

LAQ PQ QUESTIONS

1. If $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{c} = 4\vec{i} + 5\vec{j} - 2\vec{k}$ and $\vec{d} = \vec{i} + 3\vec{j} + 5\vec{k}$ then compute the following i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and ii) $(\vec{a} \times \vec{b}) \cdot \vec{c} - (\vec{a} \times \vec{d}) \cdot \vec{b}$
2. If A,B,C are angles of a triangle, prove that $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$
3. If A,B,C are angles of a triangle, prove that $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$
4. If A, B, C are angles in a triangle, then prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$

SAQ PQ QUESTIONS

FUNCTIONS

1. If $f = \{(4,5), (5,6), (6,-4)\}$; $g = \{(4,-4), (6,5), (8,5)\}$ find
(i) $f+g$ (iii) $2f+4g$ (v) fg (ix) f^2

MATHEMATICAL INDUCTION

2. Show that $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \text{upto } n \text{ terms} = \frac{n}{3n+1} \quad \forall n \in \mathbb{N}$

MATRICES

3. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$
4. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$
5. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then show that $\text{adj } A = 3A'$. Also find A^{-1}

ADDITION OF VECTORS

6. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then test for collinearity of the following points whose position vectors are given by $3\vec{a} - 4\vec{b} + 3\vec{c}, -4\vec{a} + 5\vec{b} - 6\vec{c}, 4\vec{a} - 7\vec{b} + 6\vec{c}$
7. Show that the points whose P.V are $-2\vec{a} + 3\vec{b} + 5\vec{c}, \vec{a} + 2\vec{b} + 3\vec{c}, 7\vec{a} - \vec{c}$ are collinear, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors.
8. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then prove that the points with position vectors $2\vec{a} + 3\vec{b} - \vec{c}, \vec{a} - 2\vec{b} + 3\vec{c}, 3\vec{a} + 4\vec{b} - 2\vec{c}, \vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.
9. Find t , for which the vectors $2\vec{i} - 3\vec{j} + \vec{k}, \vec{i} + 2\vec{j} - 3\vec{k}, \vec{j} - t\vec{k}$ are coplanar.
10. For non coplanar vectors, $\vec{a}, \vec{b}$ and $\vec{c}$ determine p for which the vector $\vec{a} + \vec{b} + \vec{c}, \vec{a} + p\vec{b} + 2\vec{c}$ and $-\vec{a} + \vec{b} + \vec{c}$ are coplanar.
11. $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors. Prove that the following four points are coplanar: (ii) $6\vec{a} + 2\vec{b} - \vec{c}, 2\vec{a} - \vec{b} + 3\vec{c}, -\vec{a} + 2\vec{b} - 4\vec{c}, -12\vec{a} - \vec{b} - 3\vec{c}$

PRODUCT OF VECTORS

12. Find a unit vector perpendicular to the plane containing the vectors $\vec{a} = 4\vec{i} + 3\vec{j} - \vec{k}, \vec{b} = 2\vec{i} - 6\vec{j} - 3\vec{k}$
13. Find the volume of the parallelopiped formed by the coterminous edges $\vec{i} + \vec{j} + \vec{k}, \vec{i} - \vec{j}$ and $\vec{i} + 2\vec{j} - \vec{k}$

TRIGONOMETRIC EQUATIONS

14. Solve $\sin\theta + \sin 5\theta = \sin 3\theta, 0 < \theta < \pi$.
15. Find the general solution of $5\cos^2\theta + 7\sin^2\theta = 6$
16. Find the general solution of $2\cos^2\theta + 11\sin\theta = 7$
17. Solve $\tan\theta + \sec\theta = \sqrt{3}, 0 \leq \theta \leq 2\pi$

INVERSE TRIGONOMETRIC FUNCTIONS

18. Find the value of $\sin\left(\cos^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13}\right)$
19. Find the value of $\tan\left(\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{5}{\sqrt{34}}\right)$
20. Find $\cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right)$
21. Prove that $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

22. Prove that $\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19} = \frac{\pi}{4}$

23. Prove that $\sin^{-1} \frac{4}{5} + 2\tan^{-1} \frac{1}{3} = \frac{\pi}{2}$

24. Prove that $\cos\left(2\tan^{-1} \frac{1}{7}\right) = \sin\left(2\tan^{-1} \frac{3}{4}\right)$

VSAQ PQ QUESTIONS

SEQUENCES & SERIES

- Find the 8th term of the A.P. 3, 5, 7, 9,.....
- Find the sum of $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \dots \infty$
- Find the sum $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \dots \text{to } \infty$
- The arithmetic mean between two positive numbers is 34 and their geometric mean is 16. find the numbers.

MATRICES

- Find the trace of $\begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$
- If $\begin{bmatrix} x-1 & 2 & y-5 \\ z & 0 & 2 \\ 1 & -1 & 1+a \end{bmatrix} = \begin{bmatrix} 1-x & 2 & -y \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ then find the values of x,y,z & a.
- If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, find A^2 .
- Find the product $\begin{bmatrix} 2 & 1 & 4 \\ 6 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- If $A = \begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 8 \\ 2 & -8 & x \end{bmatrix}$ is a skew symmetric matrix, find the value of x

ADDITION OF VECTORS

- Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} + \vec{j}$. Find a unit vector in the direction of $\vec{a} + \vec{b}$

LAQ PQ SOLUTIONS

PRODUCT OF VECTORS

1. If $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{c} = 4\vec{i} + 5\vec{j} - 2\vec{k}$ and $\vec{d} = \vec{i} + 3\vec{j} + 5\vec{k}$ then compute the following i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and ii) $(\vec{a} \times \vec{b}) \cdot \vec{c} - (\vec{a} \times \vec{d}) \cdot \vec{b}$

Sol: i) To find $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

Given that $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j} + 2\vec{k}$, $\vec{c} = 4\vec{i} + 5\vec{j} - 2\vec{k}$ and $\vec{d} = \vec{i} + 3\vec{j} + 5\vec{k}$

$$\text{Now } \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -1 & 3 & 2 \end{vmatrix} = \vec{i}(-2-6) - \vec{j}(6+2) + \vec{k}(9-1) = -8\vec{i} - 8\vec{j} + 8\vec{k} = 8(-\vec{i} - \vec{j} + \vec{k})$$

$$\text{Also, } \vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & -2 \\ 1 & 3 & 5 \end{vmatrix} = \vec{i}(25+6) - \vec{j}(20+2) + \vec{k}(12-5) = 31\vec{i} - 22\vec{j} + 7\vec{k}$$

Thus we have $\vec{a} \times \vec{b} = 8(-\vec{i} - \vec{j} + \vec{k})$ and $\vec{c} \times \vec{d} = 31\vec{i} - 22\vec{j} + 7\vec{k}$. Then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 8 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 1 \\ 31 & -22 & 7 \end{vmatrix} = 8[\vec{i}(-7+22) - \vec{j}(-7-31) + \vec{k}(22+31)]$$

$$\therefore (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 8(15\vec{i} + 38\vec{j} + 53\vec{k})$$

ii) To find $(\vec{a} \times \vec{b}) \cdot \vec{c} - (\vec{a} \times \vec{d}) \cdot \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -1 & 3 & 2 \end{vmatrix} = \vec{i}(-2-6) - \vec{j}(6+2) + \vec{k}(9-1) = -8\vec{i} - 8\vec{j} + 8\vec{k} = 8(-\vec{i} - \vec{j} + \vec{k})$$

$$\text{Now } (\vec{a} \times \vec{b}) \cdot \vec{c} = 8(-\vec{i} - \vec{j} + \vec{k}) \cdot (4\vec{i} + 5\vec{j} - 2\vec{k}) = 8(-4 - 5 - 2) = -88 \dots (1)$$

$$\vec{a} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 1 & 3 & 5 \end{vmatrix} = \vec{i}(-5-6) - \vec{j}(15-2) + \vec{k}(9+1) = -11\vec{i} - 13\vec{j} + 10\vec{k}$$

$$\text{Also } (\vec{a} \times \vec{d}) \cdot \vec{b} = (-11\vec{i} - 13\vec{j} + 10\vec{k}) \cdot (-\vec{i} + 3\vec{j} + 2\vec{k}) = 11 - 39 + 20 = -8 \dots (2)$$

$$\therefore \text{From (1) \& (2) } (\vec{a} \times \vec{b}) \cdot \vec{c} - (\vec{a} \times \vec{d}) \cdot \vec{b} = (-88) - (-8) = -80$$

TRIGONOMETRIC RATIOS

2. If A,B,C are angles of a triangle, prove that

$$\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$$

Sol: L.H.S = $\sin 2A + \sin 2B - \sin 2C$

$$= 2 \sin \left(\frac{2A + 2B}{2} \right) \cos \left(\frac{2A - 2B}{2} \right) - \sin 2C \quad \left[\because \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \right]$$

$$= 2 \sin(A+B) \cos(A-B) - \sin 2C$$

$$= 2 \sin(180^\circ - C) \cos(A-B) - \sin 2C \quad [\because A+B+C=180^\circ]$$

$$= 2 \sin C \cos(A-B) - 2 \sin C \cos C \quad [\because \sin 2\theta = 2 \sin \theta \cos \theta]$$

$$= 2 \sin C [\cos(A-B) - \cos C]$$

$$= 2 \sin C [\cos(A-B) + \cos(A+B)]$$

$$= 2 \sin C (2 \cos A \cos B) \quad [\because \cos(A-B) + \cos(A+B) = 2 \cos A \cos B]$$

$$= 4 \cos A \cos B \sin C = \text{R.H.S}$$

3. If A,B,C are angles of a triangle, prove that $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$

Sol: Given A,B,C are angles of a triangle, then $A+B+C=180^\circ \Rightarrow \frac{A+B+C}{2} = 90^\circ$

$$\text{L.H.S} = \sin A + \sin B - \sin C = (\sin A + \sin B) - \sin C$$

$$= 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) - \sin C \quad \left[\because \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \right]$$

$$= 2 \cos \frac{C}{2} \cos \left(\frac{A-B}{2} \right) - \sin C \quad \left[\because \frac{A+B}{2} = 90^\circ - \frac{C}{2}, \sin \left(90^\circ - \frac{C}{2} \right) = \cos \frac{C}{2} \right]$$

$$= 2 \cos \frac{C}{2} \cos \left(\frac{A-B}{2} \right) - 2 \sin \frac{C}{2} \cos \frac{C}{2} \quad \left[\because \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right]$$

$$= 2 \cos \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \sin \frac{C}{2} \right] = 2 \cos \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \cos \left(\frac{A+B}{2} \right) \right] \quad \left[\because \sin \frac{C}{2} = \cos \left(90^\circ - \frac{C}{2} \right) = \cos \frac{A+B}{2} \right]$$

$$= 2 \cos \frac{C}{2} \left[2 \sin \frac{A}{2} \sin \frac{B}{2} \right] \quad [\because \cos(A-B) - \cos(A+B) = 2 \sin A \sin B]$$

$$= 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} = \text{RHS}$$

4. If A, B, C are angles in a triangle, then prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$

Sol: Given A, B, C are angles of a triangle, then $A + B + C = 180^\circ$

$$\begin{aligned}
 \text{L.H.S} &= \sin^2 A + (\sin^2 B - \sin^2 C) \\
 &= \sin^2 A + \sin(B+C) \sin(B-C) \\
 &= \sin^2 A + \sin(180^\circ - A) \sin(B-C) \\
 &= \sin^2 A + \sin A \sin(B-C) = \sin A [\sin A + \sin(B-C)] \\
 &= \sin A [\sin(180^\circ - (B+C)) + \sin(B-C)] \\
 &= \sin A [\sin(B+C) + \sin(B-C)] \\
 &= \sin A (2 \sin B \cos C) \quad [\because (\sin(A+B)) + \sin(A-B) = 2 \sin A \cos B] \\
 &= 2 \sin A \sin B \cos C = \text{R.H.S}
 \end{aligned}$$

SAQ PQ SOLUTIONS

FUNCTIONS

1. If $f=\{(4,5),(5,6),(6,-4)\}; g=\{(4,-4), (6,5), (8,5)\}$ find

(i) $f+g$ (iii) $2f+4g$ (v) fg (ix) f^2

Sol: Here Domain of f is $A=\{4,5,6\}$ and Domain of g is $B=\{4,6,8\}$

Domain of $f \pm g = A \cap B = \{4,6\}$. So, we have to consider only the two preimages 4,6.

$$(i) f+g = \{(4,5-4), (6,-4+5)\} = \{(4,1), (6,1)\}$$

$$(iii) 2f = \{(4,2(5)), (5,2(6)), (6,2(-4))\} = \{(4,10), (5,12), (6,-8)\}$$

$$4g = \{(4,4(-4)), (6,4(5)), (8,4(5))\} = \{(4,-16), (6,20), (8,20)\}$$

$$\therefore 2f+4g = \{(4,10-16), (6,-8+20)\} = \{(4,-6), (6,12)\} \quad [\because \text{Domain of } f \pm g = A \cap B = \{4,6\}]$$

$$(v) fg = \{(4,(5)(-4)), (6,(-4)(5))\} = \{(4,-20), (6,-20)\}$$

$$(ix) \text{Domain of } f^2 = A = \{(4,25), (5,36), (6,16)\}$$

MATHEMATICAL INDUCTION

2. **Show that** $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \text{upto } n \text{ terms} = \frac{n}{3n+1} \quad \forall n \in \mathbb{N}$

Sol: The first terms of the series 1,4,7... are in A.P with $a=1, d=3$

$$\therefore T_n = a + (n-1)d \Rightarrow T_n = 1 + (n-1)3 = 1 + 3n - 3 = 3n - 2$$

The second terms of the series 4,7... are in A.P with $a=4, d=3 \therefore T_n = 4 + (n-1)3 = 3n + 1$

Hence, the n^{th} term of the given series is $T_n = \frac{1}{(3n-2)(3n+1)}$

$$\text{Let } S(n) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

Step-1: (a) L.H.S of $S(1) = \frac{1}{1.4} = \frac{1}{4}$; R.H.S of $S(1) = \frac{1}{3.1+1} = \frac{1}{4}$

$\therefore$ L.H.S of $S(1) = \text{R.H.S of } S(1) \Rightarrow S(1)$ is true

Step-2: Assume that $S(k)$ is true for $k \in \mathbb{N}$

$$S(k) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \dots(1)$$

Step-3: We show that $S(k+1)$ is true

$$S(k+1) : \left(\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right) + \frac{1}{(3(k+1)-2)(3(k+1)+1)} = \frac{k+1}{3(k+1)+1}$$

$$\text{Now, L.H.S of } S(k+1) = \left[\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right] + \frac{1}{(3(k+1)-2)(3(k+1)+1)}$$

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}, [\text{From (1)}] = \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2 + 4k + 1}{(3k+1)(3k+4)}$$

$$\Rightarrow \frac{(k+1)(3k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} = \frac{k+1}{3k+3+1} = \frac{k+1}{3(k+1)+1} = \text{R.H.S of } S(k+1)$$

$\therefore$ L.H.S of $S(k+1) = \text{R.H.S of } S(k+1) \Rightarrow S(k+1)$ is true whenever $S(k)$ is true

Hence, by the principle of Mathematical induction, the given statement is true $\forall n \in \mathbb{N}$

MATRICES

3. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

Sol: $\det A = 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1(1-0) - 0 + 2(4-3) = 1 + 2 = 3 \neq 0 \Rightarrow A \text{ is non-singular}$

$$\text{Cofactor matrix of } A = \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -5 & -2 \\ -2 & 4 & 1 \end{bmatrix}$$

$$\therefore \text{Adj } A = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det A} (\text{Adj } A) = \frac{1}{3} \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 4/3 & -2/3 \\ -2/3 & -5/3 & 4/3 \\ 1/3 & -2/3 & 1/3 \end{bmatrix}$$

4. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

Sol: $\det A = 2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2(0-2) - 1(1-2) + 2(2-0) = -4 + 1 + 4 = 1 \neq 0$

$\Rightarrow A$ is non-singular

$$\text{Cofactor matrix of } A = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -2 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \text{Adj } A = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det A} (\text{Adj } A) = \frac{1}{1} \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

5. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then show that $\text{adj } A = 3A'$. Also find A^{-1}

Sol: Given that $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \det A = -1 \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix}$

$$= -(1-4) + 2(2+4) - 2(-4-2) = 3 + 2(6) + 2(6) = 3 + 12 + 12 = 27$$

Cofactor matrix of A

$$= \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} \\ \begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} \Rightarrow \text{Adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \text{-----(1)}$$

$$\text{Also, } A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \Rightarrow 3A^T = 3 \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \text{---(2)}$$

$\therefore$ from (1) & (2); $\text{Adj } A = 3A^T$

$$\therefore A^{-1} = \frac{1}{\det A} (\text{Adj } A) = \frac{1}{27} \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} = \frac{3}{27} \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

ADDITION OF VECTORS

6. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then test for collinearity of the following points whose position vectors are given by $3\vec{a} - 4\vec{b} + 3\vec{c}, -4\vec{a} + 5\vec{b} - 6\vec{c}, 4\vec{a} - 7\vec{b} + 6\vec{c}$

Let $\vec{OP} = 3\vec{a} - 4\vec{b} + 3\vec{c}$, $\vec{OQ} = -4\vec{a} + 5\vec{b} - 6\vec{c}$, $\vec{OR} = 4\vec{a} - 7\vec{b} + 6\vec{c}$. Then

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (-4\vec{a} + 5\vec{b} - 6\vec{c}) - (3\vec{a} - 4\vec{b} + 3\vec{c}) = -7\vec{a} + 9\vec{b} - 9\vec{c};$$

$$\vec{PR} = \vec{OR} - \vec{OP} = (4\vec{a} - 7\vec{b} + 6\vec{c}) - (3\vec{a} - 4\vec{b} + 3\vec{c}) = \vec{a} - 3\vec{b} + 3\vec{c}$$

$\Rightarrow \vec{PQ} = t\vec{PR}$ is not possible for any scalar $t \in \mathbb{R}$ $\therefore$ The given points are not collinear.

7. Show that the points whose P.V are $-2\vec{a} + 3\vec{b} + 5\vec{c}, \vec{a} + 2\vec{b} + 3\vec{c}, 7\vec{a} - \vec{c}$ are collinear, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors.

Sol: P.Vs of P,Q,R are $\vec{OP} = -2\vec{a} + 3\vec{b} + 5\vec{c}$, $\vec{OQ} = \vec{a} + 2\vec{b} + 3\vec{c}$, $\vec{OR} = 7\vec{a} - \vec{c}$. Then

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (\vec{a} + 2\vec{b} + 3\vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c}) = 3\vec{a} - \vec{b} - 2\vec{c}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = (7\vec{a} - \vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c}) = 9\vec{a} - 3\vec{b} - 6\vec{c} = 3(3\vec{a} - \vec{b} - 2\vec{c}) = 3\vec{PQ}$$

$\therefore \vec{PR} = 3\vec{PQ}$, which is in the form $\vec{PR} = t\vec{PQ}$, $t \in \mathbb{R}$

Hence the vectors $\vec{PQ}, \vec{PR}$ are collinear $\Rightarrow$ the points P,Q,R are collinear.

8. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then prove that the points with position vectors $2\vec{a} + 3\vec{b} - \vec{c}, \vec{a} - 2\vec{b} + 3\vec{c}, 3\vec{a} + 4\vec{b} - 2\vec{c}, \vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Sol: Let $\vec{OP} = 2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{OQ} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\vec{OR} = 3\vec{a} + 4\vec{b} - 2\vec{c}$, $\vec{OS} = \vec{a} - 6\vec{b} + 6\vec{c}$ Then

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (\vec{a} - 2\vec{b} + 3\vec{c}) - (2\vec{a} + 3\vec{b} - \vec{c}) = -\vec{a} - 5\vec{b} + 4\vec{c}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = (3\vec{a} + 4\vec{b} - 2\vec{c}) - (2\vec{a} + 3\vec{b} - \vec{c}) = \vec{a} + \vec{b} - \vec{c}$$

$$\vec{PS} = \vec{OS} - \vec{OP} = (\vec{a} - 6\vec{b} + 6\vec{c}) - (2\vec{a} + 3\vec{b} - \vec{c}) = -\vec{a} - 9\vec{b} + 7\vec{c}$$

$$\text{Now, } [\vec{PQ} \vec{PR} \vec{PS}] = \begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ -1 & -9 & 7 \end{vmatrix} [\vec{a} \vec{b} \vec{c}]$$

$$= [-1(7-9) + 5(7-1) + 4(-9+1)] [\vec{a} \vec{b} \vec{c}] = [2+30-32] [\vec{a} \vec{b} \vec{c}] = 0 [\vec{a} \vec{b} \vec{c}] = 0$$

Hence, the 3 vectors $\vec{PQ}, \vec{PR}, \vec{PS}$ are coplanar.

$\therefore$ The four points P,Q,R,S are coplanar.

9. Find t , for which the vectors $2\bar{i} - 3\bar{j} + \bar{k}$, $\bar{i} + 2\bar{j} - 3\bar{k}$, $\bar{j} - t\bar{k}$ are coplanar.

Sol: Let $\bar{a} = 2\bar{i} - 3\bar{j} + \bar{k}$, $\bar{b} = \bar{i} + 2\bar{j} - 3\bar{k}$, $\bar{c} = \bar{j} - t\bar{k}$

If $\bar{a}, \bar{b}, \bar{c}$ are coplanar then $[\bar{a}, \bar{b}, \bar{c}] = 0$

$$\Rightarrow \begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 0 & 1 & -t \end{vmatrix} = 0 \Rightarrow 2(-2t + 3) + 3(-t) + (1) = 0 \Rightarrow -7t = -7 \Rightarrow t = 1$$

10. For non coplanar vectors, $\bar{a}, \bar{b}$ and $\bar{c}$ determine p for which the vector $\bar{a} + \bar{b} + \bar{c}$, $\bar{a} + p\bar{b} + 2\bar{c}$ and $-\bar{a} + \bar{b} + \bar{c}$ are coplanar.

Sol: Given that $\bar{a} + \bar{b} + \bar{c}$, $\bar{a} + p\bar{b} + 2\bar{c}$, $-\bar{a} + \bar{b} + \bar{c}$ are co-planar.

$$\Rightarrow [\bar{a} + \bar{b} + \bar{c} \quad \bar{a} + p\bar{b} + 2\bar{c} \quad -\bar{a} + \bar{b} + \bar{c}] = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & p & 2 \\ -1 & 1 & 1 \end{vmatrix} [\bar{a}\bar{b}\bar{c}] = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & p & 2 \\ -1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow 1(p-2) - 1(1+2) + 1(1+p) = 0$$

$$\Rightarrow p - 2 - 3 + 1 + p = 0 \Rightarrow 2p = 4 \Rightarrow p = 2$$

11. $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors. Prove that the following four points are coplanar. (ii)
 $6\bar{a} + 2\bar{b} - \bar{c}$, $2\bar{a} - \bar{b} + 3\bar{c}$, $-\bar{a} + 2\bar{b} - 4\bar{c}$, $-12\bar{a} - \bar{b} - 3\bar{c}$

Sol: $\overline{OP} = 6\bar{a} + 2\bar{b} - \bar{c}$, $\overline{OQ} = 2\bar{a} - \bar{b} + 3\bar{c}$, $\overline{OR} = -\bar{a} + 2\bar{b} - 4\bar{c}$, $\overline{OS} = -12\bar{a} - \bar{b} - 3\bar{c}$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = (2\bar{a} - \bar{b} + 3\bar{c}) - (6\bar{a} + 2\bar{b} - \bar{c}) = -4\bar{a} - 3\bar{b} + 4\bar{c}$$

$$\overline{PR} = \overline{OR} - \overline{OP} = (-\bar{a} + 2\bar{b} - 4\bar{c}) - (6\bar{a} + 2\bar{b} - \bar{c}) = -7\bar{a} - 3\bar{c}$$

$$\overline{PS} = \overline{OS} - \overline{OP} = (-12\bar{a} - \bar{b} - 3\bar{c}) - (6\bar{a} + 2\bar{b} - \bar{c}) = -18\bar{a} - 3\bar{b} - 2\bar{c}$$

Let us suppose that one vector can be expressed as a linear combination of other two.

$$\text{Let } x(\overline{PR}) + y(\overline{PS}) = \overline{PQ} \Rightarrow x(-7\bar{a} - 3\bar{c}) + y(-18\bar{a} - 3\bar{b} - 2\bar{c}) = -4\bar{a} - 3\bar{b} + 4\bar{c}$$

$$\Rightarrow (-7x - 18y)\bar{a} - 3y\bar{b} + (-3x - 2y)\bar{c} = -4\bar{a} - 3\bar{b} + 4\bar{c}$$

Now we equate the coefficients of $\bar{a}, \bar{b}, \bar{c}$ as $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors

$$-7x - 18y = -4 \Rightarrow 7x + 18y = 4 \dots\dots(1), \quad -3y = -3 \Rightarrow y = 1 \dots\dots(2),$$

$$-3x - 2y = 4 \Rightarrow 3x + 2y = -4 \dots\dots(3)$$

$$\text{Put } y = 1 \text{ in (3) then } 3x + 2(1) = -4 \Rightarrow 3x = -6 \Rightarrow x = -2$$

$$\text{Now putting } x = -2 \text{ and } y = 1 \text{ in (1) we have } 7(-2) + 18(1) - 4 = 0.$$

So $\overline{PQ}, \overline{PR}, \overline{PS}$ are coplanar. Hence the given points P, Q, R, S are coplanar.

$\Rightarrow$ The points A, B, C, D are coplanar

PRODUCT OF VECTORS

12. Find a unit vector perpendicular to the plane containing the vectors

$$\vec{a} = 4\vec{i} + 3\vec{j} - \vec{k}, \vec{b} = 2\vec{i} - 6\vec{j} - 3\vec{k}$$

Sol: $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix} = \vec{i}(-9-6) - \vec{j}(-12+2) + \vec{k}(-24-6) = -15\vec{i} + 10\vec{j} - 30\vec{k} = 5(-3\vec{i} + 2\vec{j} - 6\vec{k})$

$$\Rightarrow |\vec{a} \times \vec{b}| = 5\sqrt{9+4+36} = 5\sqrt{49} = 5 \times 7 = 35$$

$\therefore$ Unit vector perpendicular to $\vec{a}, \vec{b}$ is

$$\pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \pm \left(\frac{5(-3\vec{i} + 2\vec{j} - 6\vec{k})}{35} \right) = \pm \left(\frac{3\vec{i} - 2\vec{j} + 6\vec{k}}{7} \right)$$

13. Find the volume of the parallelopiped formed by the coterminous edges $\vec{i} + \vec{j} + \vec{k}$, $\vec{i} - \vec{j}$ and $\vec{i} + 2\vec{j} - \vec{k}$

Sol: Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - \vec{j}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$. Then

$$[\vec{a} \ \vec{b} \ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix} = 1(1-0) - 1(-1-0) + 1(2+1) = 1+1+3 = 5$$

$$\therefore \text{Volume of the parallelopiped} = |[\vec{a} \ \vec{b} \ \vec{c}]| = |5| = 5 \text{ cu. units}$$

TRIGONOMETRIC EQUATIONS

14. Solve $\sin\theta + \sin 5\theta = \sin 3\theta$, $0 < \theta < \pi$.

Sol: Given that $\sin\theta + \sin 5\theta = \sin 3\theta \Rightarrow \sin\theta + (\sin 5\theta - \sin 3\theta) = 0$

$$\Rightarrow \sin\theta + 2\cos\left(\frac{5\theta+3\theta}{2}\right)\sin\left(\frac{5\theta-3\theta}{2}\right) = 0$$

$$\Rightarrow \sin\theta + 2\cos 4\theta \cdot \sin\theta = 0 \Rightarrow \sin\theta(1 + 2\cos 4\theta) = 0$$

$$\Rightarrow \sin\theta = 0 \text{ (or) } 1 + 2\cos 4\theta = 0 \Rightarrow \cos 4\theta = -\frac{1}{2}$$

(i) If $\sin\theta = 0$ then $\theta = n\pi$, $n \in \mathbb{Z}$

(ii) $\cos 4\theta = -\frac{1}{2} = \cos \frac{2\pi}{3} \Rightarrow 4\theta = 2n\pi \pm \frac{2\pi}{3} \Rightarrow \theta = \frac{2n\pi}{4} \pm \frac{2\pi}{12} \Rightarrow \theta = \frac{n\pi}{2} \pm \frac{\pi}{6}$, $n \in \mathbb{Z}$

But $0 < \theta < \pi$

$$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$$

15. Find the general solution of $5\cos^2\theta + 7\sin^2\theta = 6$

Sol: $5\cos^2\theta + 7\sin^2\theta = 6$

$$\Rightarrow 5\cos^2\theta + 7(1 - \cos^2\theta) = 6 \Rightarrow 5\cos^2\theta + 7 - 7\cos^2\theta = 6$$

$$\Rightarrow -2\cos^2\theta = -1 \Rightarrow \cos^2\theta = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2 = \cos^2\left(\frac{\pi}{4}\right)$$

Here, Principal value is $\alpha = \frac{\pi}{4}$ $\therefore$ General solution is $\theta = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$

16. Find the general solution of $2\cos^2\theta + 11\sin\theta = 7$

Sol: Given that $2\cos^2\theta + 11\sin\theta = 7$

$$\Rightarrow 2(1 - \sin^2\theta) + 11\sin\theta - 7 = 0 \Rightarrow 2 - 2\sin^2\theta + 11\sin\theta - 7 = 0$$

$$\Rightarrow 2\sin^2\theta - 11\sin\theta + 5 = 0 \Rightarrow 2\sin^2\theta - 10\sin\theta - \sin\theta + 5 = 0$$

$$\Rightarrow 2\sin\theta(\sin\theta - 5) - 1(\sin\theta - 5) = 0 \Rightarrow (2\sin\theta - 1)(\sin\theta - 5) = 0$$

$$\Rightarrow 2\sin\theta - 1 = 0 \text{ or } \sin\theta - 5 = 0$$

$$\Rightarrow \sin\theta = 1/2 \text{ or } \sin\theta = 5 > 1, \text{ and hence this has no solution}$$

$$\therefore \sin\theta = 1/2 = \sin \pi/6. \text{ Here P.V is } \alpha = \pi/6.$$

G.S is given by $\theta = n\pi + (-1)^n \alpha, n \in \mathbb{Z}$

$$\Rightarrow \theta = \{n\pi + (-1)^n \pi/6, n \in \mathbb{Z}\}$$

17. Solve $\tan\theta + \sec\theta = \sqrt{3}, 0 \leq \theta \leq 2\pi$

Sol: Given equation is $\tan\theta + \sec\theta = \sqrt{3} \Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} = \sqrt{3}, \cos\theta \neq 0$

$$\Rightarrow \sin\theta + 1 = \sqrt{3}\cos\theta \Rightarrow \sqrt{3}\cos\theta - \sin\theta = 1$$

Dividing the above equation by $\sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$, we get

$$\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2} \Rightarrow \cos\theta \cdot \cos\frac{\pi}{6} - \sin\theta \cdot \sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \cos\left(\theta + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$$

Here, P.V is $\alpha = \frac{\pi}{3}$

$$\therefore \text{General solution is } \theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}, n \in \mathbb{Z}$$

If $n=0$ then $\theta = 0 \pm \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}, \frac{\pi}{2}$ ($\theta = \pi/2$ is not possible, since $\cos\theta \neq 0$)

If $n=1$ then $\theta = 2\pi \pm \frac{\pi}{3} - \frac{\pi}{6} = 2\pi - \frac{\pi}{6} \pm \frac{\pi}{3} = \frac{11\pi}{6} \pm \frac{\pi}{3} = \frac{13\pi}{6}, \frac{3\pi}{2}$ ($\theta = \frac{3\pi}{2}$ is not possible, since $\cos\theta \neq 0$)

$$\therefore \text{the only solution in the interval } [0, 2\pi] \text{ is } \pi/6$$

INVERSE TRIGONOMETRIC FUNCTIONS

18. Find the value of $\sin\left(\cos^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13}\right)$

Sol: Let $\cos^{-1}\frac{3}{5} = \alpha \Rightarrow \cos \alpha = \frac{3}{5} \Rightarrow \sin \alpha = \frac{4}{5}$; $\cos^{-1}\left(\frac{12}{13}\right) = \beta \Rightarrow \cos \beta = \frac{12}{13} \Rightarrow \sin \beta = \frac{5}{13}$

$$\text{Now, } \sin\left(\cos^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13}\right) = \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta = \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13} = \frac{63}{65}$$

19. Find the value of $\tan\left(\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{5}{\sqrt{34}}\right)$

Sol: Let $\sin^{-1}\left(\frac{3}{5}\right) = \alpha \Rightarrow \sin \alpha = \frac{3}{5} \Rightarrow \tan \alpha = \frac{3}{4}$

$$\cos^{-1}\frac{5}{\sqrt{34}} = \beta \Rightarrow \cos \beta = \frac{5}{\sqrt{34}} \Rightarrow \tan \beta = \frac{3}{5}$$

$$\text{Now, } \tan\left(\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{5}{\sqrt{34}}\right) = \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \cdot \frac{3}{5}} = \frac{\frac{15+12}{20}}{1 - \frac{9}{20}} = \frac{27}{11}$$

20. Find $\cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right)$

Sol: Let $\sin^{-1}\frac{3}{5} = \alpha \Rightarrow \sin \alpha = \frac{3}{5} \Rightarrow \cos \alpha = \frac{4}{5}$

$$\sin^{-1}\frac{5}{13} = \beta \Rightarrow \sin \beta = \frac{5}{13} \Rightarrow \cos \beta = \frac{12}{13}$$

$$\text{Now, } \cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right) = \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{33}{65}$$

21. Prove that $\sin^{-1} \frac{3}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$

Sol: Let $\sin^{-1} \frac{3}{5} = \alpha$ and $\cos^{-1} \frac{12}{13} = \beta$

Required To Prove (RTP): $\alpha + \beta = \cos^{-1} \frac{33}{65}$

$$\Rightarrow \cos(\alpha + \beta) = \frac{33}{65} \sin^{-1} \frac{3}{5} = \alpha \Rightarrow \sin \alpha = \frac{3}{5} \Rightarrow \cos \alpha = \frac{4}{5}$$

$$\cos^{-1} \frac{12}{13} = \beta \Rightarrow \cos \beta = \frac{12}{13} \Rightarrow \sin \beta = \frac{5}{13}$$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

22. Prove that $\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19} = \frac{\pi}{4}$

Sol: $\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} = \tan^{-1} \left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \cdot \frac{3}{5}} \right) = \tan^{-1} \left(\frac{\frac{15+12}{20}}{\frac{20-9}{20}} \right) = \tan^{-1} \left(\frac{27}{11} \right)$

$$\therefore \text{L.H.S} = \tan^{-1} \left(\frac{27}{11} \right) - \tan^{-1} \left(\frac{8}{19} \right) = \tan^{-1} \left(\frac{\frac{27}{11} - \frac{8}{19}}{1 + \frac{27}{11} \cdot \frac{8}{19}} \right) \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right) \right]$$

$$= \tan^{-1} \left(\frac{513-88}{209+216} \right) = \tan^{-1} \left(\frac{425}{425} \right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S}$$

23. Prove that $\sin^{-1} \frac{4}{5} + 2\tan^{-1} \frac{1}{3} = \frac{\pi}{2}$

Sol: Let $\tan^{-1} \frac{1}{3} = \theta \Rightarrow \tan \theta = \frac{1}{3}$

$$\text{Now, } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \cdot \frac{1}{3}}{1 - \frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$$

$$\therefore \tan 2\theta = \frac{3}{4} \Rightarrow \cos 2\theta = \frac{4}{5} \Rightarrow 2\theta = \cos^{-1} \frac{4}{5}$$

$$\text{Now, } \sin^{-1} \frac{4}{5} + 2\tan^{-1} \frac{1}{3} = \sin^{-1} \frac{4}{5} + \cos^{-1} \frac{4}{5} = \frac{\pi}{2}, (\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2})$$

24. Prove that $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(2\tan^{-1}\frac{3}{4}\right)$

Sol: Let $\tan^{-1}\frac{1}{7} = \alpha \Rightarrow \tan \alpha = \frac{1}{7}$ and $\tan^{-1}\frac{3}{4} = \beta \Rightarrow \tan \beta = \frac{3}{4}$

$$\text{Now, } \cos\left(2\tan^{-1}\frac{1}{7}\right) = \cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{1 - \frac{1}{49}}{1 + \frac{1}{49}} = \frac{48}{50} = \frac{24}{25} \dots\dots(1)$$

$$\text{Also, } \sin\left(2\tan^{-1}\frac{3}{4}\right) = \sin 2\beta = \frac{2 \tan \beta}{1 + \tan^2 \beta} = \frac{2 \cdot \frac{3}{4}}{1 + \frac{9}{16}} = \frac{3}{2} \times \frac{16}{25} = \frac{24}{25} \dots\dots(2)$$

$$\therefore \text{ from (1) and (2), we have, } \cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(2\tan^{-1}\frac{3}{4}\right)$$

VSAQ PQ SOLUTIONS

SEQUENCES & SERIES

1. Find the 8th term of the A.P. 3, 5, 7, 9,.....

Sol: First term $a = 3$ and common difference $d = 5 - 3 = 2$

$$n^{\text{th}} \text{ term } T_n = a + (n - 1)d = 3 + (n - 1)2 = 2n + 1$$

$$\therefore 8^{\text{th}} \text{ term } T_8 = 2(8) + 1 = 17$$

2. Find the sum of $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \dots$

Sol: Let $S = \frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \dots$

$$\text{Grouping alternately, } S = S_1 + S_2 = \left(\frac{2}{5} + \frac{2}{5^3} + \frac{2}{5^5} + \dots \right) + \left(\frac{3}{5^2} + \frac{3}{5^4} + \dots \right)$$

$$\text{For first series } S_1: a = \frac{2}{5}, r = \frac{1}{25} \text{ and for second series } S_2: a = \frac{3}{25}, r = \frac{1}{25}. \quad S_{\infty} = \frac{a}{(1-r)}$$

$$\therefore S_1 = \frac{\left(\frac{2}{5}\right)}{\left(1 - \frac{1}{25}\right)} = \frac{\left(\frac{2}{5}\right)}{\left(\frac{24}{25}\right)} = \left(\frac{2}{5}\right) \left(\frac{25}{24}\right) = \frac{10}{24} = \frac{5}{12}; \quad S_2 = \frac{\left(\frac{3}{25}\right)}{\left(1 - \frac{1}{25}\right)} = \frac{\left(\frac{3}{25}\right)}{\left(\frac{24}{25}\right)} = \left(\frac{3}{25}\right) \left(\frac{25}{24}\right) = \frac{3}{24} = \frac{1}{8}$$

$$\therefore S = S_1 + S_2 = \frac{5}{12} + \frac{1}{8} = \frac{10}{24} + \frac{3}{24} = \frac{13}{24}$$

3. Find the sum $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \dots$ to ∞

Sol: Note that the given series is an AGP with $a=1, d=1, r=1/2$

$$\text{The sum of the infinite AGP is } \frac{a}{1-r} + \frac{rd}{(1-r)^2} \text{ when } |r| < 1.$$

$$\therefore \text{Sum } S = \frac{1}{1 - \frac{1}{2}} + \frac{\frac{1}{2}(1)}{\left(1 - \frac{1}{2}\right)^2} = 2 + 2 = 4$$

4. The arithmetic mean between two positive numbers is 34 and their geometric mean is 16. find the numbers.

Sol: Let the required numbers be a and b.

$$\text{Given that } A.M = \frac{a+b}{2} = 34 \Rightarrow a+b = 68 \dots (1) \quad \text{and } G.M = \sqrt{ab} = 16 \Rightarrow ab = 256$$

$$\text{We know that } (a-b)^2 = (a+b)^2 - 4ab = (68)^2 - 4(256) = 4624 - 1024 = 3600 = 60^2$$

$$\therefore a-b = 60 \dots (2)$$

$$\text{Adding (1) and (2) we get } 2a = 128 \Rightarrow a = 64$$

$$\text{Now (1)} \Rightarrow b = 68 - a = 68 - 64 = 4 \quad \therefore \text{the numbers are 64 and 4.}$$

MATRICES

5. Find the trace of $\begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

Sol: The trace of a square matrix is the sum of elements in the principal diagonal.

$$\therefore \text{the trace of the given matrix is } 1 + (-1) + 1 = 1$$

6. If $\begin{bmatrix} x-1 & 2 & y-5 \\ z & 0 & 2 \\ 1 & -1 & 1+a \end{bmatrix} = \begin{bmatrix} 1-x & 2 & -y \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ then find the values of x, y, z & a.

Sol: Equating the corresponding elements, we have

$$x-1=1-x \Rightarrow x+x=1+1 \Rightarrow 2x=2 \Rightarrow x=1; \quad y-5=-y \Rightarrow y+y=5 \Rightarrow 2y=5 \Rightarrow y=5/2; \quad z=2;$$

$$1+a=1 \Rightarrow a=0$$

$$\therefore x=1, y=5/2, z=2, a=0$$

7. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, find A^2 .

Sol:

$$A^2 = A \times A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i \times i + 0 \times 0 & i \times 0 + 0 \times i \\ 0 \times i + i \times 0 & 0 \times 0 + i \times i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$$

8. Find the product $\begin{bmatrix} 2 & 1 & 4 \\ 6 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Sol: $GE = \begin{bmatrix} 2 & 1 & 4 \\ 6 & -2 & 3 \end{bmatrix}_{(2 \times 3)} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_{(3 \times 1)} = \begin{bmatrix} 2(1) + 1(2) + 4(1) \\ 6(1) + (-2)(2) + 3(1) \end{bmatrix}_{(2 \times 1)} = \begin{bmatrix} 2 + 2 + 4 \\ 6 - 4 + 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

9. If $A = \begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 8 \\ 2 & -8 & x \end{bmatrix}$ is a skew symmetric matrix, find the value of x

Sol: A is a skew symmetric matrix and x is an element of the diagonal. Hence $x=0$

ADDITION OF VECTORS

10. Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} + \vec{j}$. Find a unit vector in the direction of $\vec{a} + \vec{b}$

Sol: $\vec{a} + \vec{b} = (\vec{i} + 2\vec{j} + 3\vec{k}) + (3\vec{i} + \vec{j}) = 4\vec{i} + 3\vec{j} + 3\vec{k}$

$$\Rightarrow |\vec{a} + \vec{b}| = \sqrt{4^2 + 3^2 + 3^2} = \sqrt{34}$$

$$\therefore \text{Unit vector in the direction of } \vec{a} + \vec{b} = \frac{(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} = \frac{(4\vec{i} + 3\vec{j} + 3\vec{k})}{\sqrt{34}}$$