $\leftrightarrow$ (i)

d) In the word MATHEMATICS, the repeated letters are M, A, T. $\therefore$ (d) $\leftrightarrow$ (iii)

Dropping the repetition letters we have M, A, T, H, E, I, C, S

So, the right matching : (i) $\rightarrow$ (c); (ii) $\rightarrow$ (a); (iii) $\rightarrow$ (d); (iv) $\rightarrow$ (b)

RELATIONS AND FUNCTIONS

3. Let $f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a function from Z to Z defined by $f(x) = ax + b$, for some integers a, b . Determine a, b . Also find $f(x)$.

Sol: Given $f(x) = ax + b$

$$\text{Now, } (1, 1) \in f \Rightarrow f(1) = 1 \Rightarrow f(1) = a(1) + b = 1 \Rightarrow a + b = 1 \dots (i)$$

$$\text{Also } (2, 3) \in f \Rightarrow f(2) = 3 \Rightarrow f(2) = a(2) + b = 3 \Rightarrow 2a + b = 3 \dots (ii)$$

Subtracting (ii) from (i), we get $a = 2$.

$$\text{Putting this value of } a \text{ in (i), we get } 2 + b = 1 \Rightarrow b = -1 \quad \therefore a = 2, b = -1.$$

$$\therefore f(x) = 2x - 1$$

4. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that
(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$. (ii) $A \times C$ is a subset of $B \times D$.

Sol. i) Given $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$

$$B \cap C = \{1, 2, 3, 4\} \cap \{5, 6\} = \emptyset$$

$$\therefore A \times (B \cap C) = A \times \emptyset = \emptyset \dots (i)$$

$$\text{Also } A \times B = \{1, 2\} \times \{1, 2, 3, 4\}$$

$$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4)\} \dots (a)$$

$$\text{and } A \times C = \{1, 2\} \times \{5, 6\} = \{(1, 5), (1, 6), (2, 5), (2, 6)\} \dots (b)$$

From (a) & (b), $(A \times B) \cap (A \times C)$

$$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4)\} \cap \{(1, 5), (1, 6), (2, 5), (2, 6)\}$$

$$= \emptyset \dots (ii)$$

From (i) and (ii), $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

ii) $A \times C = \{1, 2\} \times \{5, 6\} = \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

$$B \times D = \{1, 2, 3, 4\} \times \{5, 6, 7, 8\}$$

$$= \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$$

Here, every element of $A \times C$ is also an element of $B \times D$.

$\therefore A \times C$ is a subset of $B \times D$.

LINEAR INEQUALITIES

5. Solve the inequality & show the graph of the solution on number line: $\frac{x}{2} \geq \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

Sol: Given that $\frac{x}{2} \geq \frac{(5x-2)}{3} - \frac{(7x-3)}{5} \Rightarrow \frac{x}{2} \geq \frac{5(5x-2)-3(7x-3)}{15} \Rightarrow \frac{x}{2} \geq \frac{25x-10-21x+9}{15}$

$$\Rightarrow \frac{x}{2} \geq \frac{4x-1}{15} \Rightarrow 15x \geq 2(4x-1) \Rightarrow 15x \geq 8x-2 \Rightarrow 15x-8x \geq -2$$

$$\Rightarrow 7x \geq -2 \Rightarrow x \geq -\frac{2}{7}. \text{ We have } -\frac{2}{7} = -0.28 \quad \therefore x \geq -0.28 \Rightarrow x \in [-0.28, \infty)$$

Number line graph of $[-0.28, \infty)$:



6. IQ of a person is given by the formula $IQ = \frac{MA}{CA} \times 100$, where MA is mental age and CA is chronological age. If $80 \leq IQ \leq 140$ for a group of 12 years old children, find the range of their mental age.

Sol: It is given that for a group of 12 years old children, $80 \leq IQ \leq 140 \dots (i)$

For a group of 12 years old children, $CA = 12$ years $\therefore IQ = \frac{MA}{12} \times 100$

Putting this value of IQ in (i), we have

$$80 \leq \left(\frac{MA}{12} \times 100 \right) \leq 140 \Rightarrow 80 \times \frac{12}{100} \leq MA \leq 140 \times \frac{12}{100} \Rightarrow 9.6 \leq MA \leq 16.8$$

$\therefore$ the range of mental age of the group of 12 years old children is $9.6 \leq MA \leq 16.8$

7. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

Sol: Of the two consecutive even positive integers let x be the smaller one and then the other is $x+2$.

Given that the both the integers are larger than 5, $x > 5$ and $x+2 > 5$

$$\Rightarrow x > 5 \text{ and } x > 3 \Rightarrow x > 5 \dots (i)$$

Also, the sum of the two integers is less than 23.

$$x + (x+2) < 23 \Rightarrow 2x + 2 < 23 \Rightarrow 2x < 23 - 2 \Rightarrow 2x < 21 \Rightarrow x < \frac{21}{2} \Rightarrow x < 10.5 \dots (ii)$$

From (i) and (ii), we obtain $5 < x < 10.5$

Since x is an even number, x takes the values, 6, 8, and 10.

$\therefore$ the required possible pairs whose sum is less than 23 are (6, 8), (8, 10), and (10, 12).

PERMUTATIONS AND COMBINATIONS

8. Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that
- (i) all vowels occur together (ii) all vowels do not occur together.

Sol: (i) The word DAUGHTER contains 8 different letters in which A,U,E are the 3 vowels. Treat (AUE) as a single object
This single object together with 5 remaining letters will be counted as 6 objects.
The number of ways of arranging these 6 objects is ${}^6P_6 = 6!$.
Also, the three vowels A, U, E can be arranged among themselves in $3!$ ways.
Hence the number of ways that all vowels occur together is $6! \times 3!$

(ii) All possible ways of arrangements of 8 letters taken all at a time is $8!$.
The number of permutations in which the vowels are always together $= 6! \times 3!$
The required number of ways in which all vowels do not occur together is
 $8! - (6! \times 3!) = (8 \times 7 \times 6!) - (6! \times 3!) = 6! (8 \times 7 - 3!) = 6! (56 - 6) = 6! \times 50 = 720 \times 50 = 36,000$

9. In how many ways can the letters of the word PERMUTATIONS be arranged if the
- (i) Words start with P and end with S, (ii) Vowels are all together,
(iii) There are always 4 letters between P and S?

Sol: The word PERMUTATIONS contain 12 letters in which 2 Ts are alike.

- (i) **Words start with P and end with S:**

If P is fixed at the left end and S at the right end then 10 letters are left.

Hence, the required number of arrangements $= \frac{10!}{2!} = 18,14,400$

- (ii) **Vowels are all together:**

There are 5 vowels E U A I O in the given word.

Since they have to always occur together, they are treated as a single object

This single object together with the remaining 7 objects will account for 8 objects.

These 8 objects in which there are 2 Ts can be arranged in $\frac{8!}{2!}$ ways.

Also, the 5 different vowels can be arranged among themselves in $5!$ ways.

$\therefore$ By multiplication principle, required number of arrangements

$$= \frac{8!}{2!} \times 5! = 24,19,200$$

BINOMIAL THEOREM

10. If n is a positive integer, then prove that $C_0 + C_1 \cdot \frac{x}{2} + C_2 \cdot \frac{x^2}{3} + \dots + C_n \cdot \frac{x^n}{n+1} = \frac{(1+x)^{n+1} - 1}{(n+1)x}$

Sol: L.H.S = $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \dots + \frac{C_n}{n+1}x^n = C_0 + \frac{{}^nC_1}{2}x + \frac{{}^nC_2}{3}x^2 + \dots + \frac{{}^nC_n}{n+1}x^n$

$$= 1 + \frac{n}{(1)2}x + \frac{n(n-1)}{(1.2)3}x^2 + \dots + \frac{1}{n+1}x^n$$

$$= \frac{1}{(n+1)x} \left[(n+1)x + \frac{(n+1)nx^2}{1.2} + \frac{(n+1)n(n-1)}{1.2.3}x^3 + \dots + x^{n+1} \right] \text{ (multiplying and dividing by } (n+1)x \text{)}$$

$$= \frac{1}{(n+1)x} \left[{}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+1}C_3 x^3 + \dots + {}^{n+1}C_{n+1} x^{n+1} \right]$$

$$= \left[\frac{{}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+1}C_3 x^3 + \dots + {}^{n+1}C_{n+1} x^{n+1} - (n+1)C_0}{(n+1)x} \right] = \frac{(1+x)^{n+1} - 1}{(n+1)x} = \text{R.H.S}$$

STRAIGHT LINES

11. By using the concept of equation of a line, prove that $(3, 0)$, $(-2, -2)$ $(8, 2)$ are collinear.

Sol: Let $A = (3, 0)$ $B = (-2, -2)$, $C = (8, 2)$

The equation of the line passing through points $A(3, 0)$ and $B(-2, -2)$ is

$$(y-0) = \frac{(-2-0)}{(-2-3)}(x-3) \Rightarrow y = \frac{-2}{-5}(x-3) \Rightarrow 5y = 2x-6 \Rightarrow 2x-5y-6=0$$

At the point $(8, 2)$ we have L.H.S. = $2(8) - 5(2) - 6 = 16 - 10 - 6 = 0 = \text{R.H.S.}$

$\therefore$ the points $(3, 0)$, $(-2, -2)$, $(8, 2)$ are collinear.

12. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

Sol: Let the equation of the line having equal intercepts on the axes be $\frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a \dots(i)$

On solving equations $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ we have

$$\left(\frac{7(1) - (-3)(-3)}{4(-3) - 2(7)}, \frac{(-3)(2) - 4(1)}{4(-3) - 2(7)} \right) = \left(\frac{7-9}{-12-14}, \frac{-6-4}{-12-14} \right) = \left(\frac{-2}{-26}, \frac{-10}{-26} \right) = \left(\frac{1}{13}, \frac{5}{13} \right)$$

$\therefore$ the point of intersection of the two given lines is $\left(\frac{1}{13}, \frac{5}{13} \right)$

But $x+y=a$ passes through the point $\left(\frac{1}{13}, \frac{5}{13} \right) \Rightarrow \frac{1}{13} + \frac{5}{13} = a \Rightarrow a = \frac{6}{13}$

$\therefore (i) \Rightarrow x + y = \frac{6}{13} \Rightarrow 13x + 13y - 6 = 0$

$\therefore$ the required equation of the line is $13x + 13y - 6 = 0$.

DERIVATIVES**13. Find the derivative of $\sin x + \cos x$ from the first principle****Sol:** Let $f(x) = \sin x + \cos x \Rightarrow f(x+h) = \sin(x+h) + \cos(x+h)$.

$$\begin{aligned}
 \therefore f'(x) &= \frac{f(x+h) - f(x)}{h}, \text{ [From the first principle] } \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h) + \cos(x+h) - \sin x - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cosh + \cos x \sinh + \cos x \cosh - \sin x \sinh - \sin x - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin h (\cos x - \sin x) + \sin x (\cosh - 1) + \cos x (\cosh - 1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin h}{h} (\cos x - \sin x) + \lim_{h \rightarrow 0} \sin x \frac{(\cosh - 1)}{h} + \lim_{h \rightarrow 0} \cos x \frac{(\cosh - 1)}{h} \\
 &= 1(\cos x - \sin x) + \sin x \frac{(1-1)}{h} + \lim_{h \rightarrow 0} \cos x \frac{(1-1)}{h} = (\cos x - \sin x) + 0 - 0 = \cos x - \sin x
 \end{aligned}$$

14. Find the derivative of $\frac{1}{x^2}$ **Sol:** Let $f(x) = \frac{1}{x^2}$. From the first principle, we have

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{(x+h)^2} - \frac{1}{x^2} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - (x^2 + h^2 + 2hx)}{x^2(x+h)^2} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2(x+h)^2} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h - 2x}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-h - 2x}{x^2(x+h)^2} \right] = \frac{0 - 2x}{x^2(x+0)^2} = \frac{-2}{x^3}
 \end{aligned}$$

15. Find the derivative of $\frac{x+1}{x-1}$

Sol: Let $f(x) = \frac{x+1}{x-1}$. From the first principle we have

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1} \right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-2}{(x-1)(x+h-1)} \right] = \frac{-2}{(x-1)(x+0-1)} = \frac{-2}{(x-1)^2}
 \end{aligned}$$