10 VECTOR ALGEBRA

1 OMQ + 1 VSAQ + 1 SAQ + 1 LAQ [1 M + 2M +4M + 8M = 15 M]

10(a) BASIC CONCEPTS OF VECTORS

CONCEPTS & FORMULAS

1) Scalars: Phsical quantities having only magnitude but no direction.

Ex: Length, mass, time, money, distance, speed, temperature, work,line segment, ...

2) Vectors: Physical quantities having both magnitude and direction.

Ex: Displacement, velocity, acceleration, force, momentum, directed line segment...

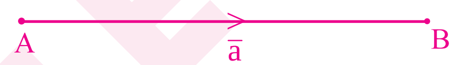
3.1) Geometrical representation of Vectors :

Vectors are represented by 'directed line segments' like $\overline{AB}, \overline{OP}$.

If the vector $\overline{AB} = \bar{a}$ then

(i) A is called the initial point of the vector $\bar{a}$

(ii) B is called the terminal point of the vector $\bar{a}$



(iii) The magnitude of $\bar{a}$ is the length AB.

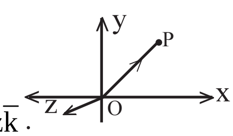
(iv) The direction of $\bar{a}$ is from the initial point A to the terminal point B.

Examples of Geometrical Vectors: $\overline{AB}, \overline{AC}, \overline{PQ}, \overline{OA}, \overline{OB}, \overline{OC}, \bar{a}, \bar{b}, \bar{c}, \bar{r}$.......

3.2) Analytical representation of Vectors:

Vectors in 3D space are represented analytically as $\bar{a} = (a_1, a_2, a_3)$, $\bar{r} = (x,y,z)$ etc.,

4) Position Vector(PV) of a point in Component form:



If P is a point in space then the position vector of P w.r.t O is $\overline{OP}$

If P(x,y,z) is a point and O(0,0,0) is the origin then $\overline{OP} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

where, $\bar{i}, \bar{j}, \bar{k}$ are the unit vectors along the positive x-axis, y-axis, z-axis.

5.1) Scalar components: If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then x,y,z are called scalar components of $\bar{r}$.

Scalar components (d.r's) denote the projection of $\bar{r}$ on the X-axis, Y-axis, Z-axis.

5.2) Vector components : If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $x\bar{i}, y\bar{j}, z\bar{k}$ are called vector components of $\bar{r}$.

Examples: $\overline{OA} = \bar{i} + 2\bar{j} + 3\bar{k}; \overline{PQ} = 2\bar{i} - 3\bar{j} + 4\bar{k}; \bar{r} = \bar{i} - \bar{j}$..........

Note 1 : If $\overline{OP} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then its magnitude is $|\overline{OP}| = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

Note 2: If the PV of a point A w.r.t O is denoted by $\bar{a}$ then $\overline{OA} = \bar{a}$. It is also written as $A(\bar{a})$

Note 3: The concept PV of a point relates every **point** in space to a **vector** and vice versa.

Note 4: Representation of 'a point' and a vector are interconvertible.

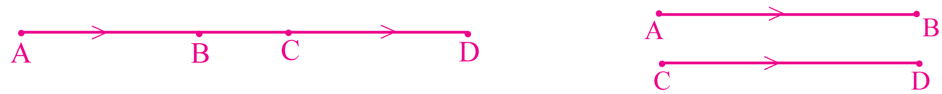
For example, point (2,3,4) is also expressed as $2\bar{i} + 3\bar{j} + 4\bar{k}$ and vice-versa.

So ,while dealing the porblems with vectors first ensure whether the given entity is point or vector.

6) Equality of vectors: Two vectors $\bar{a}, \bar{b}$ are said to be equal, written as $\bar{a} = \bar{b}$, if they have the same magnitude and direction, regardless the position of the initial points.

Note 1: If $\overline{AB} = \overline{CD}$ then (i) $|\overline{AB}| = |\overline{CD}|$ (Magnitudes are equal)

(ii) $\overline{AB} \parallel \overline{CD}$ (Same direction and same or parallel line of support)



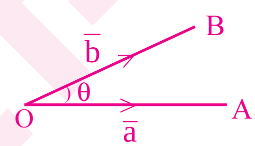
Note 2: If $\bar{a} = (a_1, a_2, a_3)$, $\bar{b} = (b_1, b_2, b_3)$ then $\bar{a} = \bar{b} \Leftrightarrow a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2$

7) Scalar Multiple: If a vector expressed in the form $m\bar{a}$ is called a scalar multiple of $\bar{a}$.

Note 1 : If $m>0$ then the direction of $m\bar{a}$ is same that of $\bar{a}$ and if $m<0$ then the direction of $m\bar{a}$ is opposite that of $\bar{a}$.

Note 2 : (i) $m(n\bar{a}) = (mn)\bar{a}$ (ii) $(m+n)\bar{a} = m\bar{a} + n\bar{a}$ (iii) $m(\bar{a} + \bar{b}) = m\bar{a} + m\bar{b}$ where $m, n \in R$

8) Angle between two vectors: If $\bar{a}, \bar{b}$ are two non zero vectors such that $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ then the measure of $\angle AOB = \theta$ such that $0^0 \le \theta \le 180^0$ is known as the angle between the vectors $\bar{a}$ and $\bar{b}$.



Angle between $\bar{a}, \bar{b}$ is denoted by $\langle \bar{a}, \bar{b} \rangle$ or $\langle \overline{OA}, \overline{OB} \rangle$.

Note 1: Angle between two vectors $\bar{a}, \bar{b}$ is determined only when they have the same initial point

Note 2: $\langle \bar{a}, \bar{b} \rangle = \langle \bar{b}, \bar{a} \rangle = \langle -\bar{a}, -\bar{b} \rangle = 180^o - \langle \bar{a}, -\bar{b} \rangle = \langle l\bar{a}, m\bar{b} \rangle$ for $l, m > 0$

Note 3: If $\langle \bar{a}, \bar{b} \rangle = 90^o$ then $\bar{a}$, $\bar{b}$ are called perpendicular vectors.

9) Types of Vectors:

9.1) Zero vector (Null vector): A vector of zero magnitude and arbitrary direction is called **zero vector** and it is denoted by $\bar{0}$ or **0**.

Note 1: If A is any point in space then $\overline{AA} = \bar{0}$ **Note 2:** 0 is a scalar while $\bar{0}$ is a vector.

Note 3: If $\bar{a} = \bar{b}$ then writing $\bar{a} - \bar{b} = 0$ is wrong expression and $\bar{a} - \bar{b} = \bar{0}$ is the right expression

9.2) Unit vector: A vector of magnitude one unit is called a unit vector. If $\bar{a}$ is a non-zero vector then the unit vector in the direction of $\bar{a}$ is denoted by $\hat{a}$ (read as, a cap or a crown or a tip).

Formula: The unit vector in the direction of the non-zero vector $\bar{a}$ is $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$

Note 1: Unit vectors parallel to $\bar{a}$ are $\pm\dfrac{\bar{a}}{|\bar{a}|}$. Here, Unit vector opposite to $\bar{a}$ is $-\dfrac{\bar{a}}{|\bar{a}|}$

Note 2: The vector of magnitude m in the direction of $\bar{a}$ is $m\hat{a} = m\left(\dfrac{\bar{a}}{|\bar{a}|}\right)$

Note 3: Any given vector $\bar{a}$ can be written as $\bar{a} = |\bar{a}|\hat{a}$

9.3) Coinitial Vectors : Vectors having the same initial point .**Ex:** $\overline{OA}, \overline{OB}, \overline{OC}$

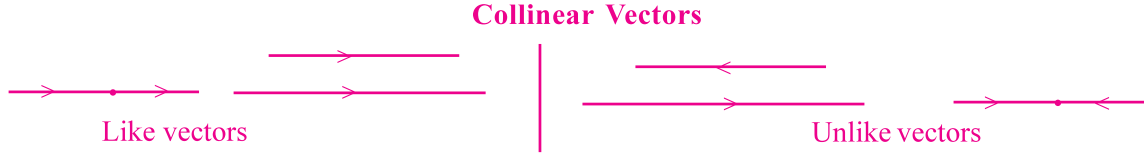
Note: The angle between 2 non zero vectors is determined only when they are co-initial .

9.4) Collinear vectors(Like, unlike vectors):

Vectors lying on the same or parallel lines are called **Collinear vectors.**
Collinear vectors having the same direction are called **Like vectors.**
Collinear vectors having opposite directions are called **Unlike vectors.**

Note : If the vectors $\bar{a}, \bar{b}$ are collinear then $\bar{a} = t\bar{b}$ for some $t \in R$.

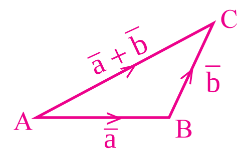
Collinear Vectors



10(b) ADDITION OF VECTORS

CONCEPTS & FORMULAS

1) Triangle Law: If two vectors $\bar{a}, \bar{b}$ are represented by 2 sides $\overline{AB}, \overline{BC}$ of a triangle taken in an order then their resultant(sum) $\bar{a}+\bar{b}$ is represented by the third side $\overline{AC}$ of the triangle taken in the reverse order. Thus $\overline{AB}+\overline{BC}=\overline{AC}$.



Here, the terminal point B of the first vector $\overline{AB}$ coincides with the initial point B of the second vector $\overline{BC}$.

Note 1: In ΔABC, $\overline{AB}+\overline{BC}+\overline{CA}=(\overline{AB}+\overline{BC})+\overline{CA}=\overline{AC}+\overline{CA}=\overline{AA}=\bar{0}$

Note 2: Hence, the condition for 3 vectors $\bar{a}, \bar{b}, \bar{c}$ to be the sides of a triangle is $\bar{a}+\bar{b}+\bar{c}=\bar{0}$

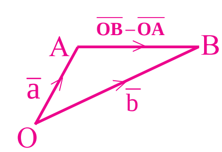
Thus, the vector sum of three sides of a triangle taken in order is $\bar{0}$

Note 3: If $\bar{a}, \bar{b}$ any two sides of the triangle then the 3rd side is given by $\bar{c}=\bar{a}+\bar{b}$ (or) $-(\bar{a}+\bar{b})$

Note 4: If $\bar{a}=(a_1,a_2,a_3)$; $\bar{b}=(b_1,b_2,b_3)$ then $\bar{a}+\bar{b}=(a_1+b_1, a_2+b_2, a_3+b_3)$

Note 5: If $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$, $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ then $\bar{a}+\bar{b}=(a_1+b_1)\bar{i}+(a_2+b_2)\bar{j}+(a_3+b_3)\bar{k}$

2) Vector joining two points: If A,B are 2 points in space and $\overline{OA}, \overline{OB}$ are their Position vectors then



$\overline{AB}=\overline{OB}-\overline{OA}=$ PV of B $-$ PV of A. $[\because \overline{OA}+\overline{AB}=\overline{OB}]$

Note 1: $\overline{AB}=\overline{PB}-\overline{PA}$ where P is any point.

Note 2: Similarly, $\overline{PQ}=\overline{OQ}-\overline{OP}, \overline{PR}=\overline{OR}-\overline{OP}, \overline{PS}=\overline{OS}-\overline{OP}$.

Note 3: If $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ are any two points then $\overline{AB}=\overline{OB}-\overline{OA}$

$=(x_2\bar{i}+y_2\bar{j}+z_2\bar{k})-(x_1\bar{i}+y_1\bar{j}+z_1\bar{k})=(x_2-x_1)\bar{i}+(y_2-y_1)\bar{j}+(z_2-z_1)\bar{k}$

Hence $|\overline{AB}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

3) SECTION FORMULA:

If the position vectors of the points A,B w.r.to O are $\bar{a}, \bar{b}$ respectively and the point P. collinear with A,B divides AB internally in the ratio m:n then the P.V of P is $\overline{OP}=\dfrac{m\bar{b}+n\bar{a}}{m+n}$

Note: If the division is external then $\overline{OP}=\dfrac{m\bar{b}-n\bar{a}}{m-n}$, $m \neq n$

☛ If P is the midpoint of the line segment joining $A(\bar{a}).B(\bar{b})$ then the PV of the midpoint of $\overline{AB}$ is $\overline{OP}=\dfrac{\bar{a}+\bar{b}}{2}=\dfrac{\overline{OA}+\overline{OB}}{2}$

4) Direction cosines (D.c's) of a vector :

4.1) Direction Cosines: If α,β,γ are angles made by a vector $\bar{r}$ with x-axis, y-axis and z-axis then cosα, cosβ, cosγ are called direction cosines of $\bar{r}$. They are denoted by *l*,m,n.

Thus *l*= cosα, m=cosβ, n=cosγ

4.2) If $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ is a vector in space and $|\bar{r}|=r=\sqrt{x^2+y^2+z^2}$ then

(i) Direction cosines $(l,m,n)=\left(\frac{x}{|\bar{r}|},\frac{y}{|\bar{r}|},\frac{z}{|\bar{r}|}\right)$= Scalar components of the unit vector $\hat{r}$

4.3) D.C's of $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ are $l=\cos\alpha=\frac{x}{r};\ m=\cos\beta=\frac{y}{r};\ n=\cos\gamma=\frac{z}{r},\quad r=|\bar{r}|$

4.4) $l^2+m^2+n^2=1 \quad \left[\because l^2+m^2+n^2=\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=\frac{x^2+y^2+z^2}{r^2}=\frac{r^2}{r^2}=1\right]$

5) Direction ratios (D.r's) of a vector:

Direction Ratios (D.r's): The numbers(x,y,z) proportional to the direction cosines *lr*, *mr*, *nr* are called as *direction ratios* of vector $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$, and are denoted by *a*, *b* , *c* respectively.

If $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ is a vector then x,y,z are the direction ratios of $\bar{r}$ =(a,b,c)=(x,y,z)= (*l*r, mr, nr)

Note 1: If *l*,m,n are d.c's and x,y,z are d.r's of vector $\bar{r}$ then

(i) $l^2+m^2+n^2=1$ (ii) $x^2+y^2+z^2\neq 1$, in general

Note 2: Direction ratios x,y,z = Scalar components of vector $\bar{r}$
whereas the direction cosines *l*,m,n= sacalar components of the unit vector $\hat{r}$

Note 3: D.C's or D.R's of a vector fix the direction of the given vector in terms of cosines of angles made by the vector with the 3 coordinate axes.

Ex: Find the D.r's & Dc's of the vector $\bar{a}=\bar{i}+\bar{j}-2\bar{k}$.

Sol: Given vector $\bar{a}=\bar{i}+\bar{j}-2\bar{k}$

(i) D.r's of $\bar{a}=(a,b,c)=(x,y,z)=(1,1,-2)$ = Scalar components of $\bar{a}$

(ii) $|\bar{a}|=\sqrt{1^2+1^2+(-2)^2}=\sqrt{1+1+4}=\sqrt{6}$

$\therefore$ Unit vector $\hat{a}=\frac{\bar{a}}{|\bar{a}|}=\frac{1}{\sqrt{6}}(\bar{i}+\bar{j}-2\bar{k})=\frac{\bar{i}}{\sqrt{6}}+\frac{\bar{j}}{\sqrt{6}}-\frac{2\bar{k}}{\sqrt{6}}$

D.C's of $\bar{a}$ = (*l*,m,n) $=\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}}\right)$ = Scalar components of $\hat{a}$

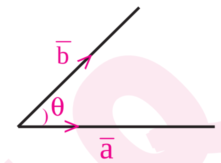
10(c) SCALAR (DOT) PRODUCT

CONCEPTS & FORMULAS

1) Scalar Product (Dot Product): Let $\bar{a},\bar{b}$ be two non-zero vectors and θ be the angle between them such that $0^\circ\le\theta\le180^\circ$ then their scalar product is $\bar{a}.\bar{b}=|\bar{a}||\bar{b}|\cos\theta$.

2) Angle between two vectors:

$$\bar{a}.\bar{b}=|\bar{a}||\bar{b}|\cos\theta \Rightarrow \cos\theta=\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\Rightarrow\theta=\cos^{-1}\left(\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right)$$



3) Scalar product on the ortho normal traid $(\bar{i},\bar{j},\bar{k})$:

(i) $\bar{i}.\bar{i}=1, \bar{j}.\bar{j}=1, \bar{k}.\bar{k}=1$ $[\because \bar{i}.\bar{i}=|\bar{i}||\bar{i}|\cos0^\circ=(1)(1)(1)=1]$

(ii) $\bar{i}.\bar{j}=0, \bar{j}.\bar{k}=0, \bar{k}.\bar{i}=0$ $[\because \bar{i}.\bar{j}=|\bar{i}||\bar{j}|\cos90^\circ=(1)(1)(0)=0]$

4) Important Remarks on Scalar Product:

Note 1: The dot product of two vectors is a scalar quantity. (real number)

Note 2: $\bar{a}.\bar{b}=\bar{b}.\bar{a}$ (i.e. the dot product of two vectors is commutative)

Note 3: $\bar{a}.\bar{b}>0 \Leftrightarrow 0^\circ\le\theta<90^\circ$ i.e, θ is acute.

Note 4: $\bar{a}.\bar{b}<0 \Leftrightarrow 90^\circ<\theta\le180^\circ$ i.e, θ is obtuse.

Note 5: $\bar{a}.\bar{b}=|\bar{a}||\bar{b}| \Leftrightarrow \theta=0$, ($\bar{a},\bar{b}$ are like vectors)

Note 6: $\bar{a}.\bar{b}=0\Leftrightarrow\bar{a}\perp\bar{b}$, ($\bar{a}$ is perpendiuclar to $\bar{b}$) (or) either $\bar{a}$ or $\bar{b}$ is a null vector.

Note 7: $\bar{a}.\bar{b}=-|\bar{a}||\bar{b}| \Leftrightarrow \theta=\pi$, ($\bar{a},\bar{b}$ are unlike vectors.)

Note 8: $\bar{a}.\bar{a}\ge0$

Note 9: $(l\bar{a}).(m\bar{b})=lm(\bar{a}.\bar{b})$ for the scalars $l,m\in R$.

Note 10: $\bar{a}.(\bar{b}+\bar{c})=\bar{a}.\bar{b}+\bar{a}.\bar{c}$

Note 11: $|\bar{a}.\bar{b}|\le|\bar{a}||\bar{b}|$ ($\because |\cos\theta|\le1$)

Note 12: $|\bar{a}+\bar{b}|\le|\bar{a}|+|\bar{b}|$ (Triangular in equality)

Note 13: $\bar{a}.\bar{a}=|\bar{a}||\bar{a}|\cos0^\circ=|\bar{a}|^2$ .Thus $|\bar{a}|^2=\bar{a}^2=\bar{a}.\bar{a}$. Hence we have following formulae.

(i) $|\bar{a}+\bar{b}|^2=|\bar{a}|^2+|\bar{b}|^2+2\bar{a}.\bar{b}$

(ii) $|\bar{a}-\bar{b}|^2=|\bar{a}|^2+|\bar{b}|^2-2\bar{a}.\bar{b}$.

(iii) $|\bar{a}+\bar{b}|^2+|\bar{a}-\bar{b}|^2=2(|\bar{a}|^2+|\bar{b}|^2)$

(iv) $|\bar{a}+\bar{b}|^2-|\bar{a}-\bar{b}|^2=4\bar{a}.\bar{b}$

(v) $(\bar{a}+\bar{b}).(\bar{a}-\bar{b})=|\bar{a}|^2-|\bar{b}|^2$

(vi) $|\bar{a}+\bar{b}+\bar{c}|^2=|\bar{a}|^2+|\bar{b}|^2+|\bar{c}|^2+2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a})$

5) Analytic expression for the Dot product:

5.1) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ **and** $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ **then** $\bar{a}.\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$.

5.2) $|\bar{a}|^2 = \bar{a}.\bar{a} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) = a_1^2 + a_2^2 + a_3^2$

5.3) If θ is the angle between the vectors $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

then $\cos\theta = \dfrac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}$

5.4) If $\bar{a}$ is perpendicular to $\bar{b}$ then $\bar{a}.\bar{b} = \bar{0} \Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

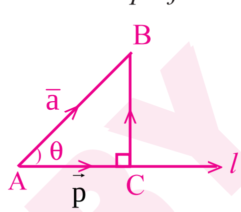
6) Projection of a vector on a line

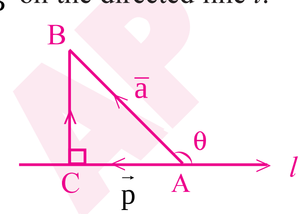
(Geometrical application of Scalar Product) :

Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a directed line l, then

the projection of $\overrightarrow{AB}$ on l is a vector $\vec{p}$ with magnitude $|\overrightarrow{AB}||\cos\theta|$

The direction of $\vec{p}$ is the same or opposite to that of the line l, depending upon whether $\cos\theta$ is positive or negative. The vector $\vec{p}$ is called the *projection vector*, and its magnitude $|\vec{p}|$ is simply called as the *projection* of the vector $\overrightarrow{AB}$ on the directed line l.





Formula:

Projection of the vector $\vec{a}$ on the vector $\vec{b}$ is $\dfrac{\vec{a}.\vec{b}}{|b|}$ (or) $\vec{a}.\dfrac{\vec{b}}{|b|}$ (or) $\vec{a}.\hat{b}$

Projection of the vector $\vec{a}$ on the line l with $\hat{p}$ as the unit vector along l is $\vec{a}\cdot\hat{p}$.

Note: If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then the scalar components (d.r's) a_1, a_2, a_3 are nothing but the projections of $\vec{a}$ along x-axis, y-axis, z-axis respectively.

10(d) VECTOR (CROSS) PRODUCT

CONCEPTS & FORMULAS

1) Vector Product: Let $\bar{a}, \bar{b}$ be two non-zero and non-collinear vectors and θ be the angle between them then the vector product of $\bar{a}, \bar{b}$ is $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n}$

Here $\hat{n}$ is the unit vector perpendicular to the plane containing $\bar{a}$ and $\bar{b}$

The direction of $\hat{n}$ is determined according to the Right Handed Screw Rule.

Note 1: The cross product $\bar{a}\times\bar{b}$ is a vector perpendicular to plane of $\bar{a}, \bar{b}$.

$\bar{a}, \bar{b}$ and $\bar{a}\times\bar{b}$ form a right handed system of vectors. Hence $\bar{a}\times\bar{b}$ is parallel to $\hat{n}$

***Note 2:** $|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$. ($\because |\sin\theta| = \sin\theta$ for $0 \le \theta \le 180^\circ$ and $|\hat{n}| = 1$)

2) Angle between two vectors using Cross Product:

If θ is the angle between $\bar{a}, \bar{b}$ then $\sin\theta = \dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}$ $\quad [\because |\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta]$

3) Unit vector perpendicular to the plane containing a,b:

We know $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n} \Rightarrow \hat{n} = \dfrac{\bar{a}\times\bar{b}}{|\bar{a}||\bar{b}|\sin\theta} = \dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$ $\quad [\because |\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta]$

Note 1: The unit vectors perpendicular to the plane containing $\bar{a}$ and $\bar{b} = \pm\dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$

Note 2: The vector of magnitude k perpendicular to plane of $\bar{a}, \bar{b} = k\left(\dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}\right)$

4) Cross product on the ortho normal traid $(\bar{i}, \bar{j}, \bar{k})$

(i) $\bar{i}\times\bar{i} = \bar{0}, \bar{j}\times\bar{j} = \bar{0}, \bar{k}\times\bar{k} = \bar{0}$ $\quad [\because \bar{i}\times\bar{i} = |\bar{i}||\bar{i}|\sin 0^\circ = (1)(1)0 = 0]$

(ii) $\bar{i}\times\bar{j} = \bar{k}, \bar{j}\times\bar{k} = \bar{i}, \bar{k}\times\bar{i} = \bar{j}$ $\quad [\because \bar{i}\times\bar{j} = |\bar{i}||\bar{j}|\sin 90^\circ\,\hat{n} = (1)(1)(1)\bar{k} = \bar{k}]$

(iii) $\bar{j}\times\bar{i} = -\bar{k}, \bar{k}\times\bar{j} = -\bar{i}, \bar{i}\times\bar{k} = -\bar{j}$

5) Important Remarks on Vector Product:

Note 1: $\bar{a}\times\bar{b} \ne \bar{b}\times\bar{a}$ (Cross product is not commutative). But $|\bar{a}\times\bar{b}| = |\bar{b}\times\bar{a}|$

Note 2: $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$ (Anti commutative law)

Note 3: $\bar{a}\times\bar{b} = \bar{0} \Leftrightarrow \bar{a} \parallel \bar{b}$ are collinear or $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$

Note 4: $\bar{a}\times\bar{a} = \bar{0} \Leftrightarrow \bar{a}, \bar{b}$ are parallel.

Note 5: $\bar{a} \perp \bar{b} \Leftrightarrow \bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\hat{n} \Leftrightarrow |\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|$

Note 6: $(\bar{a}\times\bar{b})\times\hat{n} = 0$ ($\because \bar{a}\times\bar{b}$ and $\hat{n}$ are parallel)

Note 7: $(\bar{a}\times\bar{b}).\bar{a} = 0, (\bar{a}\times\bar{b}).\bar{b} = 0$, ($\because \bar{a}\times\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$)

Note 8: $(l\bar{a})\times(m\bar{b}) = lm(\bar{a}\times\bar{b})$ for the scalars $l, m \in R$.

Note 9: $\bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b} + \bar{a}\times\bar{c}$ (Cross product is distributive over addition)

Note 10: $|\bar{a}\times\bar{b}| \le |\bar{a}||\bar{b}|$ ($\because |\sin\theta| \le 1$)

Note 11: If either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ when θ is not defined. Here, we take $\bar{a}\times\bar{b} = \bar{0}$

Note 12: A vector perpendicular to both $\bar{a}, \bar{b}$ is $\bar{a}\times\bar{b}$

6) Analytic expression for cross product:

If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}, \bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ then $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Here, $\bar{a}\times\bar{b} = \bar{i}(a_2b_3 - a_3b_2) - \bar{j}(a_1b_3 - a_3b_1) + \bar{k}(a_1b_2 - a_2b_1)$

7) Area of triangle, parallelogram (Geometrical application of Cross product):

7.1. Area of triangle:

- Area of the triangle with two adjacent sides $\bar{a}, \bar{b}$ is $\frac{1}{2}|\bar{a}\times\bar{b}|$
- Area of ΔABC is $\Delta = \frac{1}{2}|\overline{AB}\times\overline{AC}| = \frac{1}{2}|\overline{BC}\times\overline{BA}| = \frac{1}{2}|\overline{CA}\times\overline{CB}|$
- If A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are the vertices of ΔABC then $\Delta = \frac{1}{2}|\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a}|$

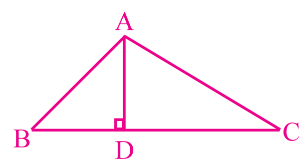
7.2. Area of parallelogram:

- Area of the parallelogram with adjacent sides $\bar{a}, \bar{b}$ is $|\bar{a}\times\bar{b}|$
- Area of the quadrilateral ABCD with diagonals AC,BD is $\frac{1}{2}(\overline{AC}\times\overline{BD})$
- Area of the quadrilateral (or parallelogram) with diagonals $\bar{d}_1, \bar{d}_2$ is $\frac{1}{2}|\bar{d}_1\times\bar{d}_2|$

8) The length of the perpendicular from a point to a line in space:

Consider ΔABC and AD be the length of the perpendicular from A on to BC

Now, Area of $\Delta ABC = \frac{1}{2}$(base)(height) $= \frac{1}{2}|\overline{BC}||\overline{AD}|$



$$\Rightarrow |\overline{AD}| = \frac{2(\text{Area of } \Delta ABC)}{|\overline{BC}|} = \frac{2.\frac{1}{2}|\overline{BC}\times\overline{BA}|}{|\overline{BC}|} = \frac{|\overline{BC}\times\overline{BA}|}{|\overline{BC}|}$$