VSAQ PQ QUESTIONS

MATRICES

1. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

2. Find X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

3. Construct a 2 × 2 matrix whose elements are $a_{ij} = \frac{i}{j}$

4. Construct a 3 × 2 matrix whose elements are $a_{ij} = \frac{1}{2}|i - 3j|$

5. **Find AB, if** $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

6. **Let** $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$. Find BA

7. Compute $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

8. Compute $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

9. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$

DETERMINANTS

10. Evaluate $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

11. Find the values of x, if $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

12. Write Minors and Cofactors of the elements of $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

13. Find the adjoint of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

14. Find the inverse of the matrix $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

VECTOR ALGEBRA

15. Classify the following as scalar and vector quantities.

i) work done ii) displacement

16. Compute the magnitude of the vector $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$

17. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

18. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$. Find the unit vector in the direction of $\vec{a} + \vec{b}$.

19. Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

20. If $\vec{a}$ is a unit vector and $(\vec{x} - \vec{a})\cdot(\vec{x} + \vec{a}) = 8$, then find $|\vec{x}|$.

21. Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear.

22. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$

23. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

3-D GEOMETRY

24. If a line makes angle 90°, 60° and 30° with the positive direction of x, y and z-axis respectively, find its direction cosines.

25. Find the angle between the pair of lines given by $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

26. Find the angle between the pair of lines $\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

27. Show that the pair of lines $\frac{4-x}{3} = \frac{2y+1}{2} = \frac{z-1}{5}$ and $\frac{2x+1}{2} = \frac{3-y}{2} = \frac{3z-1}{3}$ are perpendicular to each other.

28. Find the values of p so that the lines
$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

PROBABILITY

29. Determine P(E|F) when a coin is tossed three times, where
E : at most two tails , F : at least one tail.

CONTINUITY & DIFFERENTIABILITY

30. Differentiate sin (log x), x > 0 w.r.t. x.

31. Differentiate $e^{\sin^{-1}x}$ w.r.t. x .

32. Find $\frac{dy}{dx}$ in $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

33. Find $\frac{dy}{dx}$ of $x = a\sec\theta,\ y = b\tan\theta$

34. Find $\frac{dy}{dx}$ of $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

35. Find the second order derivative of x^{20}

APPLICATION OF DERIVATIVES

36. Show that the function given by f (x) = 7x – 3 is increasing on R.

37. Prove that the logarithmic function is increasing on $(0, \infty)$.

38. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

39. Show that the function given by f (x) = sin x is increasing in $\left(0, \frac{\pi}{2}\right)$

40. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is
(a) increasing (b) decreasing

41. The radius of an air bubble is increasing at the rate of 1/2 cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

INTEGRALS

42. Find integral of $\frac{x^3}{\sqrt{1-x^8}}$

43. Find the $\int \mathbf{cosecx(cosecx + cotx)dx}$

44. Find integral of $\frac{\sin x}{1+\cos x}$

45. Find integral of $\frac{\cos x}{\sqrt{1+\sin x}}$

46. Find integral of $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

47. Find integral of $\frac{(1+\log x)^2}{x}$

48. Evaluate $\int \frac{dx}{x^2 - 16}$

49. Evaluate $\int_{1}^{2} x^5 dx$

50. Find $\int_{0}^{1} e^x dx$

51. Evaluate $\int_0^1 \frac{x}{x^2+1}dx$

52. Evaluate $\int_2^8 |x-5|\,dx$

APPLICATION OF DERIVATIVES

53. Find the order and degree of $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$

54. Find the order and degree of $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$

55. Determine order and degree of $y''' + 2y'' + y' = 0$

56. Determine order and degree of $y'' + 2y' + \sin y = 0$

57. Find the general solution of $\frac{dy}{dx} = (1+x^2)(1+y^2)$

VSAQ PQ SOLUTIONS

MATRICES

1. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

Sol: We know that if a matrix A is of the order $m\times n$, then A has mn elements.

Thus, to find all the possible orders of a matrix having 24 elements, we have to find all the ordered pairs of natural numbers whose product is 24.

The ordered pairs are: (1,24),(24,1),(2,12),(12,2),(3,8),(8,3),(4,6), (6,4)

Hence, the possible orders of a matrix having 24 elements are:

$(1\times24),(24\times1),(2\times12),(12\times2),(3\times8),(8\times3),(4\times6),(6\times4)$.

13 is a prime, so we get only 2 ordered pairs with product 13.

They are (1×13) and (13×1)

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2. Find X and Y, if $X+Y=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$

Sol: We have $(X+Y)+(X-Y)=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}+\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$.

$$2X=\begin{bmatrix}5+3 & 2+6\\ 0+0 & 9-1\end{bmatrix}=\begin{bmatrix}8 & 8\\ 0 & 8\end{bmatrix}$$

$$\therefore X=\frac{1}{2}\begin{bmatrix}8 & 8\\ 0 & 8\end{bmatrix}=\begin{bmatrix}4 & 4\\ 0 & 4\end{bmatrix}$$

Also $(X+Y)-(X-Y)=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}-\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$

$$\Rightarrow 2Y=\begin{bmatrix}5-3 & 2-6\\ 0 & 9+1\end{bmatrix}=\begin{bmatrix}2 & -4\\ 0 & 10\end{bmatrix}$$

$$\Rightarrow Y=\frac{1}{2}\begin{bmatrix}2 & -4\\ 0 & 10\end{bmatrix}=\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}$$

3. Construct a 2 × 2 matrix, A = $[a_{ij}]$, whose elements are given by $a_{ij} = \frac{i}{j}$

Sol: A 2×2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Given that $a_{ij} = \frac{i}{j}$; $i, j = 1, 2$

$\therefore a_{11} = \frac{1}{1} = 1$; $a_{12} = \frac{1}{2}$; $a_{21} = \frac{2}{1} = 2$; $a_{22} = \frac{2}{2} = 1$

Thus, the required matrix is $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$

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4. Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$

Sol: A 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Given that $a_{ij} = \frac{1}{2}|i - 3j|, i = 1, 2, 3$ and $j = 1, 2$.

$\therefore a_{11} = \frac{1}{2}|1 - 3\times 1| = 1$; $a_{12} = \frac{1}{2}|1 - 3\times 2| = \frac{5}{2}$

$a_{21} = \frac{1}{2}|2 - 3\times 1| = \frac{1}{2}$; $a_{22} = \frac{1}{2}|2 - 3\times 2| = 2$

$a_{31} = \frac{1}{2}|3 - 3\times 1| = 0$; $a_{32} = \frac{1}{2}|3 - 3\times 2| = \frac{3}{2}$

The required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

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5. Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

Sol: We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0\times 3 + (-1)0 & 0(5) + (-1)(0) \\ 0(3) + 2(0) & (0)(5) + 2(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Observation: Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

6. **Let** $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$. **Find BA**

Sol: $BA = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1(2)+3(3) & 1(4)+3(2) \\ -2(2)+5(3) & -2(4)+5(2) \end{bmatrix} = \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

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7. **Compute** $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

Sol: $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)-2(2) & 1(2)-2(3) & 1(3)-2(1) \\ 2(1)+3(2) & 2(2)+3(3) & 2(3)+3(1) \end{bmatrix} = \begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

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8. **Compute** $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

Sol: $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3(2)-1(1)+3(3) & 3(-3)-1(0)+3(1) \\ -1(2)+0(1)+2(3) & -1(-3)+0(0)+2(1) \end{bmatrix} = \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

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9. **For what values of x :** $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$

Sol: Given that $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$

$\Rightarrow \begin{bmatrix} 1+4+1 & 2+0+0 & 0+2+2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = [0]$

$\Rightarrow \begin{bmatrix} 6 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = [0] \Rightarrow [6(0)+2(2)+4(x)] = [0]$

$\Rightarrow [4+4x] = [0] \Rightarrow 4+4x = 0 \Rightarrow 4x = -4 \Rightarrow x = -1$

DETERMINANTS

10. Evaluate $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

Sol: $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = (x^2-x+1)(x+1)-(x-1)(x+1)$ $\left[\because \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc\right]$

$= x^3+\cancel{x^2}-\cancel{x^2}+\cancel{x}-\cancel{x}+1-(x^2-1) = x^3+1-x^2+1 = x^3-x^2+2$

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11. Find the values of x, if $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Sol: Given that $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ $\left[\because \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc\right]$

$\Rightarrow 2\times1-5\times4 = 2x\times x-6\times4 \Rightarrow 2-20 = 2x^2-24 \Rightarrow 2x^2 = 6 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

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12. Write Minors and Cofactors of the elements of $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Sol: The given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Minor of the element a_{ij} is M_{ij}

M_{11}= Minor of the element $a_{11} = d$; M_{12}= Minor of the element $a_{12} = b$;

M_{21} = Minor of the element $a_{21} = c$; M_{22}= Minor of the element $a_{22} = a$

Now, cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1}(d) = d$; $A_{12} = (-1)^{1+2}(b) = -b$

$A_{21} = (-1)^{2+1}(c) = -c$; $A_{22} = (-1)^{2+2}(a) = a$

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13. Find the adjoint of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow A_{11} = 4;\quad A_{12} = -3\quad A_{21} = -2;\quad A_{22} = 1$

$\therefore \text{adj}A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

14. Find the inverse of the matrix $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Then, $|A| = (-1\times 2) - (5\times -3) = -2 + 15 = 13$

Now, $A_{11} = 2;\ \ A_{12} = 3$

$A_{21} = -5;\ \ A_{22} = -1$

Hence, $\text{adj}\,A = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$ $\quad\therefore A^{-1} = \dfrac{1}{|A|}\text{adj}\,A = \dfrac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

VECTOR ALGEBRA

15. Classify the following as scalar and vector quantities.

i) work done ii) displacement

Sol: i) Work done - scalar. ii) displacement - vector

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16. Compute the magnitude of the vector $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$

Sol: Given that $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$

$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2} = \sqrt{4 + 49 + 9} = \sqrt{62}.$

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17. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Sol: We know that a unit vector in the direction of the vector

$\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{1+1+4}} \Rightarrow \quad \hat{a} = \dfrac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}} = \dfrac{1}{\sqrt{6}}\hat{i} + \dfrac{1}{\sqrt{6}}\hat{j} + \dfrac{2}{\sqrt{6}}\hat{k}.$

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18. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$. Find the unit vector in the direction of $\vec{a} + \vec{b}$.

Sol: Given vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$;

$\therefore \vec{a} + \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k} - \hat{i} + \hat{j} - \hat{k} = \hat{i} + 0\hat{j} + \hat{k} \quad \therefore |\vec{a} + \vec{b}| = \sqrt{(1)^2 + (0)^2 + (1)^2} = \sqrt{2}$

$\therefore$ A unit vector in the direction of $\vec{a} + \vec{b}$ is $\dfrac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \dfrac{\hat{i} + 0\hat{j} + \hat{k}}{\sqrt{2}} = \dfrac{\hat{i} + \hat{k}}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\hat{i} + \dfrac{1}{\sqrt{2}}\hat{k}.$

19. Find a vector in the direction of vector $\vec{a}=\hat{i}-2\hat{j}$ that has magnitude 7 units.

Sol: The unit vector in the direction of the given vector $\vec{a}$ is $\hat{a}=\frac{1}{|\vec{a}|}\vec{a}=\frac{1}{\sqrt{5}}(\hat{i}-2\hat{j})=\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{j}$

Now the vector having magnitude 7 and in the direction of $\vec{a}$ is $7\hat{a}$

$$\therefore 7\hat{a}=7\left(\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{j}\right)=\frac{7}{\sqrt{5}}\hat{i}-\frac{14}{\sqrt{5}}\hat{j}$$

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20. If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8$, then find $|\vec{x}|$.

Sol: Given $\vec{a}$ is a unit vector $\Rightarrow |\vec{a}|=1 \Rightarrow (\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8$

$\Rightarrow \vec{x}\cdot\vec{x}+\cancel{\vec{x}\cdot\vec{a}}-\cancel{\vec{a}\cdot\vec{x}}-\vec{a}\cdot\vec{a}=8 \Rightarrow |\vec{x}|^2-1=8 \Rightarrow |\vec{x}|^2=9$

$\therefore |\vec{x}|=3$ (as magnitude of a vector is non negative).

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21. Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k})$, $B(\hat{i}+2\hat{j}+3\hat{k})$ and $C(7\hat{i}-\hat{k})$ are collinear.

Sol: We have $\overline{AB}=(1+2)\hat{i}+(2-3)\hat{j}+(3-5)\hat{k}=3\hat{i}-\hat{j}-2\hat{k}\Rightarrow|\overline{AB}|=\sqrt{3^2+(-1)^2+(-2)^2}=\sqrt{14}$

$\overline{BC}=(7-1)\hat{i}+(0-2)\hat{j}+(-1-3)\hat{k}=6\hat{i}-2\hat{j}-4\hat{k}\Rightarrow|\overline{BC}|=\sqrt{6^2+(-2)^2+(-4)^2}=2\sqrt{14}$

$\overline{AC}=(7+2)\hat{i}+(0-3)\hat{j}+(-1-5)\hat{k}=9\hat{i}-3\hat{j}-6\hat{k}\Rightarrow|\overline{AC}|=\sqrt{9^2+(-3)^2+(-6)^2}=3\sqrt{14}$

Here, $|\overline{AC}|=|\overline{AB}|+|\overline{BC}|$.

Hence, the points A, B and C are collinear.

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22. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$

Sol: Let $\vec{a}=i+3\hat{j}+7\hat{k}$ and $\vec{b}=7\hat{i}-\hat{j}+8\hat{k}$

Projection of vector $\vec{a}$ on $\vec{b}=\frac{\vec{a}.\vec{b}}{|\vec{b}|}=\frac{1(7)+3(-1)+7(8)}{\sqrt{7^2+(-1)^2+8^2}}=\frac{7-3+56}{\sqrt{49+1+64}}=\frac{60}{\sqrt{114}}$

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23. Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Sol: Given that $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=19\hat{j}+19\hat{k}$$

$$\therefore |\vec{a}\times\vec{b}|=\sqrt{19^2+19^2}=\sqrt{2\times(19)^2}=19\sqrt{2}$$

3-D GEOMETRY

24. If a line makes angle 90°, 60° and 30° with the positive direction of x, y and z-axis respectively, find its direction cosines.

Sol: Let l, m, n be the d . c . 's of the line

Then $l = \cos 90^0 = 0$, $m = \cos 60^0 = 1/2$, $n = \cos 30^0 = \frac{\sqrt{3}}{2}$.

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25. Find the angle between the pair of lines given by $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ **and** $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

Sol: Angle between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is given by $\cos\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

Here $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$$\therefore \cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| = \left|\frac{(\hat{i} + 2\hat{j} + 2\hat{k})\cdot(3\hat{i} + 2\hat{j} + 6\hat{k})}{\sqrt{1+4+4}\sqrt{9+4+36}}\right| = \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

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26. **Find the angle between the pair of lines**

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4} \text{ and } \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

Sol: The direction ratios of the first line are $(a_1, b_1, c_1) = (3, 5, 4)$

the direction ratios of the second line are $(a_2, b_2, c_2) = (1, 1, 2)$

If θ is the angle between them, then $\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

$$= \left|\frac{3.1 + 5.1 + 4.2}{\sqrt{3^2 + 5^2 + 4^2}\sqrt{1^2 + 1^2 + 2^2}}\right| = \frac{16}{\sqrt{50}\sqrt{6}} = \frac{16}{5\sqrt{2}\sqrt{6}} = \frac{8\sqrt{3}}{15}$$

Hence, the required angle is $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$

27. Show that the pair of lines $\frac{4-x}{3}=\frac{2y+1}{2}=\frac{z-1}{5}$ and $\frac{2x+1}{2}=\frac{3-y}{2}=\frac{3z-1}{3}$ are perpendicular to each other.

Sol: The equations of the given lines are $\frac{4-x}{3}=\frac{2y+1}{2}=\frac{z-1}{5} \Rightarrow \frac{x-4}{-3}=\frac{y+1/2}{1}=\frac{z-1}{5}$

$$\frac{2x+1}{2}=\frac{3-y}{2}=\frac{3z-1}{3} \Rightarrow \frac{x+1/2}{1}=\frac{y-3}{-2}=\frac{z-1/3}{1}$$

Here, $a_1=-3, b_1=1, c_1=5$ $\quad a_2=1, b_2=-2, c_2=1$

Two lines with direction ratios, a_1,b_1,c_1 and a_2,b_2,c_2 are perpendicular to each other,

if $a_1a_2+b_1b_2+c_1c_2=0$. $\quad$ Here, $(-3)1+1(-2)+5(1)=-3-2+5=0$

$\therefore$ the given lines are perpendicular to each other.

---

28. Find the values of p so that the lines
$\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles.

Sol: Equation of the lines in the standard form are

$$\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2} \Rightarrow \frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2} \text{ and } \frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5} \Rightarrow \frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$$

The direction ratios of the lines are given by $a_1=-3, b_1=\frac{2p}{7}, c_1=2\,;\, a_2=\frac{-3p}{7}, b_2=1, c_2=-5$
Since, both the lines are perpendicular to each other, we have

$$a_1a_2+b_1b_2+c_1c_2=0 \Rightarrow (-3)\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)1+2(-5)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}-10=0 \Rightarrow \frac{11}{7}p=10 \Rightarrow 11p=10\times 7 \Rightarrow p=\frac{70}{11}$$

PROBABILITY

29. Determine P(E|F) when a coin is tossed three times, where
E : at most two tails , F : at least one tail.

Sol: **E : at most two tails** $\Rightarrow$E={TTH, THT, HTT, THH, HTH, HHT, HHH} $\therefore n(E)=7$

F : at least one tail$\Rightarrow$F={THH, HTH, HHT, TTH, THT, HTT, TTT} $\therefore n(F)=7$

Hence $E\cap F$ = {TTH, THT, HTT, THH, HTH, HHT} $\Rightarrow n(E\cap F)=6$

Now $P(E)=\frac{7}{8}, P(F)=\frac{n(F)}{n(S)}=\frac{7}{8}, P(E\cap F)=\frac{6}{8}=\frac{3}{4}$

$$\therefore P(E/F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{3}{4}}{\frac{7}{8}}=\frac{3}{4}\times\frac{8}{7}=\frac{6}{7}.$$

CONTINUITY & DIFFERENTIABILITY

30. Differentiate sin (log x), x > 0 w.r.t. x:

Sol: Let y = sin (log x). Using chain rule, we have $\frac{dy}{dx}=\cos(\log x)\cdot\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$

---

31. Differentiate $e^{\sin^{-1}x}$ w.r.t. x.

Sol: Let $y=e^{\sin^{-1}x}$

$\therefore \frac{dy}{dx}=e^{\sin^{-1}x}\frac{d}{dx}\sin^{-1}x=e^{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}},\ x\in(-1,1)$ $\left[\because \frac{d}{dx}e^{f(x)}=e^{f(x)}\frac{d}{dx}f(x)\right]$

---

32. Find $\frac{dy}{dx}$ in $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$

Sol: Given that $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$ Put x=tanθ.

$\therefore\ y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)$

$y=2\theta=2\tan^{-1}x\ (\because\ x=\tan\theta\Rightarrow\theta=\tan^{-1}x)\Rightarrow \frac{dy}{dx}=2\frac{1}{1+x^2}=\frac{2}{1+x^2}$

---

33. Find $\frac{dy}{dx}$ of $x=a\sec\theta, y=b\tan\theta$

Sol: Given that $x=a\sec\theta$ and $y=b\tan\theta$. Differentiating both eqns. wrt. θ, we have

$\frac{dx}{d\theta}=a\sec\theta\tan\theta$ and $\frac{dy}{d\theta}=b\sec^2\theta$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b\sec\theta}{a\tan\theta}=\frac{b\cdot\frac{1}{\cos\theta}}{a\cdot\frac{\sin\theta}{\cos\theta}}=\frac{b}{\cos\theta}\cdot\frac{\cos\theta}{a\sin\theta}=\frac{b}{a\sin\theta}=\frac{b}{a}\text{cosec}\,\theta$$

34. Find $\frac{dy}{dx}$ of $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

Sol: $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$. Differentiating both eqns. w.r.t. θ we have

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta) = a\left[\frac{d}{d\theta}\theta - \frac{d}{d\theta}\sin\theta\right] = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}(1+\cos\theta) = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a(0-\sin\theta) = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} = -\frac{\sin\theta}{1-\cos\theta} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}.$$

---

35. Find the second order derivative of x^{20}

Sol: Let $y = x^{20}$ $\therefore \frac{dy}{dx} = 20x^{19}$. Again differentiating w.r.t x, we have

$$\frac{d^2y}{dx^2} = 20\times 19x^{18} = 380x^{18}$$

APPLICATION OF DERIVATIVES

36. Show that the function given by f (x) = 7x – 3 is increasing on R.

Sol: Let x_1 and x_2 be any two numbers in **R**.

Then $x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$

Thus, by Definition it follows that f is strictly increasing on **R**.

---

37. Prove that the logarithmic function is increasing on (0, ∞).

Sol: The given function is $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$

For, $x > 0,\ f'(x) = \frac{1}{x} > 0$

Thus, the logarithmic function is strictly increasing in interval $(0,\infty)$

---

38. Prove that the function given by f (x) = $x^3 - 3x^2 + 3x - 100$ is increasing in R.

Sol: Given that $f(x) = x^3 - 3x^2 + 3x - 100$

$\Rightarrow f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$

For $x \in R, (x-1)^2 \geq 0$ .So, f'(x) is always positive in R.

Thus,the function is increasing in R.

39. Show that the function given by f (x) = sin x is increasing in $\left(0, \frac{\pi}{2}\right)$

Sol: Given that $f(x) = \sin x \Rightarrow f'(x) = \cos x$

For $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \cos x > 0 \Rightarrow f'(x) > 0$. Thus, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

---

40. Find the intervals in which the function f given by f (x) = $2x^3 - 3x^2 - 36x + 7$ is (a) increasing (b) decreasing

Sol: Given that $f(x) = 2x^3 - 3x^2 - 36x + 7$

$\Rightarrow f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x+2)(x-3)$

$\therefore f'(x) = 0 \Rightarrow x = -2, 3$

In $(-\infty, -2)$ and $(3, \infty)$, $f'(x) > 0$. In $(-2, 3)$, $f'(x) < 0$

$\therefore$ f is strictly increasing in $(-\infty, -2)$, $(3, \infty)$ and strictly decreasing in $(-2, 3)$

---

41. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Sol: For the sphere, we take radius = r, Volume = V.

Given that $\frac{dr}{dt} = \frac{1}{2}$ cm/s, we have to find $\frac{dV}{dt}$ at r=1

Volume $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt} = 4\pi(1)^2 \frac{1}{2} = 2\pi$ cm³/s

INTEGRALS

42. Find integral of $\frac{x^3}{\sqrt{1-x^8}}$

Sol: Put $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\therefore \int \frac{x^3}{\sqrt{1-x^8}} dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{4}\sin^{-1} t + C = \frac{1}{4}\sin^{-1}(x^4) + C$$

---

43. Find the $\int$ cosecx(cosecx + cotx)dx [BMP]

Sol: $\int (\operatorname{cosec} x(\operatorname{cosec} x + \cot x) dx = \int \operatorname{cosec}^2 x dx + \int \operatorname{cosec} x \cot x dx = -\cot x - \operatorname{cosec} x + C$

44. Find integral of $\frac{\sin x}{1+\cos x}$

Sol: Put $1 + \cos x = t \Rightarrow -\sin x\,dx = dt$

$$\Rightarrow \int \frac{\sin x}{1+\cos x}dx = \int -\frac{dt}{t} = -\log|t| + C = -\log|1+\cos x| + C$$

---

45. Find integral of $\frac{\cos x}{\sqrt{1+\sin x}}$

Sol: Put $1+\sin x = t \Rightarrow \cos x\,dx = dt$

$$\therefore \int \frac{\cos x}{\sqrt{1+\sin x}}dx = \int \frac{dt}{\sqrt{t}} = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{t} + C = 2\sqrt{1+\sin x} + C$$

---

46. Find integral of $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

Sol: Put $\sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$

$$\therefore \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int t\,dt = \frac{t^2}{2} + C = \frac{(\sin^{-1}x)^2}{2} + C$$

---

47. Find integral of $\frac{(1+\log x)^2}{x}$

Sol: Put $1+\log x = t \Rightarrow \frac{1}{x}dx = dt$

$$\therefore \int \frac{(1+\log x)^2}{x}dx = \int t^2dt = \frac{t^3}{3} + C = \frac{(1+\log x)^3}{3} + C$$

---

48. Find $\int \frac{dx}{x^2-16}$

Sol: We have $\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C$

49. **Evaluate** $\int_1^2 x^5 dx$

Sol: $I = \int_1^2 x^5 dx = \left[\frac{x^6}{6}\right]_1^2 = \frac{2^6}{6} - \frac{1}{6} = \frac{64-1}{6} = \frac{63}{6} = \frac{21}{2}$

---

50. **Find** $\int_0^1 e^x dx$

Sol: $I = \int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = e - 1$

---

51. **Evaluate** $\int_0^1 \frac{x}{x^2+1} dx$

Sol; Put $x^2 + 1 = t \Rightarrow 2x dx = dt$ When $x = 0$ we have $t = 1$ and when $x = 1$ we have $t = 2$

$\therefore \int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}[\log|t|]_1^2 = \frac{1}{2}[\log 2 - \log 1] = \frac{1}{2}\log 2$

---

52. **Evaluate** $\int_2^8 |x-5| dx$

Sol: Let $I = \int_2^8 |x-5| dx$

As $(x-5) \le 0$ on $[2,5]$ and $(x-5) \ge 0$ on $[5,8]$

$I = \int_2^5 -(x-5)dx + \int_5^8 (x-5)dx$ $\left(\because \int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$

$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8 = -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right] = 9$

APPLICATION OF DERIVATIVES

53. Find the order and degree of $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$

Sol: $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x \Rightarrow \frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y - \log x = 0$

Highest order derivative present in differential equation is $\frac{d^2y}{dx^2}$. Its order is two.

Highest power raised to $\frac{d^2y}{dx^2}$ is one. Its degree is one.

---

54. Find the order and degree of $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$

Sol: $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x \Rightarrow \left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y - \sin x = 0$

Highest order derivative in differential equation is $\frac{dy}{dx}$. Its order is one.

Highest power raised to $\frac{dy}{dx}$ is three. Its degree is three.

---

55. Determine order and degree of $y''' + 2y'' + y' = 0$

Sol: Given D.E is $y''' + 2y'' + y' = 0$

Highest order derivative present in the differential equation is y''' . Its order is 3.

It is a polynomial equation in y''' ,y'' and y'. The highest power of y''' is 1.

Degree of the D.E is1

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56. Determine order and degree of $y'' + 2y' + \sin y = 0$

Sol: Given D.E is $y'' + 2y' + \sin y = 0$

Highest order derivative present in the differential equation is y''. Its order is two.

This is a polynomial equation in y'' and y' the highest power of y'' is one.

Degree of the D.E is 1

---

57. Find the general solution of $\frac{dy}{dx}=(1+x^2)(1+y^2)$

Sol: Given D.E. is $\frac{dy}{dx}=(1+x^2)(1+y^2) \Rightarrow \frac{dy}{1+y^2}=(1+x^2)dx$

Integrating both sides, we get $\int \frac{dy}{1+y^2}=\int (1+x^2)dx$

$$\Rightarrow \tan^{-1} y=\int dx+\int x^2 dx \Rightarrow \tan^{-1} y = x+\frac{x^3}{x}+C$$