7 INTEGRALS

1 OMQ + 1 VSAQ + 1 SAQ + 1 LAQ [1 M + 2M + 4M + 8M= 15 M]

7(a) INTEGRAL AS ANTI-DERIVATIVE

CONCEPTS & FORMULAS

1) Integral as Anti-derivative :

If $\frac{d}{dx}F(x)=f(x)$, then we know that, f(x) is called the derivative of F(x) with respect to x.

Inversely mentioning, F(x) is known as **anti derivative** or **primitive** or **integral** of f(x) w.r.to x.

We observe that, a given function possess several anti derivatives.

Ex: $\frac{d}{dx}(x^2+3x)=2x+3,\quad \frac{d}{dx}(x^2+3x+4)=2x+3,\quad \frac{d}{dx}(x^2+3x-5)=2x+3$

Thus, the anti derivatives of 2x+3 are x^2+3x or x^2+3x+4 or x^2+3x-5....., ingeneral x^2+3x+c, $c\in R$

2) Indefinite Integral: If F(x) is an anti derivative of f(x), then **F(x)+c**, $c\in R$ is called the **indefinite integral of f(x).**

3) Integration: Integration is an 'inverse process' (Anti - differentiation) of differentiation .

4) Notation: If the integral of f(x) is F(x) then it is denoted by $I=\int f(x)dx=F(x)+c$

The symbol $\int$ is called the integral operator, which is simply read as **integral.**

x is the variable of integration, **c** or **C** is called the **constant of integration**.

dx the differential, just conveys that the integration is to be done w.r.t **x**.

f(x), the function to be integrated is called the **integrand (integrable function)**.

F(x) is the **integral** and **F(x)+c** is the **indefinite integral** of f(x).

Rem 1: $\int(af(x)\pm bg(x))dx=a\int f(x)dx\pm b\int g(x)dx$, where a,b are any fixed constants.

Rem 2: In Indefinite Integrals, constant of integration 'c' can be added accordingly.

Rem 3: The domain restrictions on integrands are not mentioned in this book, as we focus mainly on the process of integration techniques.

5) The process of integration: Just like the process of differentiation, the process of integration for various kinds of functions like Algebraic rational / irrational functions, Trigonometric rational functions etc., involves various methods / techniques / rules of integration like; using certain standard forms, substitution method, integration by resolving into partial fractions, integration by parts,etc.,

ILLUSTRATIVE TABLE ON INTEGRATION

INTEGRATION FORMULAS	ILLUSTRATIONS
1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$	$\int x^2 dx = \frac{x^3}{3} + c;\ \int x^3 dx = \frac{x^4}{4} + c;\ \int x\,dx = \frac{x^2}{2} + c$
2) $\int k dx = kx + c$, k is constant	$\int 5dx = 5x + c;\ \int 1dx = x + c;\ \int \frac{1}{2}dx = \frac{1}{2}x + c$
3) $\int x dx = \frac{x^2}{2} + c$	$\int 3x\,dx = \frac{3x^2}{2} + c;\ \int 2x\,dx = \cancel{2}\left(\frac{x^2}{\cancel{2}}\right) + c = x^2 + c$
4) $\int \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} + c$	$\int (\sqrt{x} + x + 4)\,dx = \frac{2}{3} x^{\frac{3}{2}} + \frac{x^2}{2} + 4x + c$
5) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$	$\int \left(\frac{1}{\sqrt{x}} + \sqrt{x} + x + 5\right) dx = 2\sqrt{x} + \frac{2}{3} x\sqrt{x} + \frac{x^2}{2} + 5x + c$
6) $\int \frac{1}{x^2} dx = -\frac{1}{x} + c$	$\int \left(\frac{1}{x^2} + x^2\right) dx = -\frac{1}{x} + \frac{x^3}{3} + c$
7) $\int \frac{1}{x} dx = \log_e \lvert x \rvert + c$	$\int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx = \log_e \lvert x \rvert - \frac{1}{x} + c$
8) $\int a^x dx = a^x \log_a e + c,\ a \neq 1$	$\int (2^x - 3^x + 4^x) dx = 2^x \log_2 e - 3^x \log_3 e + 4^x \log_4 e + c$
9) $\int e^x dx = e^x + c$	$\int (e^x + 5^x + x^5) dx = e^x + 5^x \log_5 e + \frac{x^6}{6} + c$
10) $\int \sin x dx = -\cos x + c$	$\int (\sin x - x + 1) dx = -\cos x - \frac{x^2}{2} + x + c$
11) $\int \cos x dx = \sin x + c$	$\int (\cos x - \sin x) dx = \sin x + \cos x + c$
12) $\int \tan x dx = \log \lvert \sec x \rvert + c$	$\int (\tan x - \sin x + \cos x)\,dx = \log\lvert\sec x\rvert + \cos x + \sin x + c$
13) $\int \cot x dx = \log \lvert \sin x \rvert + c$	$\int (\cot x + \tan x) dx = \log \lvert \sin x \rvert + \log \lvert \sec x \rvert + c$
14) $\int \sec x\,dx = \log \lvert \sec x + \tan x \rvert + c$	$\int (\sec x + \tan x)\,dx = \log \lvert \sec x + \tan x \rvert + \log \lvert \sec x \rvert + c$
15) $\int \csc x dx = \log \lvert \csc x - \cot x \rvert + c$	$\int (\csc x - \cot x) dx = \log \lvert \csc x - \cot x \rvert - \log \lvert \sin x \rvert + c$
16) $\int \csc^2 x\,dx = -\cot x + c$	$\int (\csc^2 x - \cot x)\,dx = -\cot x - \log\lvert\sin x\rvert + c$
17) $\int \sec^2 x\,dx = \tan x + c$	$\int (\sec^2 x - \csc^2 x)\,dx = \tan x + \cot x + c$
18) $\int \sec x \tan x\,dx = \sec x + c$	$\int (\sec x \tan x + \sec x + \tan x) dx = \sec x + \log\lvert\sec x + \tan x\rvert + \log\lvert\sec x\rvert + c$
19) $\int \csc x \cot x\,dx = -\csc x + c$	$\int (\csc x \cot x - \sec x \tan x)\,dx = -\csc x - \sec x + c$

7(b) INTEGRATION BY SUBSTITUTION

CONCEPTS & FORMULAS

Certain integrands can be reduced into standard forms by using suitable substitutions given there.

MODEL FORMULA	ILLUSTRATIONS
Type-1: $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+c$ • **Identification:** ax + b occurs in place of x • **Answer:** $\frac{1}{a}$(Usual Integral) • **Substitution:** ax + b = t	1) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log\|ax+b\|+c$ 2) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$ 3) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$ 4) $\int (2x+3)^4 dx = \frac{1}{2}\frac{(2x+3)^5}{5}+c$
Type-2: $\int \frac{f'(x)}{f(x)}dx = \log f(x)+c$ • **Identification:** Derivative of Dr. comes in Nr. • **Answer:** log(Denominator) • $\int \frac{\text{derivative of } f(x)}{f(x)}dx = \log f\|x\|+c$	1) $\int \frac{2x+3}{x^2+3x+4}dx = \log\|x^2+3x+4\|+c$ 2) $\int \cot x dx = \int \frac{\cos x}{\sin x}dx = \log\|\sin x\|+c$ 3) $\int \frac{1-\sin x}{x+\cos x}dx = \log\|x+\cos x\|+c$
Type-3: $\int (f(x))^n f'(x)dx = \frac{(f(x))^{n+1}}{n+1}+c$ • **Identification:** Function with its derviative comes in the integrand • **Substitution:** f(x) = t	1) $\int (x^2+3x+4)^5(2x+3)dx = \frac{(x^2+3x+4)^6}{6}+c$ 2) $\int \sin^5 x\cos x dx = \frac{\sin^6 x}{6}+c$
Type-4: $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+c$	• $\int \frac{(2ax+b)}{\sqrt{ax^2+bx+c}}dx = 2\sqrt{ax^2+bx+c}+k$
Type-5: $\int f(g(x))g'(x)dx = F(g(x))+c$ • Inner function g(x) with its derivative g' (x) is seen in the Integrand • **Substitution:** g(x) = t	• $I = \int \cos(x^3)x^2dx$, Put $x^3 = t \Rightarrow 3x^2dx = dt \Rightarrow x^2dx = \frac{dt}{3}$ $\Rightarrow \int \cos t\frac{dt}{3} = \frac{1}{3}\int \cos t dt = \frac{1}{3}\sin t + c = \frac{1}{3}\sin x^3 + c$
Type-6: $\text{Linear}\sqrt{\text{Linear}}$; $\frac{1}{\text{Linear}\sqrt{\text{Linear}}}$ • **Substitution:** $\sqrt{\text{Linear}} = t$ $\text{Linear}\sqrt{\text{Quadratic Expression}}$ • **Substitution:** $\sqrt{\text{Quadratic}} = t$	• $\int x\sqrt{4x+3}\,dx$, Put $\sqrt{4x+3} = t$ • $\int \frac{1}{(x+2)\sqrt{x+1}}dx$ Put $\sqrt{x+1} = t$ • $(4x+2)\sqrt{x^2+x+1}$, Put $x^2+x+1 = t$

7(d) INTEGRALS OF PARTICULAR FUNCTIONS

6 STANDARD FORMULAS	ILLUSTRATIONS
i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + c$ (Resolvable Partial Fractions)	• $\int \frac{dx}{x^2-25} = \int \frac{dx}{x^2-5^2} = \frac{1}{2(5)}\log\left\lvert\frac{x-5}{x+5}\right\rvert + c$
ii) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + c$	• $\int \frac{dx}{49-x^2} = \int \frac{dx}{7^2-x^2} = \frac{1}{2(7)}\log\left\lvert\frac{7+x}{7-x}\right\rvert + c$
iii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\left(\tan^{-1}\frac{x}{a}\right) + c$ (Substitution: $x=a\tan\theta$)	1) $\int \frac{dx}{9+x^2} = \int \frac{dx}{3^2+x^2} = \frac{1}{3}\mathrm{Tan}^{-1}\frac{x}{3} + c$ 2) $\int \frac{dx}{3+4x^2} = \int \frac{dx}{\left((\sqrt{3})^2+(2x)^2\right)} = \frac{1}{\sqrt{3}}\frac{1}{2}\mathrm{Tan}^{-1}\frac{2x}{\sqrt{3}} + c$
iv) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$ (Substitution: $x=a\sin\theta$)	1) $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{\sqrt{2^2-x^2}} = \mathrm{Sin}^{-1}\frac{x}{2} + c$ 2) $\int \frac{dx}{\sqrt{9-25x^2}} = \int \frac{dx}{\sqrt{3^2-(5x)^2}} = \frac{1}{5}\mathrm{Sin}^{-1}\frac{5x}{3} + c$
v) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\lvert x+\sqrt{x^2-a^2}\rvert + c$ (Substitution: $x=a\cosh\theta$)	1) $\int \frac{dx}{\sqrt{x^2-1}} = \log\lvert x+\sqrt{x^2-1}\rvert + c$ 2) $\int \frac{6x}{\sqrt{x^2-4}} = \mathrm{Cosh}^{-1}\frac{x}{2} + c$
vi) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\lvert x+\sqrt{x^2+a^2}\rvert + c$ (Substitution: $x=a\sinh\theta$)	1) $\int \frac{dx}{\sqrt{3+x^2}} = \log\lvert x+\sqrt{3+x^2}\rvert + c$ 2) $\int \frac{dx}{\sqrt{1+4x^2}} = \int \frac{dx}{\sqrt{1^2+(2x)^2}} = \frac{1}{2}\mathrm{Sinh}^{-1}(2x) + c\,]$

II) Evaluation of $\int \frac{px+q}{ax^2+bx+c}dx,\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,\ \int (px+q)\sqrt{ax^2+bx+c}\ dx$

i.e., $\int \frac{\text{Linear}}{\text{Quadratic}}dx,\ \int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}}dx,\ \int \text{Linear}\sqrt{\text{Quadratic}}\ dx$ **forms**

Procedure: $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$ i.e, $\text{Linear} = A\frac{d}{dx}(\text{Quadratic})+B$

Now, find A,B by comparing the coefficients of x and the constant terms and integrate accordingly

III) 'COMPLETING THE SQUARE' PROCESS OF A QUADRATIC

Reducing process of ax^2+bx+c into $A^2 \pm X^2$ form.

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$

$$= a\left(x^2+2(x)\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right) = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2\right]$$

Inwords: First see that coefficient of x^2 is 1. Take its coefficient common if required.
Then, add and subtract the square of the half of the coefficient of x and simplify.

7(e) INTEGRATION BY PARTIAL FRACTIONS

HINTS ON PARTIAL FRACTIONS

1) $\dfrac{px+q}{(x-a)(x-b)} = \dfrac{A}{x-a}+\dfrac{B}{x-b}, \; a \neq b$; 2) $\dfrac{px+q}{(x-a)^2} = \dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$

3) $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \dfrac{A}{x-a}+\dfrac{B}{(x-b)}+\dfrac{C}{(x-c)}$

4) $\dfrac{px^2+qx+r}{(x-a)^2(x-b)} = \dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-b)}$

Tips on Partial Fractions:

$\dfrac{1}{x(x+a)} = \dfrac{1}{a}\left(\dfrac{1}{x}-\dfrac{1}{x+a}\right)$; $\dfrac{1}{x(x-a)} = \dfrac{1}{a}\left(\dfrac{1}{x-a}-\dfrac{1}{x}\right)$

$\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{\text{difference of factors}}\left(\dfrac{1}{\text{small factor}}-\dfrac{1}{\text{big factor}}\right)$

5) $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$ where x^2+bx+c cannot be factorised further.

Note: If the integrand is an improper rational function of the form $\dfrac{P(x)}{Q(x)}$ (deg P(x)≥ deg Q(x) then divide P(x) by Q(x), write $\dfrac{P(x)}{Q(x)} = T(x)+\dfrac{P_1(x)}{Q(x)}$ and proceed accordingly.

7(f) INTEGRATION BY PARTS

CONCEPTS & FORMULAS

We know well about the **uv** formula in **differentiation**. Similarly, if it is required to integrate a function in the form **uv**, then we apply the formula called **Integration by parts**.

1) Integration by parts: If u,v are functions of x then $\int(uv)dx = uv_1 - \int(v_1u')dx + c$ where v_1 is integral of v and u' is derivative of u.

Here, u is called the first function and v is called the second function.

In words, $\int$ (first function).(second function) dx =

(first function)(integral of second function)–$\int$(integral of second function) (diff.coeff.of first function)dx

☛ We prefer the second function whose integral is well known to us .

2) 'Integration by parts formula' can also be written in various following other ways.

(i) $\int uv = u\int v - \int\left(\int v.du\right) + c$ (or) $\int uv = u\int v - \int\left(du\int v\right) + c$

(ii) $\int udv = uv - \int vdu + c$

(iii) $\int uv' = uv - \int vu' + c$

(iv) $\int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)]dx + c$

Note 1: In applying the 'by parts formula', any convenient notation or method can be used.

Note 2: 'Integration by parts formula' can be applied directly incase of lengthy solutions.

Note 3: Constant of integration 'c' can be added accordingly.

Note 4: In the second part of R.H.S of the formula, any one of $\int(v_1u')dx$ (or) $\int(u'v_1)dx$ be used.

3) Extension Rule: If u is an algebraic function whose derivative vanishes after few steps and v is an exponential function then $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$ +c

Here u', u'', u''',... are successive derivatives of u and v_1, v_2, v_3,... are successive integrals of v.

4) ILATE Order: It is a preferable choice, to designate first function u & and second function v in the serial order of **I**nverse trigonometric, **L**ogarithmic, **A**lgebraic, **T**rigonometric, **E**xponential functions.

5) Single Integrand- Second Function 1(SF1): Incase of certain single integrand functions like $\log x, \mathrm{Sin}^{-1}x, \sqrt{a^2 - x^2}$,...... the second function (v) should be taken as '1' .

6) Model problems involving 'integration by parts':

(i) Problems which use 'integration by parts' only once

(ii) Problems which use 'integration by parts' twice or more

(iii) Problems which need to take '1' as second function

(iv) Problems reducible to 'integration by parts' form by suitable substitution

(v) 'Recurrence of Integrand' or 'Successive reduction' model problems

7) Important Result: $\int e^x(f(x) + f'(x))dx = e^x f(x) + c$

7(g) INTEGRATION OF SPECIAL FUNCTIONS

CONCEPTS & FORMULAS

1) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

• $\int\sqrt{25-x^2}\,dx=\frac{x}{2}\sqrt{5^2-x^2}+\frac{25}{2}\mathrm{Sin}^{-1}\frac{x}{5}+c$

(Substitution: x=asinθ)

2) $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+c$

• $\int\sqrt{1+x^2}\,dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\mathrm{Sinh}^{-1}x+c$

(Substitution: x=asinhθ)

3) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+c$

• $\int\sqrt{x^2-9}\,dx=\frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\mathrm{Cosh}^{-1}\frac{x}{4}+c$

(Substitution: x=acoshθ)

4) For $\int\frac{dx}{ax^2+bx+c}$ or $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ write ax^2+bx+c as follows:

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right)$$

5) Evaluation of $\int\frac{px+q}{ax^2+bx+c}dx$, $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$, $\int(px+q)\sqrt{ax^2+bx+c}\;dx$

i.e., $\int\frac{\text{Linear}}{\text{Quadratic}}dx,\int\frac{\text{Linear}}{\sqrt{\text{Quadratic}}}dx,\int\text{Linear}\sqrt{\text{Quadratic}}\;dx$ forms

Procedure: Write $\text{Linear}=A\frac{d}{dx}(\text{Quadratic})+B$

$$px+q=A\frac{d}{dx}(ax^2+bx+c)+B=A(2ax+b)+B$$

Now, find A,B by comparing the coefficients of x and the constant terms and integrate accordingly

7(h) DEFINITE INTEGRALS

CONCEPTS & FORMULAS

1) The fundamental theorem - II of integral calculus:

Let f be a continuous function defined on the interval [a, b] and F be the premitive of f then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Here, a is called the lower limit of integration and b is called the upper limit of integration

Example: $\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

2) Evaluation of Definite Integrals:

Procedure 1: Integrate the given integrand as usual without changing the limits and finally in the premitive function substitute the lower and upper limits as required.

Procedure 2: Select a proper substitution, say f(x) = t so that the given integrand reduces to a known standard form, in terms of the new variable t. Find the corresponding lower limit & upper limit in terms of the new variable t. Similarly, integrate the 'new integrand' w.r.to 't' and simplify and finally replace t with f(x).

Ex: Let us evaluate $\int_0^1 \frac{Tan^{-1}x}{1+x^2}dx$ using both the procedures.

Sol: Procedure 1: We know that $\int f(x)f'(x)dx = \frac{[f(x)]^2}{2}$

$$\therefore I = \int_0^1 \frac{Tan^{-1}x}{1+x^2}dx = \left[\frac{(Tan^{-1}x)^2}{2}\right]_0^1 = \frac{1}{2}\left[[Tan^{-1}(1)]^2 - [Tan^{-1}(0)]^2\right] = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{32}$$

Procedure 2: **Put $Tan^{-1}x = t \Rightarrow \left(\frac{1}{1+x^2}\right)dx = dt$. Also $Tan^{-1}x = t \Rightarrow x = \tan t$**

Now, the lower limit x = 0 ⇒ tant = 0 ⇒ t = 0; the upper limit x = 1 ⇒ tant = 1 ⇒ t= π/4

$$\therefore I = \int_0^{\pi/4} t dt = \left(\frac{t^2}{2}\right)_0^{\pi/4} = \frac{1}{2}[t^2]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0\right] = \frac{\pi^2}{32}$$

3) Definite Integral by 'By parts': $\int_a^b uv\,dx = [uv_1]_a^b - \int_a^b (v_1 u')dx$

Ex: Let us Evaluate $\int_0^{\pi/2} x\sin x dx$ using integration by parts.

Sol: Let u = x and v = sinx. Then v_1= –cosx and u' = 1. Now from "By Parts Rule", we have

$$I = \int_0^{\pi/2} x\sin x dx = [x(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)(1)dx = \left[\frac{-\pi}{2}\cos\frac{\pi}{2} - 0\right] + \int_0^{\pi/2} \cos x dx$$

$$= (0-0) + [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

7(j)PROPERTIES OF DEFINITE INTEGRAL

Property-0 (P_0) : $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(\theta)d\theta$

Thus the value of the definite integral does not change with the change of the variable.

Example: $\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

$\int_0^{\pi/2} \cos\theta d\theta = [\sin\theta]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

Property P_1: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof: If $\int f(x)dx = F(x)$ then $\int_a^b f(x)dx = F(b) - F(a) = -[F(a) - F(b)] = -[F(x)]_b^a = -\int_b^a f(x)dx$

Example: $\int_1^2 xdx = \left(\frac{x^2}{2}\right)_1^2 = \frac{1}{2}(2^2 - 1^2) = \frac{1}{2}(4-1) = \frac{3}{2}$

$\int_2^1 xdx = \left(\frac{x^2}{2}\right)_2^1 = \frac{1}{2}(1^2 - 2^2) = \frac{1}{2}(1-4) = -\frac{3}{2} = -\int_1^2 xdx$

Property P_2: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $a < c < b$

Proof: $\int_a^c f(x)dx + \int_c^b f(x)dx = [F(x)]_a^c + [F(x)]_c^b = \cancel{F(c)} - F(a) + F(b) - \cancel{F(c)} = F(b) - F(a) = \int_a^b f(x)dx$

Example: Let us verify that $\int_1^3 xdx = \int_1^2 xdx + \int_2^3 xdx$

Sol: $\int_1^3 xdx = \left(\frac{x^2}{2}\right)_1^3 = \frac{1}{2}(3^2 - 1^2) = \frac{1}{2}(9-1) = \frac{1}{2}(8) = 4$

$\int_1^2 xdx = \left(\frac{x^2}{2}\right)_1^2 = \frac{1}{2}(2^2 - 1^2) = \frac{1}{2}(4-1) = \frac{3}{2}$

$\int_2^3 xdx = \left(\frac{x^2}{2}\right)_2^3 = \frac{1}{2}(3^2 - 2^2) = \frac{1}{2}(9-4) = \frac{5}{2}$

$\therefore \int_1^2 xdx + \int_2^3 xdx = \frac{3}{2} + \frac{5}{2} = \frac{8}{2} = 4 = \int_1^3 xdx$ Hence verified.

Property P_3: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Example: $\int_{\pi/4}^{\pi/2} \csc^2 x dx = [-\cot x]_{\pi/4}^{\pi/2} = -\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) = -(0-1) = 1$

Now, $\int_{\pi/4}^{\pi/2} \csc^2\left(\frac{\pi}{4} + \frac{\pi}{2} - x\right)dx = \int_{\pi/4}^{\pi/2} \csc^2\left(\frac{3\pi}{4} - x\right)dx = \left(\cot\left(\frac{3\pi}{4} - x\right)\right)_{\pi/4}^{\pi/2}$

$= \cot\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) - \cot\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \cot\frac{\pi}{4} - \cot\frac{\pi}{2} = 1 - 0 = 1$. Hence verified the result.

Property P_4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Example: $\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

Now, $\int_0^{\pi/2} \cos\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = -\left(\cos\frac{\pi}{2} - \cos 0\right) = -(0-1) = 1$

Hence verified the result.

Property P_5: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Property P_6: $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Example: **Evaluate $\int_0^{\pi} \sin^3 x . \cos^3 x \, dx$.**

Sol: Let $f(x) = \sin^3 x \cos^3 x$

$\Rightarrow f(\pi - x) = \sin^3(\pi - x)\cos^3(\pi - x) = (\sin^3 x)(-\cos^3 x) = -\sin^3 x \cos^3 x = -f(x)$

$\therefore I = \int_0^{\pi} \sin^3 x . \cos^3 x \, dx = 0$

Property P_7: $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function} \\ 0 & \text{if } f(x) \text{ is an odd function} \end{cases}$

Example 1: Find $\int_{-\pi/2}^{\pi/2} \cos x dx$

Sol: Let $f(x) = \cos x \Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$. So $\cos x$ is an even function.

$\therefore I = \int_{-\pi/2}^{\pi/2} \cos x dx = 2\int_0^{\pi/2} \cos x dx = 2[\sin x]_0^{\pi/2} = 2\left(\sin\frac{\pi}{2} - \sin 0\right) = 2(1-0) = 2$

Example 2: Find $\int_{-\pi/2}^{\pi/2} \sin x dx$

Sol: Let $f(x) = \sin x \Rightarrow f(-x) = \sin(-x) = -\sin x = -f(x)$. So $\sin x$ is an odd function.

$\therefore I = \int_{-\pi/2}^{\pi/2} \sin x dx = 0$