A decorative rectangular box with a light pink background and a dark pink border. The box is adorned with intricate floral and vine patterns in dark pink, extending from the corners and sides. Inside the box, the text "BULLET MODEL PAPER" is written in a bold, dark pink, sans-serif font.

BULLET MODEL PAPER

A 'MULTI QUESTION PAPER' WITH 'BULLET ANSWERS'

SAQ & LAQ

SECTIONS

SAQ SECTION-B

Q11 : LOCUS:

- Find the locus of the third vertex of a right angled triangle, the ends of whose hypotenuse are (4,0) and (0,4)

Sol: Let the locus point P = (x, y)

Given points A = (4, 0), B = (0, 4)

Given condition: $\angle APB = 90^\circ$

$$\Rightarrow PA^2 + PB^2 = AB^2$$

$$\begin{aligned} \Rightarrow [(x-4)^2 + (y-0)^2] + [(x-0)^2 + (y-4)^2] \\ = (4-0)^2 + (0-4)^2 \\ \Rightarrow (x^2 - 8x + 16) + y^2 + x^2 + (y^2 - 8y + 16) = 16 + 16 \\ \Rightarrow 2x^2 + 2y^2 - 8x - 8y = 0 \\ \Rightarrow 2(x^2 + y^2 - 4x - 4y) = 0 \\ \Rightarrow x^2 + y^2 - 4x - 4y = 0 \end{aligned}$$

Hence, Locus of P is $x^2 + y^2 - 4x - 4y = 0$

- A(5,3) and B(3,-2) are two fixed points. Find the equation of locus of P, so that the area of triangle PAB is 9 sq.units.

A: Given that A = (5,3), B = (3,-2) and P = (x, y) be a point on the locus.

Given condition: Area of $\Delta PAB = 9$ sq.units

$$\begin{aligned} \Delta = \frac{1}{2} \begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix} = 9 \Rightarrow \frac{1}{2} \begin{vmatrix} 5-3 & 5-x \\ 3+2 & 3-y \end{vmatrix} = 9 \\ \Rightarrow \begin{vmatrix} 2 & 5-x \\ 5 & 3-y \end{vmatrix} = 2(9) \Rightarrow |2(3-y) - 5(5-x)| = 18 \\ \Rightarrow |6 - 2y - 25 + 5x| = 18 \Rightarrow |5x - 2y - 19| = 18 \\ \Rightarrow 5x - 2y - 19 = \pm 18 \\ \Rightarrow 5x - 2y - 19 = 18 \text{ (or) } 5x - 2y - 19 = -18 \\ \Rightarrow 5x - 2y - 37 = 0 \text{ (or) } 5x - 2y - 1 = 0 \\ \Rightarrow (5x - 2y - 37)(5x - 2y - 1) = 0 \end{aligned}$$

Hence, locus of P is $(5x - 2y - 37)(5x - 2y - 1) = 0$

- If the distance from 'P' to the points (2,3), (2,-3) are in the ratio 2:3, then find the equation of locus of P.

A: We take A = (2,3), B = (2,-3), P = (x,y) is a point on locus.

$$\begin{aligned} \text{Condition: } \frac{PA}{PB} = \frac{2}{3} \Rightarrow 3PA = 2PB \Rightarrow 9PA^2 = 4PB^2 \\ \Rightarrow 9[(x-2)^2 + (y-3)^2] = 4[(x-2)^2 + (y+3)^2] \\ \Rightarrow 9[(x^2 + 4 - 4x) + (y^2 + 9 - 6y)] \\ = 4[(x^2 + 4 - 4x) + (y^2 + 9 + 6y)] \\ \Rightarrow 9x^2 - 36x + 9y^2 + 81 - 54y \\ = 4x^2 + 16 - 16x + 4y^2 + 36 + 24y \\ \Rightarrow 9x^2 - 4x^2 + 9y^2 - 4y^2 - 36x + 16x - 54y - 24y + 81 - 16 = 0 \\ \Rightarrow 5x^2 + 5y^2 - 20x - 78y + 65 = 0 \\ \text{Hence, locus of P is } 5x^2 + 5y^2 - 20x - 78y + 65 = 0. \end{aligned}$$

- Find the equation of locus of a point, which forms a triangle of area 2 with the points A(1, 1), B(-2, 3)

Sol: Let P = (x, y) be a point on the locus and A = (1, 1), B = (-2, 3).

Given condition, area of $\Delta PAB = 2$

$$\begin{aligned} \Rightarrow \frac{1}{2} \begin{vmatrix} x-1 & x+2 \\ y-1 & y-3 \end{vmatrix} = 2 \\ \Rightarrow |(x-1)(y-3) - (y-1)(x+2)| = 2(2) \\ \Rightarrow |xy - 3x - y + 3 - (xy + 2y - x - 2)| = 4 \\ \Rightarrow |xy - 3x - y + 3 - xy - 2y + x + 2| = 4 \\ \Rightarrow |-2x - 3y + 5| = 4 \\ \Rightarrow |2x + 3y - 5| = 4 \Rightarrow 2x + 3y - 5 = \pm 4 \\ \Rightarrow 2x + 3y - 5 = 4 \text{ (or) } 2x + 3y - 5 = -4 \\ \Rightarrow 2x + 3y - 9 = 0 \text{ (or) } 2x + 3y - 1 = 0 \\ \Rightarrow (2x + 3y - 9)(2x + 3y - 1) = 0 \end{aligned}$$

$\therefore$ The equation of locus of P(x,y) is

$$(2x + 3y - 9)(2x + 3y - 1) = 0$$

Q12 : TRANSFORMATIONS:

- When the origin is shifted to $(-1,2)$, find the transformed equation of $x^2+y^2+2x-4y+1=0$

A: Given equation (original) is $x^2+y^2+2x-4y+1=0$(1)

We take new origin $(h,k)=(-1,2)$, then

$$x=X+h \Rightarrow x=X-1 ; y=Y+k \Rightarrow y=Y+2$$

From (1), transformed equation is

$$(X-1)^2+(Y+2)^2+2(X-1)-4(Y+2)+1=0$$

$$\Rightarrow (X^2+1-2X)+(Y^2+4+4Y)+2X-2-4Y-8+1=0$$

$$\Rightarrow X^2+Y^2-4=0$$

- When the axes are rotated through an angle α , find the transformed equation of $x\cos\alpha + y\sin\alpha = p$

Sol: Given original equation is

$$x\cos\alpha + y\sin\alpha = p \text{.....(1)}$$

Angle of rotation $\theta = \alpha$, then

$$x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos\alpha - Y\sin\alpha$$

$$y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos\alpha + X\sin\alpha$$

From (1), transformed equation is

$$(X\cos\alpha - Y\sin\alpha)\cos\alpha + (Y\cos\alpha + X\sin\alpha)\sin\alpha = p$$

$$\Rightarrow X\cos^2\alpha - Y\sin\alpha\cos\alpha$$

$$+ Y\cos\alpha\sin\alpha + X\sin^2\alpha = p$$

$$\Rightarrow X(\cos^2\alpha + \sin^2\alpha) = p \Rightarrow X(1) = p \Rightarrow X = p$$

- Find the transformed equation of $3x^2+10xy+3y^2=9$ when the axes are rotated through an angle $\pi/4$.

A: Given equation (original) is $3x^2+10xy+3y^2=9$ (1)

Angle of rotation $\theta = \pi/4 = 45^\circ$, then

$$x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos 45^\circ - Y\sin 45^\circ$$

$$= X\left(\frac{1}{\sqrt{2}}\right) - Y\left(\frac{1}{\sqrt{2}}\right) \Rightarrow x = \frac{X-Y}{\sqrt{2}}$$

$$y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos 45^\circ + X\sin 45^\circ$$

$$= Y\left(\frac{1}{\sqrt{2}}\right) + X\left(\frac{1}{\sqrt{2}}\right) \Rightarrow y = \frac{X+Y}{\sqrt{2}}$$

From (1), transformed equation is

$$3\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 3\left(\frac{X+Y}{\sqrt{2}}\right)^2 = 9$$

$$\Rightarrow 3\left(\frac{X^2+Y^2-2XY}{2}\right) + 10\left(\frac{X^2-Y^2}{2}\right) + 3\left(\frac{X^2+Y^2+2XY}{2}\right) = 9$$

$$\Rightarrow \frac{3X^2+3Y^2-6XY+10X^2-10Y^2+3X^2+3Y^2+6XY}{2} = 9$$

$$\Rightarrow 16X^2-4Y^2=2(9) \Rightarrow 8(X^2-2Y^2)=9$$

$$\Rightarrow 8X^2-2Y^2=9$$

- When the axes are rotated through an angle 45° , the transformed equation of a curve is $17x^2-16xy+17y^2=225$. Find the original equation.

A: Given transformed(new) equation is taken as $17X^2-16XY+17Y^2=225$ (1)

Angle of rotation $\theta = 45^\circ$, then

$$X = x\cos\theta + y\sin\theta = x\cos 45^\circ + y\sin 45^\circ$$

$$= x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) \Rightarrow X = \frac{x+y}{\sqrt{2}}$$

$$Y = y\cos\theta - x\sin\theta = y\cos 45^\circ - x\sin 45^\circ$$

$$= y\left(\frac{1}{\sqrt{2}}\right) - x\left(\frac{1}{\sqrt{2}}\right) \Rightarrow Y = \frac{y-x}{\sqrt{2}}$$

From (1), original equation is

$$17\left(\frac{x+y}{\sqrt{2}}\right)^2 - 16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{y-x}{\sqrt{2}}\right) + 17\left(\frac{y-x}{\sqrt{2}}\right)^2 = 225$$

$$\Rightarrow 17\left(\frac{x^2+y^2+2xy}{2}\right) - 16\left(\frac{y^2-x^2}{2}\right) + 17\left(\frac{y^2+x^2-2xy}{2}\right) = 225$$

$$\Rightarrow \frac{17x^2+17y^2+34xy-16y^2+16x^2+17x^2+17y^2-34xy}{2} = 225$$

$$\Rightarrow 50x^2+18y^2=2(225) \Rightarrow 25x^2+9y^2=225$$

$$\Rightarrow 25x^2+9y^2=225$$

Q13 & 14 : PAIR OF LINES

- If $x^2 + xy - 2y^2 + 4x - y + k = 0$ represents a pair of straight lines, find k .

Sol: Comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\text{we get } a = 1, 2h = 1 \Rightarrow h = 1/2, b = -2, 2g = 4$$

$$\Rightarrow g = 2, 2f = -1 \Rightarrow f = -1/2, c = k$$

The condition for the above equation to represent a

$$\text{pair of lines is } \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow -2k + 2\left(-\frac{1}{2}\right)(2)\left(\frac{1}{2}\right) - 1\left(-\frac{1}{2}\right)^2 + 2(2)^2 - k\left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow -2k - 1 - \frac{1}{4} + 8 - \frac{k}{4} = 0 \Rightarrow -8k - 4 - 1 + 32 - k = 0$$

$$\Rightarrow -9k + 27 = 0 \Rightarrow 9k = 27 \Rightarrow k = 3$$

Find the distances between pairs of parallel straight

$$\text{lines } 9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$$

Sol: Comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\text{we get } a = 9, 2h = -6 \Rightarrow h = -3, b = 1, 2g = 18$$

$$\Rightarrow g = 9, 2f = -6 \Rightarrow f = -3, c = 8$$

The distance between the pair of parallel lines is given

$$\begin{aligned} & \text{by } 2\sqrt{\frac{g^2 - ac}{a(a+b)}} \\ & = 2\sqrt{\frac{9^2 - (9)(8)}{9(9+1)}} = 2\sqrt{\frac{81 - 72}{9(10)}} \\ & = 2\sqrt{\frac{9}{90}} = 2\sqrt{\frac{1}{10}} = \sqrt{\frac{4}{10}} = \sqrt{\frac{2}{5}} \end{aligned}$$

- Find the condition for the chord $lx+my=1$ of the circle $x^2+y^2=a^2$ (whose centre is the origin) to subtend a right angle at the origin.

A: Given chord is $lx+my=1$... (1)

Given circle is $x^2+y^2=a^2$

$$\Rightarrow x^2+y^2-a^2=0 \dots\dots\dots(2)$$

From (1) & (2), homogenised equation is

$$\begin{aligned} & \Rightarrow x^2 + y^2 - a^2(l^2) = 0 \\ & \Rightarrow x^2 + y^2 - a^2(lx + my)^2 = 0 \\ & \Rightarrow x^2 + y^2 - a^2(l^2x^2 + m^2y^2 + 2lmxy) = 0 \\ & \Rightarrow x^2 + y^2 - a^2l^2x^2 - a^2m^2y^2 - 2a^2lmxy = 0 \\ & \Rightarrow x^2 + y^2 - a^2l^2x^2 - a^2m^2y^2 - 2a^2lmxy = 0 \\ & \Rightarrow x^2(1-a^2l^2) + y^2(1-a^2m^2) - 2a^2lmxy = 0 \end{aligned}$$

On applying perpendicular pair of lines condition,

we get Coeff. x^2 + Coeff. y^2 = 0

$$\Rightarrow (1-a^2l^2) + (1-a^2m^2) = 0$$

$$\Rightarrow 2 - a^2l^2 - a^2m^2 = 0$$

$$\Rightarrow a^2l^2 + a^2m^2 = 2 \Rightarrow a^2(l^2 + m^2) = 2$$

Find the condition for the lines joining to the points of intersection of the circle $x^2+y^2=a^2$ and the line $lx+my=1$ to coincide.

Sol: Given chord is $lx+my=1$... (1)

Given circle is $x^2+y^2=a^2 \Rightarrow x^2+y^2-a^2=0 \dots\dots\dots(2)$

From (1) & (2), the homogenised equation is

$$\begin{aligned} & x^2 + y^2 - a^2(1^2) = 0 \\ & \Rightarrow x^2 + y^2 - a^2(lx + my)^2 = 0 \\ & \Rightarrow x^2 + y^2 - a^2(l^2x^2 + m^2y^2 + 2lmxy) = 0 \\ & \Rightarrow x^2 + y^2 - a^2l^2x^2 - a^2m^2y^2 - 2a^2lmxy = 0 \end{aligned}$$

$$\Rightarrow x^2(1-a^2l^2) + y^2(1-a^2m^2) - 2a^2lmxy = 0$$

If this pair of lines are coincident then $h^2 = ab$

$$\Rightarrow (a^2lm)^2 = (1-a^2l^2)(1-a^2m^2)$$

$$\begin{aligned} & \Rightarrow a^4l^2m^2 = 1 - a^2l^2 - a^2m^2 + a^4l^2m^2 \\ & \Rightarrow a^2l^2 + a^2m^2 = 1 \end{aligned}$$

Q15 : CONTINUITY:

- Is f defined by $f(x) = \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ continuous at 0?

A: (a) Given $f(0)=1$ (1)

$$(b) \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2 \dots\dots(2) \left(\because \lim_{x \rightarrow 0} \frac{\sin kx}{x} = k \right)$$

$$\therefore \text{From (1) \& (2), } \lim_{x \rightarrow 0} f(x) \neq f(0)$$

Hence, proved that f is not continuous at $x=0$

- Is f given by $f(x) = \begin{cases} \frac{x^2-9}{x^2-2x-3} & \text{if } 0 < x < 5, x \neq 3 \\ 1.5 & \text{if } x = 3 \end{cases}$, continuous at the point 3.

A: (a) Given $f(3)=1.5$(1)

$$\begin{aligned} (b) \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2-9}{x^2-2x-3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)(x+1)} \\ &= \lim_{x \rightarrow 3} \frac{x+3}{x+1} = \frac{3+3}{3+1} = \frac{6}{4} = \frac{3}{2} = 1.5 \dots\dots(2) \end{aligned}$$

$$\text{From (1) \& (2), } \lim_{x \rightarrow 0} f(x) = f(0)$$

Hence proved that $f(x)$ is continuous at $x=3$

- If f is given by $f(x) = \begin{cases} k^2x - k & \text{if } x \geq 1 \\ 2 & \text{if } x < 1 \end{cases}$ is a continuous function on $\mathbb{R}$, then find k .

A: (a) When $x < 1$, L.H.L = $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2 = 2$ (1)

(b) When $x > 1$,

$$\begin{aligned} \text{R.H.L} &= \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} k^2x - k = k^2(1) - k = k^2 - k \\ &\dots\dots(2) \end{aligned}$$

From (1) & (2), L.H.L = R.H.L

[$\because f(x)$ is continuous at $x=1$ as it is continuous on $\mathbb{R}$]

$$\text{So, } k^2 - k = 2 \Rightarrow k^2 - k - 2 = 0 \Rightarrow (k-2)(k+1) = 0 \Rightarrow k = 2 \text{ (or) } -1$$

- Check the continuity of $f(x) = \begin{cases} \frac{1}{2}(x^2-4) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2-8x^{-3} & \text{if } x > 2 \end{cases}$ at 2.

A: (a) When $x < 2$,

$$\text{L.H.L.} = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{1}{x^2 - 4} = \frac{1}{2(4-4)} = 0 \dots (1)$$

(b) When $x > 2$, R.H.L. = $\lim_{x \rightarrow 2^+} f(x)$

$$= \lim_{x \rightarrow 2^+} (2 - 8x^{-3}) = \lim_{x \rightarrow 2^+} \left(2 - \frac{8}{x^3} \right) = 2 - \frac{8}{8} = 2 - 1 = 1 \dots (2)$$

From (1) & (2), L.H.L. $\neq$ R.H.L.

Hence proved that $f(x)$ is not continuous at 2.

Q16 : DIFFERENTIATION:

• Find the derivative of $\sin 2x$ from the first principle.

A: We take $f(x) = \sin 2x$, then $f(x+h) = \sin 2(x+h) = \sin(2x+2h)$

$$\text{From the first principle, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(2x+2h) - \sin 2x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{(2x+2h)+2x}{2} \right) \sin \left(\frac{(2x+2h)-2x}{2} \right) \right]$$

$$= 2 \lim_{h \rightarrow 0} \frac{1}{h} \cos \left(\frac{4x+2h}{2} \right) \sin \left(\frac{2h}{2} \right)$$

$$= 2 \lim_{h \rightarrow 0} \frac{1}{h} \cos \left(\frac{2(2x+h)}{2} \right) \sin(h)$$

$$= 2 \lim_{h \rightarrow 0} \frac{1}{h} \cos(2x+h) \sin(h) = 2 \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h}$$

$$= 2 \cos(2x+0)(1) = 2 \cos 2x$$

• Find the derivative of $\cos ax$ from the first Principle.

A: We take $f(x) = \cos ax$, then $f(x+h) = \cos a(x+h) = \cos(ax+ah)$

$$\text{From the first principle, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos(ax+ah) - \cos(ax)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin \left(\frac{(ax+ah)+ax}{2} \right) \sin \left(\frac{(ax+ah)-ax}{2} \right) \right]$$

$$= -2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\sin \left(\frac{2ax+ah}{2} \right) \sin \left(\frac{ah}{2} \right) \right]$$

$$= -2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\sin \left(ax + \frac{ah}{2} \right) \sin \frac{ah}{2} \right]$$

$$= -2 \lim_{h \rightarrow 0} \sin \left(ax + \frac{ah}{2} \right) \lim_{h \rightarrow 0} \frac{\sin(ah/2)}{h}$$

$$= -2 \sin(ax+0) \left(\frac{a}{2} \right) = -a \sin ax \left(\frac{a}{2} \right) = -a \sin ax$$

• Find the derivative of $\tan 2x$ from the first principle.

A: We take $f(x) = \tan(2x)$, then $f(x+h) = \tan 2(x+h) = \tan(2x+2h)$

$$\text{From the first principle, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\tan(2x+2h) - \tan(2x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(2x+2h)}{\cos(2x+2h)} - \frac{\sin(2x)}{\cos(2x)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(2x+2h)\cos(2x) - \cos(2x+2h)\sin(2x)}{\cos(2x+2h)\cos(2x)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \frac{\sin[(2x+2h)-2x]}{\cos(2x+2h)\cos(2x)}$$

$$[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$= \lim_{h \rightarrow 0} \frac{\sin 2h}{h} \lim_{h \rightarrow 0} \frac{1}{\cos(2x+2h)\cos(2x)}$$

$$= 2 \cdot \frac{1}{\cos^2(2x)} = 2 \sec^2(2x)$$

Q17 : TANGENT & NORMAL:

• Find the equations of the tangent and the normal to the curve $xy=10$ at $(2, 5)$

A: Given curve is $xy=10 \Rightarrow y = \frac{10}{x} \Rightarrow \frac{dy}{dx} = -\frac{10}{x^2}$

Slope of the tangent at $(2, 5)$ is

$$m = \left(\frac{dy}{dx} \right)_{(2,5)} = \frac{-10}{2^2} = \frac{-10}{4} = \frac{-5}{2}$$

(i) Equation of the tangent at $(2, 5)$ with slope $\frac{-5}{2}$ is
 $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 5 = \frac{-5}{2}(x - 2) \Rightarrow 2y - 10 = -5x + 10$$

$$\Rightarrow 5x + 2y - 20 = 0$$

(ii) Slope of the normal is $\frac{-1}{m} = \frac{2}{5}$

Equation of the normal at $(2, 5)$ with slope $\frac{2}{5}$ is

$$y - y_1 = \frac{1}{m}(x - x_1)$$

$$\Rightarrow y - 5 = \frac{2}{5}(x - 2) \Rightarrow 5y - 25 = 2x - 4$$

$$\Rightarrow 2x - 5y + 21 = 0$$

• Show that the curves $x^2+y^2=2$, $3x^2+y^2=4x$ have a common tangent at the point $(1, 1)$

A: Given first curve is $x^2+y^2=2 \Rightarrow 2x + 2y \frac{dy}{dx} = 0$

$$\Rightarrow 2y \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

So, slope of the tangent at $P(1, 1)$ is $m_1 = \frac{-1}{1} = -1 \dots (1)$

Given second curve is $3x^2+y^2=4x$

$$\Rightarrow 6x + 2y \cdot \frac{dy}{dx} = 4 \Rightarrow \cancel{2}(3x + y \cdot \frac{dy}{dx}) = \cancel{2}(2)$$

$$\Rightarrow 3x + y \cdot \frac{dy}{dx} = 2 \Rightarrow y \cdot \frac{dy}{dx} = 2 - 3x \Rightarrow \frac{dy}{dx} = \frac{2-3x}{y}$$

So, slope of the tangent at P(1,1) is

$$m_2 = \frac{2-3}{1} = -\frac{1}{1} = -1 \dots\dots(2)$$

From (1) & (2), $m_1 = m_2$. So slopes are equal.
Hence proved.

- **Show that at any point (x,y) on the curve $y = be^{x/a}$, the length of subtangent is a constant and the length of the subnormal is y^2/a .**

A: Let P(x,y) be point on the curve $y = be^{x/a}$
On diff. w.r.to x, we get $\frac{dy}{dx} = be^{\frac{x}{a}} \cdot \left(\frac{1}{a}\right) = \frac{b}{a} e^{\frac{x}{a}}$

$$\therefore \text{Slope } m = \left(\frac{dy}{dx}\right)_P = \frac{b}{a} e^{\frac{x}{a}}$$

(i) Length of subtangent

$$= \left| \frac{y}{m} \right| = \left| \frac{be^{\frac{x}{a}}}{\frac{b}{a} e^{\frac{x}{a}}} \right| = a = \text{constant}$$

(i) Length of subnormal

$$= |y \cdot m| = \left| be^{\frac{x}{a}} \cdot \frac{b}{a} e^{\frac{x}{a}} \right| = \left| \left(\frac{be^{\frac{x}{a}}}{a} \right)^2 \right| = \left| \frac{y^2}{a} \right|$$

Q18 : MAXIMA & MINIMA ,RATE MEASURE:

- **A particle is moving along a line according $s=f(t) = 4t^3 - 3t^2 + 5t - 1$ where s is measured in meters and t is measured in seconds. Find the velocity and acceleration at time t. At what time the acceleration is zero.**

A: Given that, $s=f(t) = 4t^3 - 3t^2 + 5t - 1$

$$\therefore \text{Velocity } v = \frac{ds}{dt} = 12t^2 - 6t + 5$$

$$\text{Acceleration } a = \frac{dv}{dt} = 24t - 6$$

$$\text{If the acceleration is 0 then } 24t - 6 = 0 \Rightarrow t = \frac{1}{4}$$

$$\text{The acceleration of the particle is zero at } t = \frac{1}{4} \text{ sec.}$$

- **The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?**

A: For the cube, we take

length of the edge = x, Volume = V and

Surface area = S

$$\text{Given } \frac{dV}{dt} = 9 \text{ and } x = 10 \text{ cm}$$

$$\text{Volume of the cube } V = x^3$$

$$\text{On diff. w.r.t 't', we get } \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow 9 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$$

$$\text{Surface area } S = 6x^2$$

$$\text{On diff. w.r.t 't', we get } \frac{dS}{dt} = 12x \frac{dx}{dt}$$

$$= 12 \times \left(\frac{3}{x^2} \right) = \frac{36}{x} = \frac{36}{10} = 3.6 \text{ cm}^2 / \text{sec}$$

- **The profit function p(x) of a company, selling x items per day is given by $p(x) = (150-x)x - 1600$. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.**

Sol: Given profit function is $p(x) = (150-x)x - 1600$

$$p'(x) = (150-x)1 + x(-1) = 150 - 2x \text{ .Also } p''(x) = -2$$

For Maximum profit, $p'(x) = 0 \Rightarrow 150 - 2x = 0 \Rightarrow x = 75$

$\therefore$ Profit p(x) is maximum when $x = 75$

$\therefore$ Maximum Profit = $(150-75)75 - 1600$

$$= 75(75) - 1600 = 5625 - 1600 = 4025 \text{ units.}$$

- **The profit function p(x) of a company, selling x items per day is given by $p(x) = (150-x)x - 1000$. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.**

Sol: Given profit function is $p(x) = (150-x)x - 1000$

$$p'(x) = (150-x)1 + x(-1) = 150 - 2x \text{ .Also } p''(x) = -2$$

For Maximum profit, $p'(x) = 0 \Rightarrow 150 - 2x = 0 \Rightarrow x = 75$

$\therefore$ Profit p(x) is maximum when $x = 75$

$\therefore$ Maximum Profit = $(150-75)75 - 1000$

$$= 75(75) - 1000 = 5625 - 1000 = 4625 \text{ units.}$$

LAQ SECTION-C

Q19 : STRAIGHT LINES:

- Find the circumcentre of the triangle whose vertices are (1,3), (-3,5), (5,-1).

A: We take S(x,y) as circumcentre and vertices A=(1,3), B=(-3,5), C=(5,-1).

Then SA=SB=SC

First we take, SA=SB

$$\Rightarrow \sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x+3)^2 + (y-5)^2}$$

Squaring on both sides, we get

$$(x-1)^2 + (y-3)^2 = (x+3)^2 + (y-5)^2$$

$$\Rightarrow (x^2 + 1 - 2x) + (y^2 + 9 - 6y) = (x^2 + 9 + 6x) + (y^2 + 25 - 10y)$$

$$\Rightarrow 6x + 2x - 10y + 6y + 25 - 9 - 9 - 1 = 0$$

$$\Rightarrow 8x - 4y + 24 = 0 \Rightarrow 2(2x - y + 6) = 0$$

$$\Rightarrow 2x - y + 6 = 0 \dots\dots\dots(1)$$

Now, we take SB=SC

$$\Rightarrow \sqrt{(x+3)^2 + (y-5)^2} = \sqrt{(x-5)^2 + (y+1)^2}$$

Squaring on both sides, we get

$$(x+3)^2 + (y-5)^2 = (x-5)^2 + (y+1)^2$$

$$\Rightarrow (x^2 + 9 + 6x) + (y^2 + 25 - 10y) = (x^2 + 25 - 10x) + (y^2 + 1 + 2y)$$

$$\Rightarrow 6x + 10x - 10y - 2y + 9 - 25 - 25 - 1 = 0$$

$$\Rightarrow 16x - 12y + 8 = 0 \Rightarrow 4(4x - 3y + 2) = 0$$

$$\Rightarrow 4x - 3y + 2 = 0 \dots\dots\dots(2)$$

Solving (1) and (2), we get S;

$$2x - y + 6 = 0;$$

$$4x - 3y + 2 = 0$$

$$\Rightarrow \frac{x}{(-1)2 - (-3)6} = \frac{y}{6(4) - 2(2)} = \frac{1}{2(-3) - 4(-1)}$$

$$\Rightarrow \frac{x}{-2 + 18} = \frac{y}{24 - 4} = \frac{1}{-6 + 4}$$

$$\Rightarrow \frac{x}{16} = \frac{y}{20} = \frac{1}{-2}$$

$$\Rightarrow x = \frac{-16}{2} = -8, y = \frac{-20}{2} = -10$$

∴ Circumcentre S (x,y) = (-8, -10)

- Find the orthocentre of the triangle whose vertices are (-2,-1), (6,-1), (2,5)

A: We take O as orthocentre and vertices A=(-2,-1), B=(6,-1), C=(2,5)

First we find altitude through A(-2,-1):

$$\text{Slope of } \overline{BC} \text{ is } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5+1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$$

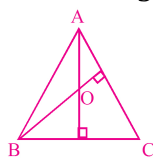
$$\text{So, slope of its perpendicular is } \frac{-1}{m} = \frac{-1}{-\frac{3}{2}} = \frac{2}{3}$$

Perpendicular through A(-2,-1) with slope $\frac{2}{3}$

$$\text{is } y - y_1 = \frac{-1}{m}(x - x_1)$$

$$\Rightarrow y + 1 = \frac{2}{3}(x + 2) \Rightarrow 3y + 3 = 2x + 4$$

$$\Rightarrow 2x - 3y + 1 = 0 \dots\dots\dots(1)$$



Now, we find altitude through B(6,-1):

Slope of $\overline{AC}$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5+1}{2+2} = \frac{6}{4} = \frac{3}{2}$$

$$\text{So, slope of its perpendicular is } \frac{-1}{m} = \frac{-1}{\frac{3}{2}} = -\frac{2}{3}$$

Perpendicular through B(6,-1) with slope $-\frac{2}{3}$

$$\text{is } y - y_1 = \frac{-1}{m}(x - x_1)$$

$$\Rightarrow y + 1 = -\frac{2}{3}(x - 6) \Rightarrow 3y + 3 = -2x + 12$$

$$\Rightarrow 2x + 3y - 9 = 0 \dots\dots\dots(2)$$

Solving (1), (2), we get 'O'; $2x - 3y + 1 = 0$
 $2x + 3y - 9 = 0$

$$(1) + (2) \Rightarrow 4x - 8 = 0 \Rightarrow 4x = 8 \Rightarrow x = 2$$

$$(1) \Rightarrow 2(2) - 3(y) + 1 = 0 \Rightarrow 3y = 5 \Rightarrow y = 5/3$$

$$\Rightarrow x = 2, y = 5/3$$

∴ Orthocentre O (x,y) = (2, 5/3).

- Find the foot of the perpendicular drawn from (3, 0) upon the straight line $5x + 12y - 41 = 0$.

Sol: Let (h, k) be the foot of the perpendicular from (3, 0) on the line $5x + 12y - 41 = 0$

Here $(x_1, y_1) = (3, 0)$, $a = 5$, $b = 12$, $c = -41$.

$$\therefore \frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2} \Rightarrow \frac{h - 3}{5} = \frac{k - 0}{12} = \frac{-[5(3) + 12(0) - 41]}{5^2 + 12^2} = \frac{26}{169} = \frac{2}{13}$$

$$\text{Now, } \frac{h - 3}{5} = \frac{2}{13} \Rightarrow 13h - 39 = 10 \Rightarrow 13h = 49 \Rightarrow h = \frac{49}{13}. \quad \text{Also } \frac{k}{12} = \frac{2}{13} \Rightarrow k = \frac{24}{13}$$

$$\therefore \text{The foot of the perpendicular is } (h, k) = \left(\frac{49}{13}, \frac{24}{13} \right)$$

- Find the image of (1, 2) in the straight line $3x + 4y - 1 = 0$

Sol: Let (h, k) be the image of (1, 2) w.r.t to $3x + 4y - 1 = 0$

Here $(x_1, y_1) = (1, 2)$, $a = 3$, $b = 4$, $c = -1$.

$$\therefore \frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

$$\Rightarrow \frac{h - 1}{3} = \frac{k - 2}{4} = \frac{-2[3(1) + 4(2) - 1]}{3^2 + 4^2} = \frac{-2(10)}{25} = -2\left(\frac{2}{5}\right) = -\frac{4}{5}$$

$$\text{Now, } \frac{h - 1}{3} = -\frac{4}{5} \Rightarrow h - 1 = -\frac{12}{5} \Rightarrow h = 1 - \frac{12}{5} = \frac{5 - 12}{5} = -\frac{7}{5}$$

$$\text{Also } \frac{k - 2}{4} = -\frac{4}{5} \Rightarrow k - 2 = -\frac{16}{5} \Rightarrow k = 2 - \frac{16}{5} = \frac{10 - 16}{5} = -\frac{6}{5}$$

$$\therefore \text{The image is } (h, k) = \left(-\frac{7}{5}, -\frac{6}{5} \right)$$

- If p and q are the lengths of the perpendiculars from the origin to the straight lines $x \sec \alpha + y \csc \alpha = a$ and $x \cos \alpha - y \sin \alpha = a \cos 2\alpha$, prove that $4p^2 + q^2 = a^2$.

Sol: Given line $x \sec \alpha + y \csc \alpha = a$

$$\Rightarrow \frac{x}{\cos \alpha} + \frac{y}{\sin \alpha} = a \Rightarrow \frac{x \sin \alpha + y \cos \alpha}{\cos \alpha \sin \alpha} = a \Rightarrow x \sin \alpha + y \cos \alpha = a \sin \alpha \cos \alpha$$

The perpendicular distance O(0,0) to $x \sin \alpha + y \cos \alpha - a \sin \alpha \cos \alpha = 0$ is

$$p = \frac{|-a \sin \alpha \cos \alpha|}{\sqrt{\sin^2 \alpha + \cos^2 \alpha}} = \frac{a \sin \alpha \cos \alpha}{1} = a \sin \alpha \cos \alpha$$

$$\text{Now } 2p = a(2 \sin \alpha \cos \alpha) = a \sin 2\alpha \Rightarrow 4p^2 = a^2 \sin^2 2\alpha \dots \dots (1)$$

The perpendicular distance from O(0,0) to $x \cos \alpha - y \sin \alpha - a \cos 2\alpha = 0$ is

$$q = \frac{|-a \cos 2\alpha|}{\sqrt{\cos^2 \alpha + \sin^2 \alpha}} = \frac{a \cos 2\alpha}{1} = a \cos 2\alpha \quad \therefore q^2 = a^2 \cos^2 2\alpha \dots \dots (2)$$

From (1) & (2), we have

$$4p^2 + q^2 = a^2 \sin^2 2\alpha + a^2 \cos^2 2\alpha = a^2 (\sin^2 2\alpha + \cos^2 2\alpha) = a^2 (1) = a^2$$

Q20: DC'S & DR'S:

- Find the angle between the lines whose d.c's are related by $l+m+n=0$ and $l^2+m^2-n^2=0$

A: Given $l+m+n=0 \Rightarrow l=-(m+n) \dots (1)$,

$$l^2+m^2-n^2=0 \dots (2)$$

Solving (1) & (2) we get

$$[-(m+n)]^2 + m^2 - n^2 = 0$$

$$\Rightarrow (m^2 + n^2 + 2mn) + m^2 - n^2 = 0$$

$$\Rightarrow 2m^2 + 2mn = 0 \Rightarrow m(m+n) = 0$$

$$\Rightarrow m^2 + mn = 0 \Rightarrow m(m+n) = 0$$

$$\Rightarrow m=0 \text{ (or) } m+n=0 \Rightarrow m=0 \text{ (or) } m=-n$$

Case (i): Put $m=0$ in (1), then $l=-(0+n)=-n$

$$\therefore l=-n$$

$$\text{Now, } l:m:n = -n:0:n = -1:0:1$$

$$\text{So, d.r's of } L_1 = (a_1, b_1, c_1) = (-1, 0, 1) \dots (3)$$

Case (ii):

$$\text{Put } m=-n \text{ in (1), then } l=-(-n+n)=0$$

$$\therefore l=0$$

$$\text{Now, } l:m:n = 0:-n:n = 0:-1:1$$

$$\text{So, d.r's of } L_2 = (a_2, b_2, c_2) = (0, -1, 1) \dots (4)$$

If θ is the angle between the lines then from (3), (4), we get

$$\begin{aligned} \cos \theta &= \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}} \\ &= \frac{|(-1)(0) + (0)(-1) + 1(1)|}{\sqrt{((-1)^2 + 0^2 + 1^2)(0^2 + (-1)^2 + 1^2)}} \\ &= \frac{1}{\sqrt{(2)(2)}} = \frac{1}{\sqrt{4}} = \frac{1}{2} = \cos 60^\circ \Rightarrow \theta = 60^\circ \end{aligned}$$

Hence angle between the lines is 60° .

- Find the direction cosines of two lines which are connected by the relations $l+m+n=0$ and $mn-2n/-2/m=0$

A: Given $l+m+n=0 \Rightarrow l=-m-n \dots (1)$,

$$mn-2n/-2/m=0 \dots (2)$$

Solving (1) & (2), we get

$$mn-2n(-m-n)-2m(-m-n)=0$$

$$\Rightarrow mn+2mn+2n^2+2m^2+2mn=0$$

$$\Rightarrow 2m^2+5mn+2n^2=0$$

$$\Rightarrow (2m+n)(m+2n)=0$$

$$\Rightarrow 2m+n=0 \text{ (or) } m+2n=0$$

$$\text{So, } 2m+n=0 \Rightarrow 2m=-n$$

$$\Rightarrow m=-n/2 \text{ (or)}$$

$$m+2n=0 \Rightarrow m=-2n$$

Case (i): Put $m=-\frac{n}{2}$ in (1), then

$$l=-\left(-\frac{n}{2}\right)-n=\frac{n}{2}-n=-\frac{n}{2} \Rightarrow l=-\frac{n}{2}$$

$$\therefore l:m:n = -\frac{n}{2}:-\frac{n}{2}:n$$

$$= -\frac{1}{2}:-\frac{1}{2}:1 = \frac{1}{2}:\frac{1}{2}:1 = 1:1:2$$

$$\text{So, d.r's of } L_1 = (a_1, b_1, c_1) = (1, 1, -2)$$

On dividing by

$$\sqrt{1^2+1^2+(-2)^2} = \sqrt{1+1+4} = \sqrt{6}, \text{ we get}$$

$$\text{d.c's of } L_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

Case (ii): Put $m=-2n$ in (1), then

$$l=-(-2n)-n=2n-n=n \Rightarrow l=n$$

$$\therefore l:m:n = n:-2n:n = 1:-2:1$$

$$\text{So, d.r's of } L_2 = (a_2, b_2, c_2) = (1, -2, 1)$$

On dividing by

$$\sqrt{1^2+(-2)^2+1^2} = \sqrt{1+4+1} = \sqrt{6}, \text{ we get}$$

$$\text{d.c's of } L_2 = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

- The vertices of a triangle are A(1,4,2), B(-2,1,2), C(2,3,-4) .

Find the angles A, B, C

Sol: Given A(1, 4, 2) B(-2, 1, 2) C(2, 3, -4)

Direction ratios (D.r's) of AB are $(-2-1, 1-4, 2-2) = (-3, -3, 0)$

D.r's of AC are $(2-1, 3-4, -4-2) = (1, -1, -6)$

If A is the angle at vertex A, then

$$\cos A = \frac{|(-3)(1) + (-3)(-1) + 0(-6)|}{\sqrt{(-3)^2 + (-3)^2 + 0^2} \sqrt{1^2 + (-1)^2 + (-6)^2}} = \frac{0}{\sqrt{18}\sqrt{38}} = 0 \quad \therefore A = \frac{\pi}{2}$$

Now, D.r's of BA are $(1 - (-2), 4 - 1, 2 - 2) = (3, 3, 0)$

D.r's of BC are $(2 - (-2), 3 - 1, -4 - 2) = (4, 2, -6)$

If B is the angle at vertex B, then

$$\cos B = \frac{|(3)(4) + (3)(2) + 0(-6)|}{\sqrt{3^2 + 3^2} \sqrt{4^2 + 2^2 + (-6)^2}} = \frac{18}{\sqrt{18}\sqrt{56}} = \frac{18}{(3\sqrt{2})(2\sqrt{14})} = \frac{3}{2\sqrt{7}} \quad \therefore B = \cos^{-1}\left(\frac{3}{2\sqrt{7}}\right)$$

Similarly, D.r's of CA are $(1 - 2, 4 - 3, 2 - (-4)) = (-1, 1, 6)$

D.r's of CB are $(-2 - 2, 1 - 3, 2 - (-4)) = (-4, -2, 6)$

If C is the angle at vertex C, then

$$\cos C = \frac{|(-1)(-4) + (1)(-2) + (6)(6)|}{\sqrt{(-1)^2 + 1^2 + 6^2} \sqrt{(-4)^2 + (-2)^2 + 6^2}} = \frac{4 - 2 + 36}{\sqrt{38}\sqrt{56}} = \frac{38}{\sqrt{38}\sqrt{56}} = \frac{\sqrt{38}}{\sqrt{56}} = \sqrt{\frac{19}{28}} \Rightarrow C = \cos^{-1}\left(\sqrt{\frac{19}{28}}\right)$$

- If (6,10,10), (1,0,-5) , (6,-10,0) vertices of a triangle , find the direction ratios of its sides. Determine whether it is right angled or isosceles.

Sol: Given A(6, 10, 10), B(1, 0, -5), C(6, -10, 0)

D.r.'s of AB = $(1-6, 0-10, -5-10) = (-5, -10, -15)$

D.r.'s of BC = $(6-1, -10+0, 0-(-5)) = (5, -10, 5)$

D.r.'s of CA = $(6-6, 10-(-10), 10+0) = (0, 20, 10)$

$$AB = \sqrt{(-5)^2 + (-10)^2 + (-15)^2} = \sqrt{350} = 5\sqrt{14} \Rightarrow AB^2 = 350$$

$$BC = \sqrt{5^2 + (-10)^2 + 5^2} = \sqrt{150} = 5\sqrt{6} \Rightarrow BC^2 = 150$$

$$CA = \sqrt{0^2 + 20^2 + 10^2} = \sqrt{500} = 10\sqrt{5} \Rightarrow CA^2 = 500$$

Check for right angle:

$$AB^2 + BC^2 = 350 + 150 = 500 = CA^2$$

$\therefore \Delta ABC$ is right angled at B.

Check for isosceles: $AB = 5\sqrt{14}$, $BC = 5\sqrt{6}$, $CA = 10\sqrt{5}$

$\therefore \Delta ABC$ is not isosceles.

Q21: DIFFERENTIATION:

- If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then

prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

A: Given $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

We take $x = \sin \alpha$, $y = \sin \beta$, then

$$\sqrt{1-\sin^2 \alpha} + \sqrt{1-\sin^2 \beta} = a(\sin \alpha - \sin \beta)$$

$$\Rightarrow \cos \alpha + \cos \beta = a(\sin \alpha - \sin \beta)$$

$$\Rightarrow \frac{\cos \alpha + \cos \beta}{\sin \alpha - \sin \beta} = a$$

$$\Rightarrow \frac{2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)}{2 \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)} = a$$

$$\left[\begin{array}{l} \because \cos C + \cos D = 2 \cos \left(\frac{C+D}{2} \right) \cos \left(\frac{C-D}{2} \right) \\ \sin C - \sin D = 2 \cos \left(\frac{C+D}{2} \right) \sin \left(\frac{C-D}{2} \right) \end{array} \right]$$

$$\Rightarrow \frac{\cos \left(\frac{\alpha - \beta}{2} \right)}{\sin \left(\frac{\alpha - \beta}{2} \right)} = a \Rightarrow \cot \left(\frac{\alpha - \beta}{2} \right) = a$$

$$\Rightarrow \frac{\alpha - \beta}{2} = \cot^{-1}(a)$$

$$\Rightarrow \alpha - \beta = 2 \cot^{-1}(a)$$

But $\sin \alpha = x \Rightarrow \alpha = \sin^{-1} x$ and $y = \sin \beta$

$$\Rightarrow \beta = \sin^{-1} y$$

$$\therefore \sin^{-1} x - \sin^{-1} y = 2 \cot^{-1}(a)$$

On diff. w.r.to x, we get $\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = 0$

$$\Rightarrow \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Hence proved.

- If $y = \tan^{-1} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right)$ then

find $\frac{dy}{dx}$

A: Given $y = \tan^{-1} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right)$

We take $x^2 = \cos 2\theta$, then

$$y = \tan^{-1} \left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}} \right)$$

$$= \tan^{-1} \left(\frac{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}} \right)$$

$$= \tan^{-1} \left(\frac{\sqrt{2} [\cos \theta + \sin \theta]}{\sqrt{2} [\cos \theta - \sin \theta]} \right)$$

$$= \tan^{-1} \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right)$$

$$= \tan^{-1} \left(\frac{\frac{\cos \theta}{\cos \theta} + \frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\cos \theta} - \frac{\sin \theta}{\cos \theta}} \right) = \tan^{-1} \left(\frac{1 + \tan \theta}{1 - \tan \theta} \right)$$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \theta \right) \right] = \frac{\pi}{4} + \theta$$

$$\therefore y = \frac{\pi}{4} + \theta$$

But $\cos 2\theta = x^2 \Rightarrow 2\theta = \cos^{-1}(x^2)$

$$\Rightarrow \theta = \frac{1}{2} \cos^{-1}(x^2)$$

So, $y = \frac{\pi}{4} + \frac{1}{2} \cos^{-1}(x^2)$

On diff. w.r.t x, we get

$$\frac{dy}{dx} = 0 + \frac{1}{2} \left(\frac{-1}{\sqrt{1-(x^2)^2}} (2x) \right),$$

$$\left[\because \frac{d}{dx} \cos^{-1} f(x) = \frac{-1}{\sqrt{1-(f(x))^2}} \cdot \frac{d}{dx} f(x) \right]$$

$$= \frac{-x}{\sqrt{1-x^4}}$$

- If $y = x\sqrt{a^2 + x^2} + a^2 \log(x + \sqrt{a^2 + x^2})$, then show that $\frac{dy}{dx} = 2\sqrt{a^2 + x^2}$

Sol: Given $y = x\sqrt{a^2 + x^2} + a^2 \log(x + \sqrt{a^2 + x^2})$;

On diff. w.r.to x, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \left[x \frac{d}{dx} \sqrt{a^2 + x^2} + \sqrt{a^2 + x^2} \frac{d}{dx} (x) \right] + a^2 \left[\frac{d}{dx} \log(x + \sqrt{a^2 + x^2}) \right] \\
 &= \left(x \frac{1}{2\sqrt{a^2 + x^2}} \frac{d}{dx} (a^2 + x^2) + (\sqrt{a^2 + x^2})(1) \right) + a^2 \left(\frac{1}{x + \sqrt{a^2 + x^2}} \frac{d}{dx} (x + \sqrt{a^2 + x^2}) \right) \\
 &= \left(\frac{x}{\cancel{2}\sqrt{a^2 + x^2}} (\cancel{2}x) + \sqrt{a^2 + x^2} \right) + \left(\frac{a^2}{x + \sqrt{a^2 + x^2}} \right) \left(1 + \frac{1}{\cancel{2}\sqrt{a^2 + x^2}} (\cancel{2}x) \right) \\
 &= \left(\frac{x^2}{\sqrt{a^2 + x^2}} + \sqrt{a^2 + x^2} \right) + \left(\frac{a^2}{x + \sqrt{a^2 + x^2}} \right) \left(1 + \frac{x}{\sqrt{a^2 + x^2}} \right) \\
 &= \left(\frac{x^2 + (a^2 + x^2)}{\sqrt{a^2 + x^2}} \right) + \left(\frac{a^2}{\cancel{x + \sqrt{a^2 + x^2}}} \right) \left(\frac{\cancel{\sqrt{a^2 + x^2}} + x}{\sqrt{a^2 + x^2}} \right) \\
 &= \frac{a^2 + 2x^2}{\sqrt{a^2 + x^2}} + \frac{a^2}{\sqrt{a^2 + x^2}} \\
 &= \frac{a^2 + 2x^2 + a^2}{\sqrt{a^2 + x^2}} \\
 &= \frac{2a^2 + 2x^2}{\sqrt{a^2 + x^2}} \\
 &= \frac{2(a^2 + x^2)}{\sqrt{a^2 + x^2}} \\
 &= 2\sqrt{a^2 + x^2} \quad [\because \frac{a}{\sqrt{a}} = \sqrt{a}]
 \end{aligned}$$

$$\therefore \boxed{\frac{dy}{dx} = 2\sqrt{a^2 + x^2}}$$