

STAR QUESTIONS PLUS

SETS AND RELATIONS

- Write the set $B = \{1, 8, 27, 64, \dots\}$ in set-builder form.
- Let A be the given set, $A = \{1, 2, 3\}$.
Then find (i) $|A|$, (ii) $|P(A)|$, (iii) List all the subsets of A and (iv) the power set of A .

FUNCTIONS

- If $A = \{1, 2, 3, 4\}$ and $f: A \rightarrow R$ is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$ then find the range of f .

SEQUENCES AND SERIES

- Find the GM between $\frac{3}{2}$ and $\frac{27}{2}$.
- In a G.P. if the 5th term is 2, then find the product of the first 9 terms.
- If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

MATRICES

- Find the minors of -1 and 3 in the matrix $\begin{bmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{bmatrix}$
- If ω is a complex cube root of 1 then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$
- Without expanding the determinant, prove that (i) $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$
- Without expanding the determinant, prove that $\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$

11. Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

12. Find the rank of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

PRODUCT OF VECTORS

13. Prove that the angle θ between any two diagonals of a cube is given by $\cos\theta=1/3$

PROPERTIES OF TRIANGLES

14. If $(a+b+c)(b+c-a)=3bc$ then find angle A.
15. If $a=4$, $b=5$, $c=7$ then find $\cos B/2$
16. If $\tan \frac{A}{2} = \frac{5}{6}$ and $\tan \frac{C}{2} = \frac{2}{5}$ then determine the relation between a, b, c .
17. If $rr_2=r_1r_3$ then find B.
18. In $\triangle ABC$, show that the sides a, b, c are in A.P., if and only if r_1, r_2, r_3 are in H.P
19. If $a=13$, $b=14$, $c=15$, show that $R = \frac{65}{8}$, $r = 4$, $r_1 = \frac{21}{2}$, $r_2 = 12$, $r_3 = 14$
20. If $r_1 = 2$, $r_2 = 3$, $r_3 = 6$ and $r = 1$, prove that $a = 3$, $b = 4$ and $c = 5$.

SOLUTIONS

SETS AND RELATIONS

1. Write the set $B = \{1, 8, 27, 64, \dots\}$ in set-builder form.

Sol: The elements of B are cubes of natural numbers.

$$\therefore B = \{x / x = n^3, \text{ where } n \text{ is a positive integer}\}$$

2. Let A be the given set, $A = \{1, 2, 3\}$.

Then find (i) $|A|$, (ii) $|P(A)|$, (iii) List all the subsets of A and (iv) the power set of A .

Sol: The given set $A = \{1, 2, 3\}$ then

(i) $|A|$ = no. of elements in $A = 3$ (Here $|A|$ denotes the cardinality of a set)

(ii) $|P(A)| = 2^n = 2^3 = 8$

(iii) All the subsets of A : $\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

(iv) Power set of A is $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

FUNCTIONS

3. If $A = \{1, 2, 3, 4\}$ and $f: A \rightarrow R$ is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$ then find the range of f .

Sol: Given $A = \{1, 2, 3, 4\}$ and $f(x) = \frac{x^2 - x + 1}{x + 1}$

$$\text{Now, } f(1) = \frac{1^2 - 1 + 1}{1 + 1} = \frac{1}{2}; f(2) = \frac{2^2 - 2 + 1}{2 + 1} = 1;$$

$$f(3) = \frac{3^2 - 3 + 1}{3 + 1} = \frac{7}{4}; f(4) = \frac{4^2 - 4 + 1}{4 + 1} = \frac{13}{5}$$

$$\therefore \text{Range of } f \text{ is } f(A) = \left\{ \frac{1}{2}, 1, \frac{7}{4}, \frac{13}{5} \right\}$$

SEQUENCES AND SERIES

4. Find the GM between $\frac{3}{2}$ and $\frac{27}{2}$.

Sol: G.M between a and b = $\sqrt{ab}$

$$\therefore \text{GM between } \frac{3}{2} \text{ and } \frac{27}{2} = \sqrt{\frac{3}{2} \times \frac{27}{2}} = \frac{9}{2}$$

5. In a G.P. if the 5th term is 2, then find the product of the first 9 terms.

Sol: Let the first nine terms of G.P, be $a/r^4, a/r^3, a/r^2, a/r, a, ar, ar^2, ar^3, ar^4$

Given 5th term = $a=2$

$$\text{Product of first nine terms} = (a/r^4)(a/r^3)(a/r^2)(a/r)(a)(ar)(ar^2)(ar^3)(ar^4) = a^9 = 2^9$$

$\therefore$ Product of first 9 terms = 512

6. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Sol: Let the roots of the quadratic equation be a and b.

$$\text{AM of the roots is } 8 \Rightarrow \text{AM} = \frac{a+b}{2} = 8 \Rightarrow a+b = 16 \dots (i)$$

$$\text{GM of the roots is } 5 \Rightarrow \text{GM} = \sqrt{ab} = 5 \Rightarrow ab = 25 \dots (ii)$$

The quadratic equation with roots a,b is given by,

$$x^2 - x(\text{Sum of roots}) + (\text{Product of roots}) = 0 \Rightarrow x^2 - x(a+b) + (ab) = 0$$

$$\Rightarrow x^2 - 16x + 25 = 0 \text{ [Using (i) and (ii)]}$$

$\therefore$ the required quadratic equation is $x^2 - 16x + 25 = 0$

MATRICES

7. Find the minors of -1 and 3 in the matrix

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{bmatrix}$$

Sol:

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{bmatrix}$$

$$\text{Minor of } -1 = \begin{vmatrix} 0 & 5 \\ -3 & 3 \end{vmatrix} = 0 + 15 = 15; \quad \text{Minor of } 3 = \begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} = -4 + 0 = -4$$

8. If ω is a complex cube root of 1 then show that

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$$

Sol:

$$\begin{aligned} \text{LHS} &= \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad (\because R_1 \rightarrow R_1+R_2+R_3) \\ &= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 = \text{R.H.S.} \quad (\because 1+\omega+\omega^2=0) \end{aligned}$$

9. Without expanding the determinant, prove that (i)

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$\begin{aligned} \text{Sol: L.H.S} &= \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \frac{1}{a} \frac{1}{b} \frac{1}{c} \begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} \begin{matrix} R_1(a) \\ R_2(b) \\ R_3(c) \end{matrix} = \frac{1}{abc} (abc) \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \quad \begin{matrix} \text{Taking out} \\ \text{abc from } C_3 \end{matrix} \\ &= - \begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix} (\because c_1 \leftrightarrow c_3) = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} (\because c_2 \leftrightarrow c_3) = \text{R.H.S} \end{aligned}$$

10. Without expanding the determinant , prove that

$$\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

Sol : L.H.S = $\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ \frac{x}{x} & \frac{y}{y} & \frac{z}{z} \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ x & y & z \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ \frac{xyz}{x} & \frac{xyz}{y} & \frac{xyz}{z} \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \text{R.H.S}$

11. Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow |A| = 1(6-4) - 2(4-0) + 3(2-0) = 2 - 8 + 6 = 0 \quad |A| = 0.$

Hence we consider a 2×2 minor, $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0 \quad \therefore \text{Rank}(A) = 2.$

12. Find the rank of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol: Any two rows of the given matrix are identical .

$\therefore$ Given matrix of order 3 is singular $\Rightarrow \text{Rank} < 3$

Similarly, every square 2×2 sub matrix is singular $\Rightarrow \text{Rank} < 2$

But all the elements of the matrix are non-zeros

$\therefore \text{Rank}(A) = 1$

PRODUCT OF VECTORS

13. Prove that the angle θ between any two diagonals of a cube is given by $\cos\theta=1/3$

Sol: Consider a unit cube with vertices O, A, B, C, L, M, N, P as shown in the diagram.

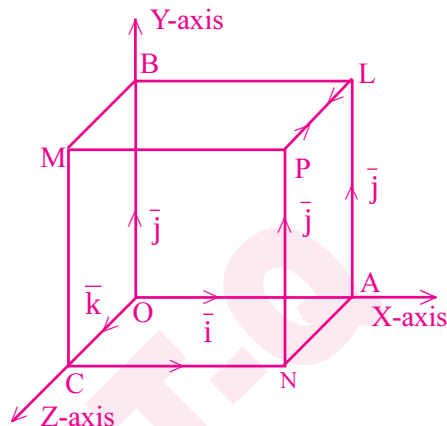
We take $\overrightarrow{OA} = \vec{i}$, $\overrightarrow{OB} = \vec{j}$, $\overrightarrow{OC} = \vec{k}$

Consider 2 diagonals $\overrightarrow{OP}$ and $\overrightarrow{CL}$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AL} + \overrightarrow{LP} = \vec{i} + \vec{j} + \vec{k} \quad \dots\dots(1)$$

$$\overrightarrow{CL} = \overrightarrow{CN} + \overrightarrow{NP} + \overrightarrow{PL} = \vec{i} + \vec{j} - \vec{k} \quad \dots\dots(2)$$

$$\begin{aligned} \therefore \cos\theta &= \frac{\overrightarrow{OP} \cdot \overrightarrow{CL}}{|\overrightarrow{OP}| |\overrightarrow{CL}|} \\ &= \frac{(1 \times 1) + (1 \times 1) + (1 \times (-1))}{\sqrt{(1^2 + 1^2 + 1^2)} \sqrt{(1^2 + 1^2 + (-1)^2)}} = \frac{1+1-1}{\sqrt{(3)} \sqrt{(3)}} = \frac{1}{3} \end{aligned}$$

**PROPERTIES OF TRIANGLES**

14. If $(a+b+c)(b+c-a)=3bc$ then find angle A.

Sol: Given that $(a+b+c)(b+c-a)=3bc \Rightarrow 2s[2(s-a)]=3bc \Rightarrow \frac{s(s-a)}{bc} = \frac{3}{4}$

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{3}{4} \Rightarrow \cos \frac{A}{2} = \frac{\sqrt{3}}{2} = \cos 30^\circ \therefore \frac{A}{2} = 30^\circ \Rightarrow A = 60^\circ$$

15. If $a=4$, $b=5$, $c=7$ then find $\cos B/2$

Sol: $2s=a+b+c=4+5+7=16 \Rightarrow s=8$

$$\therefore \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{8(8-5)}{7(4)}} = \sqrt{\frac{6}{7}}$$

16. If $\tan \frac{A}{2} = \frac{5}{6}$ and $\tan \frac{C}{2} = \frac{2}{5}$ then determine the relation between a, b, c .

Sol: Consider the product, $\tan \frac{A}{2} \cdot \tan \frac{C}{2} = \frac{s-b}{s} \cdot \frac{s-a}{s} = \frac{s-b}{s} \cdot \frac{s-a}{s} = \frac{1}{3}$

$$\Rightarrow \frac{s-b}{s} = \frac{1}{3} \Rightarrow 3s-3b=s \Rightarrow 2s=3b \Rightarrow a+b+c=3b \Rightarrow a+c=2b \Rightarrow a, b, c \text{ are in A.P}$$

17. If $rr_2=r_1r_3$ then find B.

Sol: Given that $rr_2=r_1r_3 \Rightarrow \frac{\Delta}{s} \cdot \frac{\Delta}{(s-b)} = \frac{\Delta}{(s-a)} \cdot \frac{\Delta}{(s-c)} \Rightarrow \frac{1}{s(s-b)} = \frac{1}{(s-a)(s-c)}$

$$\Rightarrow \frac{(s-a)(s-c)}{s(s-b)} = 1 \Rightarrow \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = 1$$

18. In ΔABC , show that the sides a, b, c are in A.P., if and only if r_1, r_2, r_3 are in H.P

Sol: r_1, r_2, r_3 are in H.P $\Leftrightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P

$$\Leftrightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta} \text{ are in A.P} \Leftrightarrow s-a, s-b, s-c \text{ are in A.P}$$

$$\Leftrightarrow -a, -b, -c \text{ are in A.P} \Leftrightarrow a, b, c \text{ are in A.P}$$

19. If $a=13, b=14, c=15$, show that $R = \frac{65}{8}, r = 4, r_1 = \frac{21}{2}, r_2 = 12, r_3 = 14$

Sol: Given that $a=13, b=14, c=15$

we know that $s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21.(21-13)(21-14)(21-15)}$

$$= \sqrt{21.(8)(7)(6)} = \sqrt{(7.3)(4.2)(7)(3.2)} = \sqrt{7^2 \cdot 3^2 \cdot 4^2} = 7.3.4 = 84$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4; r_1 = \frac{\Delta}{s-a} = \frac{84}{21-13} = \frac{84}{8} = \frac{21}{2}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{84}{21-14} = \frac{84}{7} = 12; r_3 = \frac{\Delta}{s-c} = \frac{84}{21-15} = \frac{84}{6} = 14$$

$$R = \frac{abc}{4\Delta} = \frac{13.14.15}{4.84} = \frac{65}{8}$$

20. If $r_1 = 2$, $r_2 = 3$, $r_3 = 6$ and $r = 1$, prove that $a = 3$, $b = 4$ and $c = 5$.

Sol: We know that $\Delta = \sqrt{rr_1r_2r_3} \Rightarrow \Delta = \sqrt{1 \times 2 \times 3 \times 6} = \sqrt{36} = 6$

$$\text{Now } r=1 \Rightarrow \frac{\Delta}{s} = 1 \Rightarrow s = \Delta = 6$$

$$r_1 = \frac{\Delta}{s-a} \Rightarrow s-a = \frac{\Delta}{r_1} = \frac{6}{2} = 3 \Rightarrow a = s-3 = 6-3 = 3$$

$$r_2 = \frac{\Delta}{s-b} \Rightarrow s-b = \frac{\Delta}{r_2} = \frac{6}{3} = 2 \Rightarrow b = s-2 = 6-2 = 4$$

$$r_3 = \frac{\Delta}{s-c} \Rightarrow s-c = \frac{\Delta}{r_3} = \frac{6}{6} = 1 \Rightarrow c = s-1 = 6-1 = 5$$

$$\Rightarrow \tan\left(\frac{B}{2}\right) = 1 = \tan 45^\circ \Rightarrow \frac{B}{2} = 45^\circ \Rightarrow B = 90^\circ = \frac{\pi}{2}$$