9 DIFFERENTIAL EQUATIONS

1 OMQ + 1 VSAQ + 1 LAQ [1 M + 2M +8M = 11 M]

9(a) ORDER & DEGREE OF D.E

CONCEPTS & FORMULAS

1) **Differential Equation (D.E):** An equation involving derivative of the dependent variable (y) w.r.to independent variable (x) is called a differential equation.

Ex: $x\frac{dy}{dx}+y=0; \frac{d^2y}{dx^2}+y=0$

2) **Order & Degree of a D.E :** Order of D.E is the order of the **highest order derivative;** and **Degree of a D.E** is **highest power of the highest order derivative.**

Ex 1: $\frac{dy}{dx}, f'(x), y'...$ are First Order Derivatives; $\frac{d^2y}{dx^2}, f''(x), y''...$ are Second Order Derivatives;

Ex 2: $\frac{dy}{dx}, \frac{d^2y}{dx^2}, f'(x), y''...$ are First degree Derivatives; $\left(\frac{dy}{dx}\right)^2, \left(\frac{d^2y}{dx^2}\right)^2, (y')^2...$ are of Second degree

Ex 3: For the derivative $\left(\frac{d^2y}{dx^2}\right)^3$, the order is 2 and degree is 3.

☛ **Highest derivative:** Among the given derivatives, it is the derivative with highest order.

Ex: Among the given derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \left(\frac{dy}{dx}\right)^3$ the highest derivative is $\frac{d^2y}{dx^2}$

3) **Conversion of an Algebraic Equation into Differential Equation & vice-versa:**
Consider an algebraic equation y = mx....(1) where m is a parameter.

Differentiating (1) w.r.t x, we get $\frac{dy}{dx}=m$ $\therefore (1) \Rightarrow y=\left(\frac{dy}{dx}\right)x \Rightarrow y-x\frac{dy}{dx}=0$..........(2)

Here, (2) is called the differential equation corresponding to the given algebraic equation (1).

Conversely, if we are given the D.E $y-x\frac{dy}{dx}=0$ then from this we can obtain the original given equation y=mx which is known as the **solution** of the given Differential Equation.

In order to get the solution from a given Differential Equation, we need to apply anti- differentiation i.e., Integration techniques.

The main focus of the chapter 'Differential Equations' is to know the 'procedures of solving various kinds of differential equations' like variables separable, homogeneous D.E, linear D.E.

4) **Order Vs No.of Parameters:** When a Differential Equation is formed from an algebraic or Trigonometric equation then the **order** of **D.E** is equal to the **number of parameters** (arbitrary constants) in the given equation.

9(b) GENERAL & PARTICULAR SOLUTION OF A D.E

CONCEPTS & FORMULAS

1) **General Solution:** The solution which contains arbitrary constants is called the ***general solution* (*primitive*)** of the differential equation.

2) **Particular Solution:** The solution free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a ***particular solution*** of the differential equation.

SOLVED EXAMPLES -9(b)

*** Ex 2: Verify that the function $y = e^{-3x}$ is a solution of the differential equation**

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Sol: Given function is $y = e^{-3x}$.

Differentiating both sides of equation with respect to x , we get $\frac{dy}{dx} = -3e^{-3x}$

Now, differentiating (1) with respect to x, we have $\frac{d^2y}{dx^2} = 9e^{-3x}$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get

L.H.S. $= 9e^{-3x} + (-3e^{-3x}) - 6e^{-3x} = 9e^{-3x} - 9e^{-3x} = 0 =$ R.H.S..

$\therefore$ the given function is a solution of the given differential equation.. Hence verified.

***Ex 3: Verify that the function $y = a\cos x + b\sin x$, where, $a, b \in R$ is a solution of the differential equation** $\frac{d^2y}{dx^2} + y = 0$

Sol: The given function is $y = a\cos x + b\sin x$ ... (1)

Differentiating both sides of equation (1) with respect to x, successively, we get

$$\frac{dy}{dx} = -a\sin x + b\cos x, \quad \frac{d^2y}{dx^2} = -a\cos x - b\sin x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

L.H.S. $= (-a\cos x - b\sin x) + (a\cos x + b\sin x) = 0 =$ R.H.S.

$\therefore$ the given function is a solution of the given differential equation.. Hence verified.

9(C) VARIABLES SEPARABLE, APPLICATIONS ON G.S & P.S

CONCEPTS & FORMULAS

1) **Variables Separable:** A differential equation is said to be in the variables separable form if it can be expressed in the form $f(y)dy = g(x)dx$ (or) $f(x)dx + g(y)dy = 0$ (or) $\frac{dy}{f(y)} = \frac{dx}{g(x)}$

Example: Consider the D.E $\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Here the two variables x and y are separated.

$\Rightarrow \frac{dx}{1+x^2} - \frac{dy}{1+y^2} = 0$. This is in the form $f(x)dx + g(y)dy = 0$. So it's Variable Separable.

2) **Solution of the D.E:** After separating the variables, integrate the terms accordingly. The constant of integration (c) can be 'modified accordingly' to express the solution interms of y in the simplest form.

While solving a D.E in the 'variable separable method', generally we keep the y & dy terms in the L.H.S in order to express the final solution in terms of the dependent variable y.

☞ Some times, the 'final solution' gets 'transformed differently' depending on how c is modified.

☞ Since c is arbitrary constant c_1, c_2, c^2, logc, – logc, kc etc., may be modified by a single 'c'.

3) **D.E reducible to variables separable:**

Certain D.Es can be reduced into the 'variables separable form' by suitable substitutions.

For example, a differential equation of the form $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to 'variables separable form' by using the substitution ax + by + c = t.

Here, after getting the solution interms of t, replace t by ax+by+c to get the final solution in x and y.

4) **Hints on Partial Fractions:** $\frac{1}{x(x+a)} = \frac{1}{a}\left(\frac{1}{x} - \frac{1}{x+a}\right)$; $\frac{1}{x(x-a)} = \frac{1}{a}\left(\frac{1}{x-a} - \frac{1}{x}\right)$

In words, $\frac{1}{(x+a)(x+b)} = \frac{1}{\text{difference of factors}}\left(\frac{1}{\text{small factor}} - \frac{1}{\text{big factor}}\right)$

9(d) HOMOGENEOUS DIFFERENTIAL EQUATIONS

CONCEPTS & FORMULAS

1) **Homogeneous function:** A function F(x,y) is said to be a homogeneous function of degree 'n' in x & y if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$, for some $\lambda \neq 0$

Ex: $F_1(x, y) = 2x-3y$; $F_2(x, y) = x^2+2xy$; $F_3(x,y)=\cos\left(\frac{x}{y}\right)$

Non Ex: $F(x, y) = \sin x+\cos y$

2) **Homogeneous differential equation:** A differential equation of the form $\frac{dy}{dx}=F(x,y)$ is said to be homogenous if F(x, y) is a homogenous function of degree zero.

Ex: $\frac{dy}{dx}=\frac{x+2y}{x-y}$; $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$; $x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$

3) **Steps for solving a homogeneous differential equation (HDE):**

Step 1: Write the given differential equation in form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ ....(1)

Step 2: Put y=vx $\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Step 3: In the L.H.S of (1), replace $\frac{dy}{dx}$ by $v+x\frac{dv}{dx}$ and in the R.H.S of (1), put y=vx

Step 4: Get $x\frac{dv}{dx}$ in terms of v. Then we get **variables separable form in x & v.**

Step 5: Integrate and simplify, after resolving into partial fractions if required.

Step 6: Replace v by y/x in the final step to get the solution interms of x and y

9(e) LINEAR DIFFERENTIAL EQUATIONS

CONCEPTS & FORMULAS

1.1) Linear Differential Equation(LDE) in y: It is in the form $\frac{dy}{dx}+Py=Q$.

1.2) Integrating Factor (I.F): Integrating factor of the above linear D.E is I.F= $e^{\int P(x)dx}$.

1.3) Solution of L.D.E in y: Solution of $\frac{dy}{dx}+Py=Q$ is $y(I.F)=\int Q(I.F)dx+C$

2.1) Linear Differential Equation(LDE) in x: It is in the form $\frac{dx}{dy}+Px=Q$

2.2) Integrating Factor (I.F): Integrating factor of the above linear D.E is I.F= $e^{\int P(y)dy}$.

2.3) Solution of L.D.E in y: Solution of $\frac{dx}{dy}+Px=Q$ is $x.e^{\int P(y)dy}=\int Q.e^{\int P(y)dy}dy$

Hint 1: Most frequently occuring Integration formula in Linear differential equations:

$$\int\frac{1}{x}dx=\log|x|+c;\qquad \int\frac{dx}{ax+b}=\frac{1}{a}\log|ax+b|+c;\qquad \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c$$

$$\int\tan x\,dx=\log_e|\sec x|+c;\qquad \int\cot x\,dx=\log_e|\sin x|+c;\qquad \int\sec x\,dx=\log|\sec x+\tan x|+c$$

Hint 2: $e^{\log_e f(x)}=e^{\log f(x)}=f(x)$ **Hint 3:** $\int xe^x dx=e^x(x-1)+c$

ILLUSTRATIONS

Find the Integrating factor of (i) $x\frac{dy}{dx}-y=2x^2$ **(ii)** $\frac{dx}{dy}-\frac{x}{y}=2y$

Sol: (i) Given D.E is $x\frac{dy}{dx}-y=2x^2\Rightarrow\frac{dy}{dx}-\frac{y}{x}=2x$ This is in the $\frac{dy}{dx}+Py=Q$ form

where, $P=-\frac{1}{x}$ and $Q=2x$

$\therefore IF=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log(x^{-1})}=x^{-1}=\frac{1}{x}$

(ii) Given D.E is $\frac{dx}{dy}-\frac{x}{y}=2y$. This is in the form $\frac{dx}{dy}+P_1x=Q_1$, where $P_1=-\frac{1}{y}$ and $Q_1=2y$.

$\therefore I.F=e^{\int-\frac{1}{y}dy}=e^{-\log y}=e^{\log(y)^{-1}}=\frac{1}{y}$