

BULLET MODEL PAPER

A 'MULTI QUESTION PAPER' WITH 'BULLET ANSWERS'

SAQ & LAQ

SECTIONS

SAQ SECTION-B

Q11. FUNCTIONS

- If $f: \mathbb{R} \rightarrow \mathbb{R}$, $g: \mathbb{R} \rightarrow \mathbb{R}$ are defined by $f(x) = 2x^2 + 3$ and $g(x) = 3x - 2$ then find (i) $(f \circ g)(x)$ (ii) $(g \circ f)(x)$ (iii) $(f \circ f)(0)$ (iv) $g \circ (f \circ f)(3)$.

Sol: Given that $f(x) = 2x^2 + 3$; $g(x) = 3x - 2$

$$\begin{aligned} \text{(i) } (f \circ g)(x) &= f[g(x)] = f(3x - 2) = 2(3x - 2)^2 + 3 \\ &= 2(9x^2 - 12x + 4) + 3 \\ &= 18x^2 - 24x + 8 + 3 = 18x^2 - 24x + 11 \end{aligned}$$

$$\begin{aligned} \text{(ii) } (g \circ f)(x) &= g[f(x)] = g(2x^2 + 3) = 3(2x^2 + 3) - 2 \\ &= 6x^2 + 9 - 2 = 6x^2 + 7 \quad [\because g(x) = 3x - 2] \end{aligned}$$

$$\begin{aligned} \text{(iii) } (f \circ f)(0) &= f[f(0)] = f(2(0)^2 + 3) = f(3) = 2(3)^2 + 3 \\ &= 18 + 3 = 21 \quad [\because f(x) = 2x^2 + 3] \end{aligned}$$

$$\begin{aligned} \text{(iv) } g \circ (f \circ f)(3) &= g[f(f(3))] = g[f(2(3)^2 + 3)] = g[f(21)] = \\ &g[2(21)^2 + 3] = g(885) = 3(885) - 2 = 2653 \end{aligned}$$

- Let $f = \{(1, a), (2, c), (4, d), (3, b)\}$ and $g^{-1} = \{(2, a), (4, b), (1, c), (3, d)\}$

then show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Sol: (i) Given $f = \{(1, a), (2, c), (4, d), (3, b)\}$

$$\Rightarrow f^{-1} = \{(a, 1), (c, 2), (d, 4), (b, 3)\}$$

$$g^{-1} = \{(2, a), (4, b), (1, c), (3, d)\} \Rightarrow g = \{(a, 2), (b, 4), (c, 1), (d, 3)\}$$

Now we find $g \circ f$ and hence $(g \circ f)^{-1}$

We have $f = \{(1, a), (2, c), (4, d), (3, b)\}$ and

$$g = \{(a, 2), (b, 4), (c, 1), (d, 3)\}$$

$$\therefore g \circ f = \{(1, 2), (2, 1), (3, 4), (4, 3)\}$$

$$\Rightarrow (g \circ f)^{-1} = \{(2, 1), (1, 2), (4, 3), (3, 4)\} \dots (i)$$

(ii) To find $f^{-1} \circ g^{-1}$

$$\text{We have } g^{-1} = \{(2, a), (4, b), (1, c), (3, d)\};$$

$$f^{-1} = \{(a, 1), (c, 2), (d, 4), (b, 3)\}$$

$$\therefore f^{-1} \circ g^{-1} = \{(2, 1), (4, 3), (1, 2), (3, 4)\} \dots (ii)$$

From (i) & (ii), it is proved that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

25. If $f = \{(1, 2), (2, -3), (3, -1)\}$ then find (i) $2f$ (ii) $2+f$ (iii) f^2 (iv) $\sqrt{f}$

Sol: Given that $f = \{(1, 2), (2, -3), (3, -1)\}$

$$\text{(i) } 2f = \{(1, 2(2)), (2, 2(-3)), (3, 2(-1))\}$$

$$= \{(1, 4), (2, -6), (3, -2)\}$$

$$\text{(ii) } (2+f) = \{(1, 2+2), (2, 2-3), (3, 2-1)\}$$

$$= \{(1, 4), (2, -1), (3, 1)\}$$

$$\text{(iii) } f^2 = \{(1, 2^2), (2, (-3)^2), (3, (-1)^2)\}$$

$$= \{(1, 4), (2, 9), (3, 1)\}$$

$$\text{(iv) } \sqrt{f} = \{(1, \sqrt{2}), (2, \sqrt{-3}), (3, \sqrt{-1})\} = \{(1, \sqrt{2})\}$$

$\therefore \sqrt{-3}, \sqrt{-1}$ are not reals.

Q12. MATHEMATICAL INDUCTION

- Using P.M.I, P.T. $1.2.3 + 2.3.4 + 3.4.5 + \dots + n$ terms

$$= \frac{n(n+1)(n+2)(n+3)}{4}, \forall n \in \mathbb{N}$$

A: n^{th} term is $T_n = n(n+1)(n+2)$.

$$\begin{aligned} S(n) : 1.2.3 + 2.3.4 + \dots + n(n+1)(n+2) &= \frac{n(n+1)(n+2)(n+3)}{4} \end{aligned}$$

Step 1: L.H.S of $S(1) = 1.2.3 = 6$

$$\text{R.H.S of } S(1) = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{2.3.4}{4} = 6$$

$\therefore$ L.H.S = R.H.S. So, $S(1)$ is true.

Step 2: Assume that $S(k)$ is true, for $k \in \mathbb{N}$

$$S(k) : 1.2.3 + 2.3.4 + \dots + k(k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} \dots (1)$$

Step 3: We show that $S(k+1)$ is true

On adding $(k+1)(k+2)(k+3)$ to both sides of (1), we get

$$\text{L.H.S} = [1.2.3 + 2.3.4 + \dots + k(k+1)(k+2)] + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4} = \text{R.H.S}$$

$\therefore$ L.H.S = R.H.S. So, $S(k+1)$ is true

Hence, by P.M.I the given statement is true, $\forall n \in \mathbb{N}$

• S.T. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + n \text{ terms} = \frac{n}{3n+1}$

A: n^{th} term is $T_n = \frac{1}{(3n-2)(3n+1)}$
 $S(n) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

Step 1: L.H.S of $S(1) = \frac{1}{1.4} = \frac{1}{4}$;

R.H.S of $S(1) = \frac{1}{3 \cdot 1 + 1} = \frac{1}{4}$

$\therefore$ L.H.S = R.H.S.

So, $S(1)$ is true

Step 2: Assume that $S(k)$ is true, for $k \in \mathbb{N}$

$S(k) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \dots (1)$

Step 3: We show that $S(k+1)$ is true

On adding $(k+1)^{\text{th}}$ term to both sides of (1), we get

$$\begin{aligned} \text{L.H.S} &= \left[\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right] + \frac{1}{(3k+1)(3k+4)} \\ &= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \\ &= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)} \\ &= \frac{(k+1)(3k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} = \frac{k+1}{3k+3+1} = \frac{k+1}{3(k+1)+1} = \text{R.H.S} \end{aligned}$$

$\therefore$ L.H.S = R.H.S. So, $S(k+1)$ is true

Hence, by P.M.I the given statement is true, $\forall n \in \mathbb{N}$

• **By Mathematical Induction, show that $49^n + 16^n - 1$ is divisible by 64 for all positive Integers n .**

A: Given $S(n) : 49^n + 16^n - 1 = 64q, q \in \mathbb{Z}$

Step 1: L.H.S of $S(1) = 49^{(1)} + 16^{(1)} - 1$
 $= 49 + 16 - 1 = 64 = 64(1)$

So, $S(1)$ is true

Step 2: Assume that $S(k)$ is true for $k \in \mathbb{N}$

$S(k) : 49^k + 16^k - 1 = 64q \dots (1)$

Step 3: We show that $S(k+1)$ is true

Writing $(k+1)^{\text{th}}$ term from (1), we get

$$\begin{aligned} \text{L.H.S} &= 49^{k+1} + 16^{(k+1)} - 1 = 49^k \cdot 49 + 16^k + 16 - 1 \\ &= (64q - 16k + 1) \cdot 49 + 16k + 15 \quad (\text{from (1)}) \\ &= 64q \cdot 49 - 16k \cdot 49 + 1 \cdot 49 + 16k + 15 \\ &= 64q \cdot 49 - 16k \cdot (49 - 1) + (49 + 15) \\ &= 64q \cdot 49 - 16k \cdot (48) + 64 = 64q \cdot 49 - 16k \cdot (4 \cdot 12) + 64 \\ &= 64q \cdot 49 - 64k \cdot (12) + 64 \\ &= 64(49q - 12k + 1) = 64(\text{an integer}) \end{aligned}$$

So, $S(k+1)$ is true whenever $S(k)$ is true

Hence, by P.M.I the given statement is true, $\forall n \in \mathbb{N}$

Q13. MATRICES

• Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Sol: L.H.S = $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$
 $= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad (\because R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1)$
 $= 1 \cdot \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}$
 $= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$
 $= (b-a)(c-a)((c+a)-(b+a))$
 $= (a-b)(b-c)(c-a) = \text{R.H.S}$

• Show that $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$

Sol: L.H.S = $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = \begin{vmatrix} bc & b+c & 1 \\ c(a-b) & a-b & 0 \\ b(a-c) & a-c & 0 \end{vmatrix} \quad (\because R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1)$
 $= (a-b)(a-c) \begin{vmatrix} bc & b+c & 1 \\ c & 1 & 0 \\ b & 1 & 0 \end{vmatrix} = (a-b)(a-c)(c-b)$
 $= (a-b)(b-c)(c-a) = \text{R.H.S}$

• S.T $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

A: L.H.S = $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$
 $= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix} \quad (\because C_1 \rightarrow C_1 + C_2 + C_3)$
 $= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix} \begin{matrix} (\because R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1) \end{matrix}$$

$$= 2(a+b+c)[(a+b+c)^2 - 0] = 2(a+b+c)^3 = \text{R.H.S}$$

- Find the adjoint and the inverse of

the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Sol: $\det A = 1 \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$

$$= 1(16-9) - 3(4-3) + 3(3-4) = 7 - 3 - 3 = 1 \neq 0$$

$\therefore A$ is invertible

Cofactor matrix of A

$$= \begin{bmatrix} \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \text{Adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det A} (\text{Adj } A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} [\because \det A = 1]$$

Q14. ADDITION OF VECTORS

- If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then test for collinearity of the following points whose position vectors are given by

$$\vec{a} - 2\vec{b} + 3\vec{c}, 2\vec{a} + 3\vec{b} - 4\vec{c}, -7\vec{b} + 10\vec{c}$$

Sol: Let $\vec{OP} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\vec{OQ} = 2\vec{a} + 3\vec{b} - 4\vec{c}$,

$$\vec{OR} = -7\vec{b} + 10\vec{c} \text{ . Then}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= (2\vec{a} + 3\vec{b} - 4\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$$

$$= \vec{a} + 5\vec{b} - 7\vec{c}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = (-7\vec{b} + 10\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$$

$$= -\vec{a} - 5\vec{b} + 7\vec{c} = -(\vec{a} + 5\vec{b} - 7\vec{c}) = -\vec{PQ}$$

$$\therefore \vec{PQ} = -1(\vec{PR}) \Rightarrow \vec{PQ} = t\vec{PR}$$

$\therefore$ Given points P, Q, R are collinear.

- Prove that the following points

$$-\vec{a} + 4\vec{b} - 3\vec{c}, 3\vec{a} + 2\vec{b} - 5\vec{c},$$

$$-3\vec{a} + 8\vec{b} - 5\vec{c}, -3\vec{a} + 2\vec{b} + \vec{c} \text{ are coplanar.}$$

A: We take $\vec{OP} = -\vec{a} + 4\vec{b} - 3\vec{c}$, $\vec{OQ} = 3\vec{a} + 2\vec{b} - 5\vec{c}$,

$$\vec{OR} = -3\vec{a} + 8\vec{b} - 5\vec{c}, \vec{OS} = -3\vec{a} + 2\vec{b} + \vec{c}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (3\vec{a} + 2\vec{b} - 5\vec{c}) - (-\vec{a} + 4\vec{b} - 3\vec{c}) = 4\vec{a} - 2\vec{b} - 2\vec{c}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = (-3\vec{a} + 8\vec{b} - 5\vec{c}) - (-\vec{a} + 4\vec{b} - 3\vec{c}) = -2\vec{a} + 4\vec{b} - 2\vec{c}$$

$$\vec{PS} = \vec{OS} - \vec{OP} = (-3\vec{a} + 2\vec{b} + \vec{c}) - (-\vec{a} + 4\vec{b} - 3\vec{c}) = -2\vec{a} - 2\vec{b} + 4\vec{c}$$

$$[\vec{PQ} \vec{PR} \vec{PS}] = \begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} [\vec{a} \vec{b} \vec{c}]$$

$$= [4(16-4) + 2(-8-4) - 2(4+8)] [\vec{a} \vec{b} \vec{c}]$$

$$= [4(12) + 2(-12) - 2(12)] [\vec{a} \vec{b} \vec{c}]$$

$$= [48 - 24 - 24] [\vec{a} \vec{b} \vec{c}] = 0 [\vec{a} \vec{b} \vec{c}] = 0$$

So, $\vec{PQ} \vec{PR} \vec{PS}$ are coplanar. Hence P, Q, R, S are coplanar.

- If the points whose position vectors are $3\vec{i} - 2\vec{j} - \vec{k}, 2\vec{i} + 3\vec{j} - 4\vec{k}, -\vec{i} + \vec{j} + 2\vec{k}, 4\vec{i} + 5\vec{j} + \lambda\vec{k}$ are coplanar, then show that $\lambda = -146/17$

A: We take $\vec{OP} = 3\vec{i} - 2\vec{j} - \vec{k}$, $\vec{OQ} = 2\vec{i} + 3\vec{j} - 4\vec{k}$,

$$\vec{OR} = -\vec{i} + \vec{j} + 2\vec{k}, \vec{OS} = 4\vec{i} + 5\vec{j} + \lambda\vec{k}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (2\vec{i} + 3\vec{j} - 4\vec{k}) - (3\vec{i} - 2\vec{j} - \vec{k}) = -\vec{i} + 5\vec{j} - 3\vec{k}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = (-\vec{i} + \vec{j} + 2\vec{k}) - (3\vec{i} - 2\vec{j} - \vec{k}) = -4\vec{i} + 3\vec{j} + 3\vec{k}$$

$$\vec{PS} = \vec{OS} - \vec{OP} = (4\vec{i} + 5\vec{j} + \lambda\vec{k}) - (3\vec{i} - 2\vec{j} - \vec{k}) = \vec{i} + 7\vec{j} + (\lambda + 1)\vec{k}$$

But $[\vec{PQ} \vec{PR} \vec{PS}] = 0$ [Since P, Q, R, S are coplanar]

$$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow (-1)[3(\lambda + 1) - 21] - 5[-4(\lambda + 1) - 3] - 3[(-28) - 3] = 0$$

$$\Rightarrow -1(3\lambda - 18) - 5(-4\lambda - 7) - 3(-31) = 0$$

$$\Rightarrow -3\lambda + 18 + 20\lambda + 35 + 93 = 0 \Rightarrow -3\lambda + 20\lambda + 35 + 93 + 18 = 0$$

$$\Rightarrow 17\lambda + 146 = 0 \Rightarrow 17\lambda = -146 \Rightarrow \lambda = -146/17$$

- Let ABCDEF is a regular hexagon with centre O, then, P.T $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = 3\vec{AD} = 6\vec{AO}$

A: Given ABCDEF is a regular hexagon with centre 'O'.

$$\therefore (\vec{AB} + \vec{AC}) + (\vec{AD}) + (\vec{AE} + \vec{AF})$$

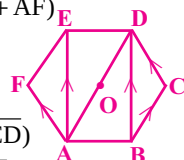
$$= (\vec{AB} + \vec{AC}) + (\vec{AD}) + (\vec{BD} + \vec{CD})$$

$$[\because \vec{AE} = \vec{BD}, \vec{AF} = \vec{CD}]$$

$$= (\vec{AB} + \vec{BD}) + \vec{AD} + (\vec{AC} + \vec{CD})$$

$$= (\vec{AD}) + (\vec{AD}) + \vec{AD} = 3\vec{AD}$$

$$= 3(2\vec{AO}) [\because \vec{AD} = 2\vec{AO}] = 6\vec{AO}$$



Q15. PRODUCT OF VECTORS

- If $\vec{a} = 2\vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{b} = 3\vec{i} - \vec{j} + 2\vec{k}$ then find the angle between $2\vec{a} + \vec{b}$ and $\vec{a} + 2\vec{b}$

Sol: Given that $\vec{a} = 2\vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{b} = 3\vec{i} - \vec{j} + 2\vec{k}$. Then

$$\begin{aligned} 2\vec{a} + \vec{b} &= 2(2\vec{i} + 2\vec{j} - 3\vec{k}) + (3\vec{i} - \vec{j} + 2\vec{k}) \\ &= (4\vec{i} + 4\vec{j} - 6\vec{k}) + (3\vec{i} - \vec{j} + 2\vec{k}) \\ &= 7\vec{i} + 3\vec{j} - 4\vec{k} \\ \Rightarrow |2\vec{a} + \vec{b}| &= \sqrt{7^2 + 3^2 + (-4)^2} \\ &= \sqrt{49 + 9 + 16} = \sqrt{74} \end{aligned}$$

Also

$$\begin{aligned} \vec{a} + 2\vec{b} &= (2\vec{i} + 2\vec{j} - 3\vec{k}) + 2(3\vec{i} - \vec{j} + 2\vec{k}) \\ &= (2\vec{i} + 2\vec{j} - 3\vec{k}) + (6\vec{i} - 2\vec{j} + 4\vec{k}) \\ &= 8\vec{i} + \vec{k} \\ \Rightarrow |\vec{a} + 2\vec{b}| &= \sqrt{8^2 + 1^2} = \sqrt{65} \end{aligned}$$

If ' θ ' is the angle between these vectors then

$$\begin{aligned} \cos \theta &= \frac{(2\vec{a} + \vec{b}) \cdot (\vec{a} + 2\vec{b})}{|2\vec{a} + \vec{b}| |\vec{a} + 2\vec{b}|} \\ &= \frac{(7\vec{i} + 3\vec{j} - 4\vec{k}) \cdot (8\vec{i} + \vec{k})}{\sqrt{74} \cdot \sqrt{65}} \\ &= \frac{56 + 0 + (-4)}{\sqrt{74} \cdot \sqrt{65}} = \frac{52}{\sqrt{74 \times 65}} = \frac{52}{\sqrt{4810}} \\ \therefore \theta &= \cos^{-1} \left(\frac{52}{\sqrt{4810}} \right) \end{aligned}$$

53. If $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, $\vec{b} = -\vec{i} + 4\vec{j} + 2\vec{k}$ then find $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ and unit vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$

Sol: Given

$$\begin{aligned} \vec{a} &= 2\vec{i} - 3\vec{j} + 5\vec{k} \text{ and } \vec{b} = -\vec{i} + 4\vec{j} + 2\vec{k} \\ \vec{a} + \vec{b} &= (2\vec{i} - 3\vec{j} + 5\vec{k}) + (-\vec{i} + 4\vec{j} + 2\vec{k}) \\ &= \vec{i} + \vec{j} + 7\vec{k} \\ \vec{a} - \vec{b} &= (2\vec{i} - 3\vec{j} + 5\vec{k}) - (-\vec{i} + 4\vec{j} + 2\vec{k}) \\ &= 3\vec{i} - 7\vec{j} + 3\vec{k} \end{aligned}$$

$$\begin{aligned} \therefore \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ -1 & 4 & 2 \end{vmatrix} \\ &= \vec{i}(3 + 49) - \vec{j}(3 - 21) + \vec{k}(-7 - 3) \\ &= 52\vec{i} + 18\vec{j} - 10\vec{k} \\ \Rightarrow |52\vec{i} + 18\vec{j} - 10\vec{k}| &= 2|26\vec{i} + 9\vec{j} - 5\vec{k}| \\ &= 2\sqrt{676 + 81 + 25} = 2\sqrt{782} \\ \therefore \text{Required Unit vector} \\ &= \pm \frac{52\vec{i} + 18\vec{j} - 10\vec{k}}{2\sqrt{782}} = \pm \frac{26\vec{i} + 9\vec{j} - 5\vec{k}}{\sqrt{782}} \end{aligned}$$

56. Find the volume of the parallelepiped formed by the coterminal edges are represented by the vectors $2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{i} - \vec{j} + 2\vec{k}$, $2\vec{i} + \vec{j} - \vec{k}$

Sol: Let $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = \vec{i} - \vec{j} + 2\vec{k}$ and $\vec{c} = 2\vec{i} + \vec{j} - \vec{k}$.

Then

$$\begin{aligned} [\vec{a} \ \vec{b} \ \vec{c}] &= \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} \\ &= 2(1 - 2) + 3(-1 - 4) + 1(1 + 2) \\ &= -2 - 15 + 3 = -14 \end{aligned}$$

$\therefore$ Volume of the parallelepiped

$$= |[\vec{a} \ \vec{b} \ \vec{c}]| = |-14| = 14 \text{ cu. units}$$

Q16. TRIGONOMETRIC EQUATIONS

- Solve $\sqrt{3} \sin \theta - \cos \theta = \sqrt{2}$

A: Given equation is $\sqrt{3} \sin \theta - \cos \theta = \sqrt{2}$

On dividing by $\sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$, we get

$$\begin{aligned} \frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta &= \frac{\sqrt{2}}{2} \Rightarrow \sin \theta \left(\frac{\sqrt{3}}{2} \right) - \cos \theta \left(\frac{1}{2} \right) = \frac{1}{\sqrt{2}} \\ \Rightarrow \sin \theta \cos 30^\circ - \cos \theta \sin 30^\circ &= \sin 45^\circ \end{aligned}$$

$$\Rightarrow \sin \theta \cos \frac{\pi}{6} - \cos \theta \sin \frac{\pi}{6} = \sin \frac{\pi}{4} \Rightarrow \sin \left(\theta - \frac{\pi}{6} \right) = \sin \frac{\pi}{4}$$

Here P.V is $\alpha = \frac{\pi}{4}$

$\therefore$ General solution is $\theta = n\pi + (-1)^n \alpha$, $n \in \mathbb{Z}$

$$\Rightarrow \theta - \frac{\pi}{6} = n\pi + (-1)^n \frac{\pi}{4} \Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{6}, n \in \mathbb{Z}$$

- Solve $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$

A: Given equation is $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$
 $\Rightarrow 2(1 - \sin^2\theta) - \sqrt{3}\sin\theta + 1 = 0$
 $\Rightarrow 2 - 2\sin^2\theta - \sqrt{3}\sin\theta + 1 = 0$
 $\Rightarrow 2\sin^2\theta + \sqrt{3}\sin\theta - 3 = 0$
 $\Rightarrow 2\sin^2\theta + 2\sqrt{3}\sin\theta - \sqrt{3}\sin\theta - (\sqrt{3})^2 = 0$
 $\Rightarrow 2\sin\theta(\sin\theta + \sqrt{3}) - \sqrt{3}(\sin\theta + \sqrt{3}) = 0$
 $\Rightarrow (2\sin\theta - \sqrt{3})(\sin\theta + \sqrt{3}) = 0$
 $\Rightarrow (2\sin\theta - \sqrt{3}) = 0$ (or) $(\sin\theta + \sqrt{3}) = 0$
 $2\sin\theta = \sqrt{3}$ (or) $\sin\theta = -\sqrt{3}$ [This has no solution]

Now, $2\sin\theta = \sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

So, $\sin\theta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$, here P.V is $\alpha = \frac{\pi}{3}$

$\therefore$ General solution is $\theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$

$\Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{3}, n \in \mathbb{Z}$

- Find the general solution of

$2\sin^2\theta - 4 = 5\cos\theta$

Sol: $2\sin^2\theta - 4 = 5\cos\theta \Rightarrow 2(1 - \cos^2\theta) - 4 = 5\cos\theta$

$\Rightarrow 2 - 2\cos^2\theta - 4 = 5\cos\theta$

$\Rightarrow -2\cos^2\theta - 5\cos\theta - 2 = 0$

$\Rightarrow 2\cos^2\theta + 5\cos\theta + 2 = 0$

$\Rightarrow 2\cos^2\theta + 4\cos\theta + \cos\theta + 2 = 0$

$\Rightarrow 2\cos\theta(\cos\theta + 2) + 1(\cos\theta + 2) = 0$

$\Rightarrow (2\cos\theta + 1)(\cos\theta + 2) = 0$

$\Rightarrow \cos\theta = -\frac{1}{2}$ (or) $\cos\theta = -2$

(Here, $\cos\theta = -2$ has no solution)

$\Rightarrow \cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$

Here, P.V is $\alpha = \frac{2\pi}{3}$

$\therefore$ General solution is $\theta = 2n\pi \pm \alpha$

$\Rightarrow \theta = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$

Q17. INVERSE TRIG.FUNCTIONS

- Find the value of $\sin\left(\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}\right)$

Sol: Let $\cos^{-1}\frac{4}{5} = \alpha \Rightarrow \cos\alpha = \frac{4}{5} \Rightarrow \sin\alpha = \frac{3}{5}$

$\cos^{-1}\left(\frac{12}{13}\right) = \beta \Rightarrow \cos\beta = \frac{12}{13} \Rightarrow \sin\beta = \frac{5}{13}$

Now, $\sin\left(\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}\right) = \sin(\alpha + \beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13} = \frac{56}{65}$

- Prove that $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{7}{25} = \sin^{-1}\frac{117}{125}$

Sol: Let $\sin^{-1}\frac{4}{5} = \alpha \Rightarrow \sin\alpha = \frac{4}{5} \Rightarrow \cos\alpha = \frac{3}{5}$

$\sin^{-1}\frac{7}{25} = \beta \Rightarrow \sin\beta = \frac{7}{25} \Rightarrow \cos\beta = \frac{24}{25}$

$\therefore \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{4}{5} \times \frac{24}{25} + \frac{3}{5} \times \frac{7}{25} = \frac{96 + 21}{125} = \frac{117}{125}$. Hence proved.

- Prove that $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

A: Let $\sin^{-1}\frac{3}{5} = \alpha \Rightarrow \sin\alpha = \frac{3}{5} \Rightarrow \cos\alpha = \frac{4}{5}$

$\cos^{-1}\frac{12}{13} = \beta \Rightarrow \sin\beta = \frac{12}{13} \Rightarrow \cos\beta = \frac{5}{13}$

$\therefore \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13} = \frac{48 - 15}{65} = \frac{33}{65}$. Hence proved.

- Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

A: Formula: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\therefore \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \tan^{-1}\left(\frac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{2} \cdot \frac{1}{5}}\right) = \tan^{-1}\left(\frac{\frac{5+2}{10}}{\frac{10-1}{10}}\right) = \tan^{-1}\frac{7}{9}$

$\therefore \text{L.H.S} = \tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8} = \tan^{-1}\left(\frac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{9} \cdot \frac{1}{8}}\right) = \tan^{-1}\left(\frac{\frac{56+9}{72}}{\frac{72-7}{72}}\right)$

$= \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S}$

Q18. PROPERTIES OF TRIANGLES

- Prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \frac{s^2}{\Delta}$

A: L.H.S = $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$
 $= \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}$
 $= \frac{s(s-a) + s(s-b) + s(s-c)}{\Delta}$
 $= \frac{s[(s-a) + (s-b) + (s-c)]}{\Delta}$
 $= \frac{s[3s - (a+b+c)]}{\Delta} = \frac{s[3s - 2s]}{\Delta}$
 $= \frac{s[s]}{\Delta} = \frac{s^2}{\Delta} = \text{R.H.S}$

- Prove that $\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4\Delta}$

A: L.H.S = $\cot A + \cot B + \cot C = \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$
 $= \frac{b^2 + c^2 - a^2}{2bc(\sin A)} + \frac{c^2 + a^2 - b^2}{2ca(\sin B)} + \frac{a^2 + b^2 - c^2}{2ab(\sin C)}$
 $= \frac{b^2 + c^2 - a^2}{4(\frac{1}{2}bc \sin A)} + \frac{c^2 + a^2 - b^2}{4(\frac{1}{2}ca \sin B)} + \frac{a^2 + b^2 - c^2}{4(\frac{1}{2}ab \sin C)}$
 $= \frac{b^2 + c^2 - a^2}{4\Delta} + \frac{c^2 + a^2 - b^2}{4\Delta} + \frac{a^2 + b^2 - c^2}{4\Delta}$
 $= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{4\Delta}$
 $= \frac{a^2 + b^2 + c^2}{4\Delta} = \text{RHS}$

- If $\cot \frac{A}{2} : \cot \frac{B}{2} : \cot \frac{C}{2} = 3 : 5 : 7$, S.T a:b:c=6:5:4

Sol: Given that $\cot \frac{A}{2} : \cot \frac{B}{2} : \cot \frac{C}{2} = 3 : 5 : 7$
 $\Rightarrow \frac{s(s-a)}{\Delta} : \frac{s(s-b)}{\Delta} : \frac{s(s-c)}{\Delta} = 3 : 5 : 7$
 $\Rightarrow (s-a) : (s-b) : (s-c) = 3 : 5 : 7$
Let $s-a=3k$ (1), $s-b=5k$ (2), $s-c=7k$ (3)
Now, (1)+(2)+(3) $\Rightarrow 3s-(a+b+c)=15k$
 $\Rightarrow 3s-2s=15k \Rightarrow s=15k$
From (1), $s-a=3k \Rightarrow 15k-a=3k \Rightarrow a=15k-3k=12k$
From (2), $s-b=5k \Rightarrow 15k-b=5k \Rightarrow b=15k-5k=10k$
From (3), $s-c=7k \Rightarrow 15k-c=7k \Rightarrow c=15k-7k=8k$
 $\therefore a : b : c = 12k : 10k : 8k = 12 : 10 : 8 = 6 : 5 : 4$

- Prove that

$$\left(\frac{1}{r} - \frac{1}{r_1} \right) \left(\frac{1}{r} - \frac{1}{r_2} \right) \left(\frac{1}{r} - \frac{1}{r_3} \right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2 s^2}$$

Sol : $\left(\frac{1}{r} - \frac{1}{r_1} \right) = \frac{s}{\Delta} - \frac{s-a}{\Delta} = \frac{s-s+a}{\Delta} = \frac{a}{\Delta}$

Similarly we get $\frac{1}{r} - \frac{1}{r_2} = \frac{b}{\Delta}$ and

$$\frac{1}{r} - \frac{1}{r_3} = \frac{c}{\Delta}$$

$$\therefore \left(\frac{1}{r} - \frac{1}{r_1} \right) \left(\frac{1}{r} - \frac{1}{r_2} \right) \left(\frac{1}{r} - \frac{1}{r_3} \right) = \frac{a}{\Delta} \cdot \frac{b}{\Delta} \cdot \frac{c}{\Delta}$$

$$= \frac{abc}{\Delta^3} = \frac{4R\Delta}{\Delta^3} = \frac{4R}{\Delta^2} = \frac{4R}{(rs)^2} = \frac{4R}{r^2 s^2}$$

- Show that $r+r_3+r_1-r_2 = 4R \cos B$.

A: L.H.S = $r+r_3+r_1-r_2 = (r_3+r_1)+(r-r_2)$
 $= \left(4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} + 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \right)$
 $+ \left(4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} - 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \right)$
 $= 4R \left(\cos \frac{B}{2} \left(\cos \frac{A}{2} \sin \frac{C}{2} + \sin \frac{A}{2} \cos \frac{C}{2} \right) \right.$
 $\left. - \sin \frac{B}{2} \left(\cos \frac{A}{2} \cos \frac{C}{2} - \sin \frac{A}{2} \sin \frac{C}{2} \right) \right)$
 $= 4R \left(\cos \frac{B}{2} \sin \left(\frac{A+C}{2} \right) - \sin \frac{B}{2} \cos \left(\frac{A+C}{2} \right) \right)$
 $= 4R \left(\cos \frac{B}{2} \cos \frac{B}{2} - \sin \frac{B}{2} \sin \frac{B}{2} \right)$
 $= 4R \left[\cos^2 \frac{B}{2} - \sin^2 \frac{B}{2} \right] = 4R \cos B = \text{R.H.S}$

LAQ SECTION-C

Q19. MATRICES

- By using Cramer's rule solve
 $2x - y + 3z = 9, x + y + z = 6, x - y + z = 2$

A: Given equations in the matrix equation form: $AX = D$,

$$\text{where } A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$\Delta = \det A = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+1) + 1(1-1) + 3(-1-1)$$

$$= 2(2) + 1(0) + 3(-2) = 4 + 0 - 6 = -2$$

$$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 9(1+1) + 1(6-2) + 3(-6-2)$$

$$= 9(2) + 1(4) + 3(-8) = 18 + 4 - 24 = -2;$$

$$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2(6-2) - 9(1-1) + 3(2-6)$$

$$= 2(4) - 9(0) + 3(-4) = 8 - 0 - 12 = -4;$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2(2+6) + 1(2-6) + 9(-1-1)$$

$$= 2(8) + 1(-4) + 9(-2) = 16 - 4 - 18 = -6$$

By Cramer's rule,

$$x = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1; y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2} = 3$$

$\therefore x=1, y=2, z=3$ are the solutions of the given equations.

- Solve the equations $3x+4y+5z=18, 2x-y+8z=13, 5x-2y+7z=20$ by Matrix inversion method.

A: Given equations in the matrix equation form: $AX=D$,

$$\text{where } A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

In the Matrix Inversion Method, solution is $X=A^{-1}D$

First we find A^{-1}

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix} = 3(-7+16) - 4(14-40) + 5(-4+5)$$

$$= 3(9) - 4(-26) + 5(1) = 27 + 104 + 5 = 136$$

$$\begin{aligned} \text{Adj}A &= \begin{bmatrix} + \begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} & - \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} & + \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} \\ - \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} & + \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} & - \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} \\ + \begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} & - \begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} & + \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} \end{bmatrix}^T \\ &= \begin{bmatrix} -7+16 & -(14-40) & -4+5 \\ -(28+10) & (21-25) & -(-6-20) \\ (32+5) & -(24-10) & (-3-8) \end{bmatrix}^T = \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}^T \\ \Rightarrow \text{Adj}A &= \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}, A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{1}{136} \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix} \\ \therefore X &= A^{-1}D = \frac{1}{136} \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix} \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix} \\ &= \frac{1}{136} \begin{bmatrix} 9 \times 18 - 38 \times 13 + 37 \times 20 \\ 26 \times 18 - 4 \times 13 - 14 \times 20 \\ 1 \times 18 + 26 \times 13 - 11 \times 20 \end{bmatrix} \\ &= \frac{1}{136} \begin{bmatrix} 162 - 494 + 740 \\ 468 - 52 - 280 \\ 18 + 338 - 220 \end{bmatrix} = \frac{1}{136} \begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \\ \therefore X &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \text{ Hence, the solution is } x=3, y=1, z=1 \end{aligned}$$

- Solve the system of equations $2x-y+3z=9, x+y+z=6, x-y+z=2$ using Gauss Jordan method.

A: Given equations in the matrix equation form: $AX=D$,

$$\text{where } A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}; D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{Augmented Matrix } [AD] = \begin{bmatrix} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 9 \\ 1 & -1 & 1 & 2 \end{bmatrix} (R_1 \rightarrow R_{12})$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & -3 \\ 0 & -2 & 0 & -4 \end{bmatrix} (R_2 \rightarrow R_2 - 2R_1) \\ (R_3 \rightarrow R_3 - R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 0 & -4 \end{bmatrix} (R_2 - R_3) \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -2 & -6 \end{bmatrix} (R_3 - 2R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} (R_3 \rightarrow R_3(-\frac{1}{2}))$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} (R_1 \rightarrow R_1 - R_3) \\ (R_2 \rightarrow R_1 - R_2) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} (R_1 \rightarrow R_1 + R_2)$$

$$\therefore x = 1, y = 2, z = 3$$

Q20 . PRODUCT OF VECTORS

If $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$, $\vec{c} = \vec{i} + 3\vec{j} - 2\vec{k}$
then, verify $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

Sol: 1) To find $\vec{a} \times (\vec{b} \times \vec{c})$, first we have to find
bracket term $(\vec{b} \times \vec{c})$

$$(i) \quad \vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \vec{i}(-2+3) - \vec{j}(-4+1) + \vec{k}(6-1)$$

$$= \vec{i} + 3\vec{j} + 5\vec{k}$$

$$(ii) \quad \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix}$$

$$= \vec{i}(-10+9) - \vec{j}(5+3) + \vec{k}(3+2)$$

$$= -\vec{i} - 8\vec{j} + 5\vec{k} \dots (1)$$

2) To find $(\vec{a} \times \vec{b}) \times \vec{c}$ we have to find
bracket term $(\vec{a} \times \vec{b})$

$$(iii) \quad \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}(2+3) - \vec{j}(-1+6) + \vec{k}(1+4)$$

$$= 5\vec{i} - 5\vec{j} + 5\vec{k}$$

$$(iv) \quad (\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -5 & 5 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \vec{i}(10-5) - \vec{j}(-10-5) + \vec{k}(15+5)$$

$$= -5\vec{i} + 15\vec{j} + 20\vec{k} \dots (2)$$

$\therefore$ From (1) & (2),

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

If $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + \vec{j} + 2\vec{k}$ then
find $|(\vec{a} \times \vec{b}) \times \vec{c}|$ and $|\vec{a} \times (\vec{b} \times \vec{c})|$

A: Given $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + \vec{j} + 2\vec{k}$

1) To find $(\vec{a} \times \vec{b}) \times \vec{c}$, first we have to find
 $\vec{a} \times \vec{b}$ (term in the bracket)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}(-2-3) - \vec{j}(1-6) + \vec{k}(1+4)$$

$$= -5\vec{i} + 5\vec{j} + 5\vec{k}$$

$$\therefore (\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 5 & 5 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \vec{i}(10-5) - \vec{j}(-10-5) + \vec{k}(-5-5)$$

$$= 5\vec{i} + 15\vec{j} - 10\vec{k} = 5(\vec{i} + 3\vec{j} - 2\vec{k})$$

$$\therefore |(\vec{a} \times \vec{b}) \times \vec{c}| = 5|\vec{i} + 3\vec{j} - 2\vec{k}|$$

$$= 5\sqrt{1^2 + 3^2 + (-2)^2} = 5\sqrt{1+9+4} = 5\sqrt{14}$$

2) To find $\vec{a} \times (\vec{b} \times \vec{c})$ first we have to find
 $\vec{b} \times \vec{c}$ (term in the bracket)

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \vec{i}(2-1) - \vec{j}(4-1) + \vec{k}(2-1) = \vec{i} - 3\vec{j} + \vec{k}$$

$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 1 & -3 & 1 \end{vmatrix}$$

$$= \vec{i}(-2+9) - \vec{j}(1-3) + \vec{k}(-3+2)$$

$$= 7\vec{i} + 2\vec{j} - \vec{k}$$

$$\therefore |\vec{a} \times (\vec{b} \times \vec{c})| = |7\vec{i} + 2\vec{j} - \vec{k}|$$

$$= \sqrt{7^2 + 2^2 + (-1)^2} = \sqrt{49+4+1} = \sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$$

- Find the shortest distance between the skew lines

$$\vec{r} = (6\vec{i} + 2\vec{j} + 2\vec{k}) + t(\vec{i} - 2\vec{j} + 2\vec{k}) \text{ and}$$

$$\vec{r} = (-4\vec{i} - \vec{k}) + s(3\vec{i} - 2\vec{j} - 2\vec{k})$$

A: Given skew lines $\vec{r} = (6\vec{i} + 2\vec{j} + 2\vec{k}) + t(\vec{i} - 2\vec{j} + 2\vec{k})$;

$$\vec{r} = (-4\vec{i} - \vec{k}) + s(3\vec{i} - 2\vec{j} - 2\vec{k})$$

Formula: For the skew lines $\vec{r} = \vec{a} + t\vec{b}$, $\vec{r} = \vec{c} + s\vec{d}$

$$\text{Shortest distance(SD)} = \frac{|(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|}$$

On comparing the given skew lines with

$$\vec{r} = \vec{a} + t\vec{b}, \vec{r} = \vec{c} + s\vec{d} \text{ we get}$$

$$\vec{a} = 6\vec{i} + 2\vec{j} + 2\vec{k}, \quad \vec{b} = \vec{i} - 2\vec{j} + 2\vec{k} \text{ and}$$

$$\vec{c} = -4\vec{i} - \vec{k}, \quad \vec{d} = 3\vec{i} - 2\vec{j} - 2\vec{k}$$

$$\text{So, } \vec{a} - \vec{c} = (6\vec{i} + 2\vec{j} + 2\vec{k}) - (-4\vec{i} - \vec{k}) = 10\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = \vec{i}(4+4) - \vec{j}(-2-6) + \vec{k}(-2+6) = 8\vec{i} + 8\vec{j} + 4\vec{k}$$

$$\therefore (\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d}) = (10\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (8\vec{i} + 8\vec{j} + 4\vec{k})$$

$$= 80 + 16 + 12 = 108$$

$$\text{Also, } |\vec{b} \times \vec{d}| = \sqrt{8^2 + 8^2 + 4^2} = \sqrt{64 + 64 + 16} = \sqrt{144} = 12$$

$$\therefore \text{Shortest distance(SD)} = \frac{|(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|} = \frac{108}{12} = 9$$

Q21 . TRANSFORMATIONS

- If A,B,C are angles of a triangle, then S.T
 $\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C$

Sol: Given A,B,C are angles of a triangle, then

$$A+B+C=180^\circ$$

$$\text{L.H.S} = \sin 2A - \sin 2B + \sin 2C$$

$$= 2\cos\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2A-2B}{2}\right) + \sin 2C$$

$$\left(\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right)$$

$$= 2\cos(A+B)\sin(A-B) + \sin 2C$$

$$= 2\cos(180^\circ - C)\sin(A-B) + \sin 2C$$

$$= -2\cos C \sin(A-B) + 2\sin C \cos C$$

$$[\because \sin 2\theta = 2\sin\theta \cos\theta]$$

$$= 2\cos C [\sin C - \sin(A-B)]$$

$$= 2\cos C [\sin(180^\circ - (A+B)) - \sin(A-B)]$$

$$= 2\cos C [\sin(A+B) - \sin(A-B)]$$

$$= 2\cos C (2\cos A \sin B)$$

$$[\because \sin(A+B) - \sin(A-B) = 2\cos A \sin B]$$

$$= 4\cos A \sin B \cos C = \text{R.H.S}$$

- If A, B, C are angles in a triangle, then prove that

$$\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \sin\frac{C}{2}$$

A: L.H.S = $(\cos A + \cos B) - \cos C$

$$= 2\cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2} - \left(1 - 2\sin^2\frac{C}{2}\right)$$

$$\left[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right); \cos\theta = 1 - 2\sin^2\frac{\theta}{2}\right]$$

$$= -1 + 2\cos\left(90^\circ - \frac{C}{2}\right)\cos\frac{A-B}{2} + 2\sin^2\frac{C}{2}$$

$$= -1 + 2\sin\frac{C}{2} \cdot \cos\frac{A-B}{2} + 2\sin^2\frac{C}{2}$$

$$= -1 + 2\sin\frac{C}{2} \left(\cos\frac{A-B}{2} + \sin\frac{C}{2}\right)$$

$$= -1 + 2\sin\frac{C}{2} \left[\cos\frac{A-B}{2} + \sin\left(90^\circ - \frac{A+B}{2}\right)\right]$$

$$= -1 + 2\sin\frac{C}{2} \left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)$$

$$(\because \sin(90^\circ - \theta) = \cos\theta)$$

$$= -1 + 2 \sin \frac{C}{2} \left(2 \cos \frac{A}{2} \cdot \cos \frac{B}{2} \right)$$

$$(\because \cos(A+B) + \cos(A-B) = 2 \cos A \cos B)$$

$$= -1 + 4 \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \sin \frac{C}{2} = \text{R.H.S}$$

- If $A+B+C=\pi$, then prove that

$$\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 \left(1 + \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

A: L.H.S = $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}$

$$= \cos^2 \frac{A}{2} + \left(1 - \sin^2 \frac{B}{2} \right) + \cos^2 \frac{C}{2}$$

$$= 1 + \left(\cos^2 \frac{A}{2} - \sin^2 \frac{B}{2} \right) + \cos^2 \frac{C}{2}$$

$$= 1 + \left(\cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right) + \left(1 - \sin^2 \frac{C}{2} \right)$$

$$\left[\because \cos^2 A - \sin^2 B = \cos(A+B) \cos(A-B) \right]$$

$$= 2 + \sin \frac{C}{2} \cos \left(\frac{A-B}{2} \right) - \sin^2 \frac{C}{2}$$

$$= 2 + \sin \frac{C}{2} \cos \left(\frac{A-B}{2} \right) - \sin^2 \frac{C}{2}$$

$$= 2 + \sin \frac{C}{2} \left(\cos \left(\frac{A-B}{2} \right) - \sin \frac{C}{2} \right)$$

$$= 2 + \sin \frac{C}{2} \left(\cos \left(\frac{A-B}{2} \right) - \cos \left(\frac{A+B}{2} \right) \right)$$

$$\left[\because \sin \frac{C}{2} = \cos \left(\frac{A+B}{2} \right) \right]$$

$$= 2 + \sin \frac{C}{2} \left(2 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \right)$$

$$\left[\because \cos(A-B) - \cos(A+B) = 2 \sin A \sin B \right]$$

$$= 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 2 \left(1 + \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) = \text{R.H.S}$$