SHINING STARS'

VSAQ STAR QUESTIONS PLUS

MATRICES & DETERMINANTS

1. Solve the equation for x, y, z and t, if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

2. Given , $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$ find the values of x, y, z and w.

3. Simplify $\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

4. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that (A + A') is a symmetric matrix

5. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that (A – A') is a skew symmetric matrix

6. Find the minor of element 6 in the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

7. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A+2B)'$

CONTINUITY & DIFFERENTIABILITY

8. Find $\frac{dy}{dx}$ if $x - y = \pi$.

9. Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$

10. Find $\frac{dy}{dx}$ of $2x + 3y = \sin x$

11. Find $\frac{dy}{dx}$ of $2x + 3y = \sin y$

12. Find $\frac{dy}{dx}$ of $\sin^2 x + \cos^2 y = 1$

13. Find $\frac{dy}{dx}$ of $y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right), 0 < x < \frac{1}{\sqrt{2}}$

14. Differentiate $\log(\cos e^{x})$ w.r.t. x.

15. Differentiate $\cos(\log x + e^{x}), x > 0$ w.r.t. x.

INTEGRALS

INDEFINITE INTEGRALS & DEFINITE INTEGRALS

16. Find integral of $\frac{1}{x + x\log x}$

17. Find integral of $\sqrt{ax+b}$

18. Find integral of e^{2x+3}

19. Find integral of $\frac{e^{2x}-1}{e^{2x}+1}$

20. Find integral of $\sqrt{\sin 2x}\cos 2x$

21. Find integral of $\frac{\sin x}{1+\cos x}$

22. Find $\int \frac{dx}{x^2 - 6x + 13}$

23. Find the integral of $\frac{x^2}{1-x^6}$

24. Evaluate $\int_2^3 \frac{1}{x} dx$

25. Evaluate $\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$

26. Evaluate $\int_0^{\frac{\pi}{4}} \tan x \, dx$

27. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

28. Evaluate $\int_2^3 \frac{dx}{x^2-1}$

29. Evaluate $\int_0^{\pi/2} \sin^3 x \, dx$

MATRICES & DETERMINANTS

1. Solve the equation for x, y, z and t, if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

Sol: $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

Equating the corresponding elements of these two matrices, $2x+3=9 \Rightarrow 2x=6 \Rightarrow x=3$

$2y=12 \Rightarrow y=6$

$2z-3=15 \Rightarrow 2z=18 \Rightarrow z=9$

$2t+6=18 \Rightarrow 2t=12 \Rightarrow t=6$ $\quad \therefore$ x=3, y=6, z=9, t=6

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2. Given , $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$ **find the values of x, y, z and w.**

Sol: $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$

Equating the corresponding elements of these two matrices,

$3x = x+4 \Rightarrow 2x = 4 \Rightarrow x = 2$

$3y = 6+x+y \Rightarrow 2y = 6+x \Rightarrow 2y = 6+2 \Rightarrow 2y = 8 \Rightarrow y = 4$

$3w = 2w+3 \Rightarrow w = 3$

$3z = -1+z+w \Rightarrow 2z = w-1 \Rightarrow 2z = 3-1 \Rightarrow 2z = 2 \Rightarrow z = 1$ $\quad \therefore$ x=2, y=4, z=1, w=3

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3. Simplify $\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

Sol: $\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. **For the matrix $A=\begin{bmatrix}1&5\\6&7\end{bmatrix}$, verify that (A + A') is a symmetric matrix**

Sol: Given that $A=\begin{bmatrix}1&5\\6&7\end{bmatrix}\Rightarrow A'=\begin{bmatrix}1&6\\5&7\end{bmatrix}$

$$A+A'=\begin{bmatrix}1&5\\6&7\end{bmatrix}+\begin{bmatrix}1&6\\5&7\end{bmatrix}=\begin{bmatrix}2&11\\11&14\end{bmatrix}$$

Also $[(A+A')]'=\begin{bmatrix}2&11\\11&14\end{bmatrix}=(A+A')$

$\therefore (A+A')$ is a symmetric matrix

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5. **For the matrix $A=\begin{bmatrix}1&5\\6&7\end{bmatrix}$, verify that (A – A') is a skew symmetric matrix**

Sol: $A-A'=\begin{bmatrix}1&5\\6&7\end{bmatrix}-\begin{bmatrix}1&6\\5&7\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$

$$\therefore (A-A')'=\begin{bmatrix}0&1\\-1&0\end{bmatrix}=-\begin{bmatrix}0&-1\\1&0\end{bmatrix}=-(A-A')$$

$\therefore (A-A')$ is a skew symmetric matrix

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6. **Find the minor of element 6 in the determinant $\Delta=\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$**

Sol: Since 6 lies in the second row and third column, its minor M_{23} is given by

$M_{23}=\begin{vmatrix}1&2\\7&8\end{vmatrix}=8-14=-6$ (obtained by deleting R_2 and C_3).

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7. **If $A'=\begin{bmatrix}-2&3\\1&2\end{bmatrix}$ and $B=\begin{bmatrix}-1&0\\1&2\end{bmatrix}$, then find $(A+2B)'$**

Sol: Given that $A'=\begin{bmatrix}-2&3\\1&2\end{bmatrix}\Rightarrow A=(A')'=\begin{bmatrix}-2&1\\3&2\end{bmatrix}$

Now $A+2B=\begin{bmatrix}-2&1\\3&2\end{bmatrix}+2\begin{bmatrix}-1&0\\1&2\end{bmatrix}=\begin{bmatrix}-2&1\\3&2\end{bmatrix}+\begin{bmatrix}-2&0\\2&4\end{bmatrix}=\begin{bmatrix}-4&1\\5&6\end{bmatrix}\Rightarrow(A+2B)'=\begin{bmatrix}-4&5\\1&6\end{bmatrix}$

CONTINUITY & DIFFERENTIABILITY

8. Find $\frac{dy}{dx}$ if $x - y = \pi$.

Sol: Given that $x-y=\pi \Rightarrow \frac{d}{dx}(x-y)=\frac{d\pi}{dx}$ [Differentiating both sides w.r.t., x, we have]

$$\Rightarrow \frac{d}{dx}(x)-\frac{d}{dx}(y)=0 \Rightarrow \frac{dy}{dx}=\frac{dx}{dx}=1$$

Otherwise, given that $x-y=\pi \Rightarrow y=x-\pi \Rightarrow \frac{dy}{dx}=1-0 \Rightarrow \frac{dy}{dx}=1$ $\left[\because \frac{d}{dx}(\pi)=0\right]$

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9. Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$

Sol: Given that $y+\sin y=\cos x \Rightarrow \frac{dy}{dx}+\frac{d}{dx}(\sin y)=\frac{d}{dx}(\cos x)$

$$\Rightarrow \frac{dy}{dx}+\cos y\cdot\frac{dy}{dx}=-\sin x$$ [From chain rule, $\frac{d}{dx}\sin y=\cos y\frac{dy}{dx}$]

$$\Rightarrow \frac{dy}{dx}(1+\cos y)=-\sin x \Rightarrow \frac{dy}{dx}=-\frac{\sin x}{1+\cos y}$$; where $y \neq (2n+1)\pi$

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10. Find $\frac{dy}{dx}$ of $2x + 3y = \sin x$

Sol: Given that $2x+3y=\sin x$. Differentiating both sides w.r.t x, we have

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y) = \frac{d}{dx}\sin x$$

$$\therefore 2+3\frac{dy}{dx}=\cos x \Rightarrow 3\frac{dy}{dx}=\cos x-2 \qquad \therefore \frac{dy}{dx}=\frac{\cos x-2}{3}.$$

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11. Find $\frac{dy}{dx}$ of $2x + 3y = \sin y$

Sol: Given that $2x+3y=\sin y \Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}\sin y$

$$\therefore 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \Rightarrow \cos y\frac{dy}{dx}-3\frac{dy}{dx}=2 \Rightarrow \frac{dy}{dx}(\cos y-3)=2 \Rightarrow \frac{dy}{dx}=\frac{2}{\cos y-3}.$$

12. Find $\frac{dy}{dx}$ of $\sin^2 x + \cos^2 y = 1$

Sol: Given that $\sin^2 x + \cos^2 y = 1 \Rightarrow \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\cos y)^2 = \frac{d}{dx}(1)$

$$\therefore 2(\sin x)\frac{d}{dx}\sin x + 2(\cos y)\frac{d}{dx}\cos y = 0 \Rightarrow \quad 2\sin x\cos x + 2\cos y\left(-\sin y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 2\sin x\cos x - 2\sin y\cos y\frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0 \Rightarrow \sin 2y\frac{dy}{dx} = \sin 2x \Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}.$$

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13. Find $\frac{dy}{dx}$ of $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Sol: Put $x=\cos\theta$ then $2x^2-1=2\cos^2\theta-1=\cos 2\theta$

$$\therefore y = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(2\cos^{-1}x) = 2\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-2}{\sqrt{1-x^2}}$$

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14. Differentiate $\log(\cos e^x)$ w.r.t. x.

Sol: Let $y = \log(\cos e^x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos e^x}\frac{d}{dx}(\cos e^x) = \frac{1}{\cos e^x}(-\sin e^x)\frac{d}{dx}e^x \quad = -(\tan e^x)e^x = -e^x(\tan e^x)$$

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15. Differentiate $\cos(\log x + e^x), x > 0$ w.r.t. x.

Sol: Let $y = \cos(\log x + e^x)$

$$\therefore \frac{dy}{dx} = -\sin(\log x + e^x)\frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x).\left(\frac{1}{x} + e^x\right) = -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x)$$

INTEGRATION

INDEFINITE INTEGRALS & DEFINITE INTEGRALS

16. Find integral of $\dfrac{1}{x + x\log x}$

Sol: Put $1+\log x = t \Rightarrow \dfrac{dx}{x} = dt$

$$\therefore \int \frac{1}{x + x\log x}dx = \int \frac{1}{1+\log x}\left(\frac{dx}{x}\right) = \int \frac{1}{t}dt = \log|t| + c = \log|1+\log x| + c \qquad [\because t = 1+\log x]$$

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17. Find integral of $\sqrt{ax+b}$

Sol:
$$\int \sqrt{ax+b}\,dx = \int (ax+b)^{1/2}\,dx = \frac{(ax+b)^{\frac{1}{2}+1}}{a\left(\frac{1}{2}+1\right)} + c = \frac{(ax+b)^{\frac{3}{2}}}{\frac{3}{2}a} + c$$

$$= \frac{2}{3a}(ax+b)^{3/2} + c. \qquad \left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \text{ if } n \neq -1\right]$$

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18. Find integral of e^{2x+3}

Sol: Put $2x+3 = t \Rightarrow 2dx = dt$ $\qquad \left[\because \int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c\right]$

$$\therefore \int e^{2x+3}dx = \frac{1}{2}\int e^t dt = \frac{1}{2}(e^t) + C = \frac{1}{2}e^{(2x+3)} + C$$

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19. Find integral of $\dfrac{e^{2x}-1}{e^{2x}+1}$

Sol: $\dfrac{e^{2x}-1}{e^{2x}+1}$ Dividing Nr. and Dr. by e^x we get $\dfrac{\dfrac{e^{2x}-1}{e^x}}{\dfrac{e^{2x}+1}{e^x}} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Put $e^x + e^{-x} = t \Rightarrow (e^x - e^{-x})dx = dt$

$$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int \frac{dt}{t} = \log|t| + C = \log\left|e^x + e^{-x}\right| + C$$

20. Find integral of $\sqrt{\sin 2x \cos 2x}$

Sol: Put $\sin 2x = t \Rightarrow 2\cos 2x dx = dt$

$$\therefore \int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{2}\int \sqrt{t}dt = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}t^{\frac{3}{2}} + C = \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

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21. Find integral of $\frac{\sin x}{1+\cos x}$

Sol: Put 1 + cosx=t ⇒ –sinxdx=dt

$$\Rightarrow \int \frac{\sin x}{1+\cos x}dx = \int -\frac{dt}{t} = -\log|t| + C = -\log|1+\cos x| + C$$

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22. Find $\int \frac{dx}{x^2-6x+13}$

Sol: We have $x^2-6x+13 = (x^2-6x+3^2)-3^2+13 = (x-3)^2+4$

$$\therefore \int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+2^2}dx$$

Put x – 3 = t. Then dx = dt

$$\therefore \int \frac{dx}{x^2-6x+13} = \int \frac{dt}{t^2+2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

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23. Find the integral of $\frac{x^2}{1-x^6}$

Sol: Put $x^3 = t \Rightarrow 3x^2dx = dt$

$$\therefore \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2} = \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C = \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

24. Evaluate $\int_2^3 \frac{1}{x}dx$

Sol: Let $I=\int_2^3 \frac{1}{x}dx=[\log|x|]_2^3=\log|3|-\log|2|=\log\frac{3}{2}$

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25. Evaluate $\int_1^2 (4x^3-5x^2+6x+9)dx$

Sol: $\int_1^2 [4x^3-5x^2+6x+9]\,dx=\left[4\frac{x^4}{4}-5\frac{x^3}{3}+6\frac{x^2}{2}+9x\right]_1^2$

$=\left[x^4-\frac{5}{3}x^3+3x^2+9x\right]_1^2=\left[2^4-\frac{5}{3}(2)^3+3(2)^2+9(2)\right]-\left[1-\frac{5}{3}+3+9\right]$

$=\left[16-\frac{40}{3}+12+18\right]-\left[13-\frac{5}{3}\right]=\left[46-\frac{40}{3}\right]-\left[13-\frac{5}{3}\right]$

$=46-\frac{40}{3}-13+\frac{5}{3}=33-\frac{40}{3}+\frac{5}{3}=\frac{99-40+5}{3}=\frac{104-40}{3}=\frac{64}{3}.$

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26. Evaluate $\int_0^{\frac{\pi}{4}} \tan x\,dx$

Sol: Let $I=\int_0^{\frac{\pi}{4}} \tan x\,dx=[-\log\cos x]_0^{\pi/4}=-\log\left|\cos\frac{\pi}{4}\right|+\log|\cos 0|$

$=-\log\left|\frac{1}{\sqrt{2}}\right|+\log|1|=-\log(2)^{\frac{-1}{2}}=\frac{1}{2}\log 2$

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27. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol: Let $I=\int_0^1 \frac{dx}{\sqrt{1-x^2}}=\left[\sin^{-1}x\right]_0^1=\sin^{-1}(1)-\sin^{-1}(0)=\frac{\pi}{2}-0=\frac{\pi}{2}$

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28. Evaluate $\int_2^3 \frac{dx}{x^2-1}$

Sol: Let $I=\int_2^3 \frac{dx}{x^2-1}=\left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_2^3$

$=\frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right|-\log\left|\frac{2-1}{2+1}\right|\right]=\frac{1}{2}\left[\log\left|\frac{2}{4}\right|-\log\left|\frac{1}{3}\right|\right]=\frac{1}{2}\left[\log\frac{1}{2}-\log\frac{1}{3}\right]=\frac{1}{2}\left[\log\frac{3}{2}\right]$

29. **Evaluate** $\int_0^{\pi/2} \sin^3 x \, dx$

Sol: Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x.\sin x dx = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx = \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \cos^2 x.\sin x dx$$

$$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}} = \left[\cos\frac{\pi}{2} - \cos 0\right] + \frac{1}{3}\left[\cos^3\frac{\pi}{2} - \cos^3 0\right] = 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$$