4 DETERMINANTS

1 OMQ + 1 VSAQ + 1 SAQ + 1 LAQ [1 M + 2M +4M + 8M = 15M]

4(a) DETERMINANTS-I

[DETERMINANT OF A 2×2 MATRIX & 3×3 MATRIX]

CONCEPTS & FORMULAS

1) DETERMINANT OF A MATRIX:

• If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then determinant of A is denoted by det A or |A| or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Ex: $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3(2)-(1)(4)=6-4=2$

• If $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ then its determinant |A| is given as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3-c_2b_3)-a_2(b_1c_3-c_1b_3)+a_3(b_1c_2-c_1b_2)$$

2) If |A|=0 then A is said to be a **singular matrix** and

if |A|≠0 then A is said to be a **non-singular matrix**.

3) What is a Determinant?

Consider the system of equations ax+by=0 ...(1); cx+dy=0 ....(2)

The above system can be written in the matrix equation form as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now, (1) $\Rightarrow ax=-by \Rightarrow \frac{x}{y}=-\frac{b}{a}$

$(2)\Rightarrow cx=-dy\Rightarrow \frac{x}{y}=-\frac{d}{c}$; hence $-\frac{b}{a}=-\frac{d}{c}\Rightarrow ad=bc\Rightarrow ad-bc=0$

This expression ad–bc is the **eliminate** of the given system of equations (1) & (2) and it is called the determinant of the coefficient matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Matrix Vs Determinant

- Elements of a matrix are enclosed between square brackets, while elements of a determinant are enclosed between vertical lines.
- Matrix is an array of numbers, but determinant of a matrix is a real value.
- In matrices we deal with square and rectangular matrices, but in determinants we deal with the determinants of square matrices only.
- Distinction should be made clearly especially w.r.to scalar multiplication property.

4(b) DETERMINANTS-II

[AREA OF A TRIANGLE-COLLINEARITY, EQUATION OF A LINE]

CONCEPTS & FORMULAS

1) The area of a triangle whose vertices are (x_1,y_1), (x_2,y_2),(x_3,y_3) is given by $\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

The expansion of the above determinant is $\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1)+ x_3(y_1-y_2)]$.

Remarks:

(i) Area is a positive quantity. So we always take the absolute value (positive) of the determinant .

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

***Textual Ex: Find the area of the triangle whose vertices are (3, 8), (– 4, 2) and (5, 1).**

Sol: Let the vertices of triangle be $A(x_1,y_1)= (3, 8)$, $B(x_2,y_2)= (-4, 2)$, $C(x_3,y_3)= (5, 1)$

The area of triangle ABC is given by $\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$

$$=\frac{1}{2}[3(2-1)-8(-4-5)+1(-4-10)]=\frac{1}{2}(3+72-14)=\frac{61}{2} \text{ Sq. units.}$$

2) If $P(x,y)$ is any point on the line joining the points $A(x_1,y_1)$, $B(x_2,y_2)$ then the equation of the line

in the determinant form is given by $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix}=0$

Textual Ex: Find the equation of the line joining A(1, 3) and B (0, 0) using determinants and find k if D(k, 0) is a point such that area of triangle ABD is 3 Sq. units.

Sol: Let P (x, y) be any point on AB. Then, area of triangle ABP is zero. [$\because$ A, B, P are collinear]

$\Rightarrow \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix}=0$ On expanding along R_1, we get $\frac{1}{2}(y-3x)=0 \Rightarrow y=3x$

This is the equation of required line AB.

Also, since the area of the triangle ABD is 3 sq. units, we have

$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix}=\pm 3$ On expanding along C_2, we get $\frac{1}{2}(3k)=\pm 3 \Rightarrow k=\pm 2$.

4(c)DETERMINANTS-III

[MINOR, COFACTOR AND DETERMINANT USING COFACTORS]

CONCEPTS & FORMULAS

1) **MINOR OF AN ELEMENT**: If a_{ij} is an element which is in the i^{th} row and j^{th} column of a square matrix A, then the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A is called minor of a_{ij} and it is denoted by M_{ij}.

Ex: If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then M_{11}= Minor of $a_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}a_{33}-a_{32}a_{23}$

$M_{12}=$ Minor of $a_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=a_{21}a_{33}-a_{31}a_{23}$ and so on.

2) **COFACTOR OF AN ELEMENT:** If a_{ij} is an element which is in the i^{th} row and j^{th} column of a square matrix A, then the product of $(-1)^{i+j}$ and the minor of a_{ij} is called cofactor of a_{ij} and it is denoted by A_{ij}. Thus, $A_{ij}=(-1)^{i+j}M_{ij}$

Ex: If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $A_{11}=(-1)^{1+1}M_{11}=+1\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}a_{33}-a_{32}a_{23}$

A_{12}= Cofactor of $a_{12}=(-1)^{1+2}M_{11}=-1\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=-(a_{21}a_{33}-a_{31}a_{23})$ and so on.

3) **DETERMINANT USING COFACTORS**:

Determinant of a 3×3 matrix A is $|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$.

Thus, determinant of a matrix A is the sum of product of elements of a row (or a column) with corresponding cofactors.

4(d) DETERMINANTS-IV

[ADJOINT OF 2X2, 3X3 MATRIX, INVERSE OF A MATRIX]

CONCEPTS & FORMULAS

1) ADJOINT OF A MATRIX: The transpose of the matrix obtained by replacing the elements of a square matrix A by the corresponding cofactors is called the adjoint matrix of A.

The adjoint of the matrix A is denoted by adj A or Adj A

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Setp1: Interchange the elements in the principal diagonal.
Setp2: Change the signs of the elements in the other diagonal.

- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

2) INVERSE OF A MATRIX: A square matrix A is said to be an invertible matrix if there exists a square matrix B such that AB=BA=I and the matrix B is called inverse of A.

- If A is a nonsingular matrix then $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

☞ **Inverse of non-singular matrix** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is non-singular $\Rightarrow$ Determinant of A=ad–bc≠0

i) Minor matrix of $A = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ ii) Cofactor matrix of $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

iii) Adjoint matrix of $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ iv) Inverse matrix of $A = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ as ad–bc≠0

Ex: If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ then (i) |A| = 2×7–5×3=14–15= –1 (ii) $\text{adj}A = \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj } A) = -1\begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

4(e)DETERMINANTS-V

[SOLVING LINEAR EQUATIONS IN MATRIX METHOD, CONSISTENCY OF EQUATIONS]

CONCEPTS & FORMULAS

1) Solution of system of linear equations using inverse of a matrix:

Consider the system of equations

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned} \Leftrightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then this system of equations can be written as, $AX = B$. Hence, $X=A^{-1}B$

Case I: If $|A| \neq 0$ then the system is called consistent system and the solution is $X=A^{-1}B$

Case II: If $|A|=0$ then we find $(\text{adj}A)B$

Case (a): If $(adj\ A)\ B \neq O$, then the system is called inconsistent system.

Here the solution does not exist ,

Case (b): If $(adj\ A)\ B = O$, then system may be either consistent with infinitely many solutions or inconsistent with no solution.

This system can be dealt with a special method called Gauss Jordan Method (not in the present syllabus)