SHINING STARS'

LAQ STAR QUESTIONS PLUS

1. **For the matrix** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, **Show that** $A^3 - 6A^2 + 5A + 11\,I = O$. **Hence, find** A^{-1}

2. **If** $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

3D-GEOMETRY

3. Find the vector equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines: $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

PROBABILITY

4. A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

APPLICATION OF DERIVATIVES

5. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?
6. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
7. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.
8. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

INTEGRATION

9. Evaluate $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2 \sin^2 x}$

10. Find the integral of $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\,dx}{\cos^2 x + 4\sin^2 x}$

11. Evaluate $\int \frac{1}{1+\sin x + \cos x}dx$.

12. **Find** $\int\left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx$

13. Find the integral of $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

14. Find the integral of $\frac{1}{x^4-1}$

DETERMINANTS

***1. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, Show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}**

Sol: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$$A^3 - 6A^2 + 5A + 11I = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Thus, $A^3 - 6A^2 + 5A + 11I = O$

Now $A^3 - 6A^2 + 5A + 11I = O$

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = O \quad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0\right]$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = -11(IA^{-1})$$

$$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I).....(1)$$

Now, $A^2 - 6A + 5I = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} ....(2)$$

From (1) and (2), we get

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

***2. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}**

Sol: Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3-6A^2+9A-4I=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}-6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}+9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}-4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}-\begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}+\begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}-\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix}-\begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0$$

$$A^3-6A^2+9A-4I=O$$

$$\Rightarrow (AAA)A^{-1}-6(AA)A^{-1}+9AA^{-1}-4IA^{-1}=O \text{ [Post-multiplying by } A^{-1} \text{as} |A|\neq 0]$$

$$\Rightarrow AA(AA^{-1})-6A(AA^{-1})+9(AA^{-1})=4(IA^{-1})$$

$$\Rightarrow AAI-6AI+9I=4A^{-1} \Rightarrow A^2-6A+9I=4A^{-1} \Rightarrow A^{-1}=\frac{1}{4}(A^2-6A+9I)......(1)$$

$$A^2-6A+9I=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}-6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}+9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}-\begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix}+\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}=\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.......(2)$$

From (1) and (2), we get $A^{-1}=\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

3D-GEOMETRY

3. **Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:** $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

Sol: Let $\vec{a}=\hat{i}+2\hat{j}-4\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

The equation of the line passing through $(1,2,-4)$ and parallel to vector $\vec{b}$ is given by

$$\Rightarrow \vec{r}=\vec{a}+\lambda\vec{b} \Rightarrow \vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}).....(1)$$

The equations of the given lines are

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}.......(2) \quad \text{and} \quad \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.......(3)$$

Since, lines of the equations (1) and (2) are perpendicular to each other we have

$3b_1-16b_2+7b_3=0...(4)$

Also, Lines (1) and (3) are perpendicular to each other $\Rightarrow 3b_1+8b_2-5b_3=0.....(5)$

Solving equations (4) and (5), we have

$$\frac{b_1}{(-16)\times(-5)-8\times 7}=\frac{b_2}{7\times 3-3\times(-5)}=\frac{b_3}{3\times 8-3\times(-16)} \Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72} \Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$$

Hence, the direction ratios of $\vec{b}$ are and 2,3, 6

$\therefore \vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

Putting $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ in equation (1), we get $\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$

$\therefore$ the required vector equation of the line is $\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$

PROBABILITY

4. **A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.**

Sol: Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus, $P(S)=\frac{1}{6}, P(F)=\frac{5}{6}$

P(A wins in the first throw) = $P(S)=\frac{1}{6}$

A gets the third throw, when the first throw by A and second throw by B result into failures.

$\therefore$P(A wins in the 3rd throw) = P(FFS) = P(F) P(F) P(S) $=\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\left(\frac{5}{6}\right)^2\times\frac{1}{6}$

P(A wins in the 5th throw) = P (FFFFS) $=\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)$ and so on.

(i) $P(A\ \text{wins})=\frac{1}{6}+\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)+\ldots=\frac{\frac{1}{6}}{1-\frac{25}{36}}=\frac{6}{11}$ $\left[\because S_\infty=\frac{a}{1-r}\right]$

(ii)P(B wins)= 1–P(A wins) $=1-\frac{6}{11}=\frac{5}{11}$

APPLICATION OF DERIVATIVES

5. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Sol: For the sphere, we take radius = r and volume=V

Given $\frac{dV}{dt} = 900\ c.c/s$, r = 15 cm

Volume of the sphere $V = \frac{4}{3}\pi r^3$ On diff. w.r.t 't', we get $\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$

$\Rightarrow 900 = 4\pi(15)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{900}{4\pi \times 15 \times 15} = \frac{\cancel{900}}{\cancel{900}\pi} = \frac{1}{\pi}$ cm/s

---

6. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

Sol: For the cube, we take length of the edge =x , Volume=V and Surface area=S

Given $\frac{dV}{dt} = 8\,cm^3/s$ and x=12 cm

Volume of the cube $V=x^3$ On diff. w.r.t 't', we get $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$\Rightarrow 8 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$

Surface area $S=6x^2$

On diff. w.r.t 't', we get $\frac{dS}{dt} = 12x\frac{dx}{dt} = 12\cancel{x}\left(\frac{8}{3x^{\cancel{2}}}\right) = \frac{32}{x}$

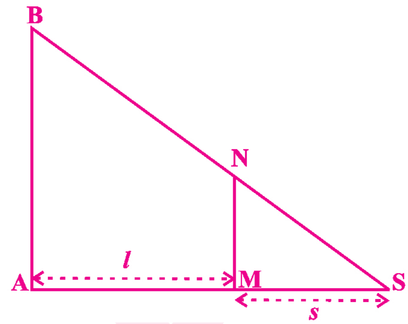
So, when x=12 cm $\Rightarrow \frac{ds}{dt} = \frac{32}{12} = \frac{8}{3}cm^2/s$

7. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.

Sol: Let AB be the lamp-post, the lamp being at the position B and let MN be the man at a particular time t and let AM = l metre.

Then, MS is the shadow of the man. Let MS = s metres.

Note that $\Delta MSN \sim \Delta ASB$



$$\frac{MS}{AS} = \frac{MN}{AB}$$

AS = 3s (as MN = 2 and AB = 6 (given))

Thus AM = 3s – s = 2s. But AM = l

So $\quad l = 2s$

$\therefore \frac{dl}{dt} = 2\frac{ds}{dt}$. $\quad$ Since $\frac{dl}{dt} = \frac{5}{2}$ km/h.

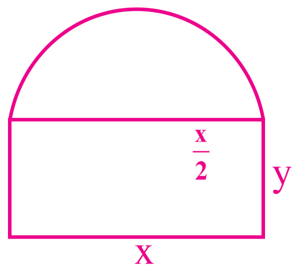
Hence, the length of the shadow increases at the rate $\frac{5}{2}$ km/h.

8. **A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.**

Sol: Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening be x/2.

It is given that the perimeter of the window is 10m.



$$\therefore x+2y+\frac{\pi x}{2}=10 \Rightarrow x\left(1+\frac{\pi}{2}\right)+2y=10$$

$$\Rightarrow 2y=10-x\left(1+\frac{\pi}{2}\right) \Rightarrow y=5-x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

Area of the window (A) is given by,

$$A=xy+\frac{\pi}{2}\left(\frac{x}{2}\right)^2=x\left[5-x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right]+\frac{\pi}{8}x^2=5x-x^2\left(\frac{1}{2}+\frac{\pi}{4}\right)+\frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx}=5-2x\left(\frac{1}{2}+\frac{\pi}{4}\right)+\frac{\pi}{4}x=5-x-\frac{\pi}{4}x$$

$$\frac{d^2A}{dx^2}=-1-\frac{\pi}{4}=-\left(1+\frac{\pi}{4}\right)<0$$

Now, $\frac{dA}{dx}=0$

$$\Rightarrow 5-x-\frac{\pi}{4}x=0 \Rightarrow x\left(1+\frac{\pi}{4}\right)=5 \Rightarrow x=\frac{5}{\left(1+\frac{\pi}{4}\right)} \Rightarrow x=\frac{20}{\pi+4}$$

When, $x=\frac{20}{\pi+4}$

Then $\frac{d^2A}{dx^2}<0$,

$\therefore$ by second derivative test, the area is the maximum when length is $\frac{20}{\pi+4}$ m

Now, $y=5-\frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right)=5-\frac{5(2+\pi)}{\pi+4}=\frac{10}{\pi+4}$

Hence, the required dimensions of the window to admit maximum light is given by length $\frac{20}{\pi+4}$ m and breadth $\frac{10}{\pi+4}$ m .

INTEGRALS

9. **Evaluate** $\int_0^{\pi} \frac{xdx}{a^2\cos^2x+b^2\sin^2x}$

Sol: Let $I=\int_0^{\pi}\frac{xdx}{a^2\cos^2x+b^2\sin^2x}=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$

$$=\pi\int_0^{\pi}\frac{dx}{a^2\cos^2x+b^2\sin^2x}-\int_0^{\pi}\frac{xdx}{a^2\cos^2x+b^2\sin^2x}$$

$$=\pi\int_0^{\pi}\frac{dx}{a^2\cos^2x+b^2\sin^2x}-I$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{dx}{a^2\cos^2x+b^2\sin^2x}$$

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2x+b^2\sin^2x}=\frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2x+b^2\sin^2x}$$

$$=\pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{a^2\cos^2x+b^2\sin^2x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2x+b^2\sin^2x}\right]$$

$$=\pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2xdx}{a^2+b^2\tan^2x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\operatorname{cosec}^2xdx}{a^2\cot^2x+b^2}\right]$$

$$=\pi\left[\int_0^{1}\frac{dt}{a^2+b^2t^2}-\int_1^{0}\frac{du}{a^2u^2+b^2}\right](\text{by taking } \tan x=t \text{ and } \cot x=u)$$

$$=\frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^1-\frac{\pi}{ab}\left[\tan^{-1}\frac{au}{b}\right]_1^0$$

$$=\frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a}+\cot^{-1}\frac{b}{a}\right]=\frac{\pi}{ab}.\frac{\pi}{2}=\frac{\pi^2}{2ab}\qquad\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$$

10. **Find the integral of** $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x dx}{\cos^2 x + 4\sin^2 x}$

Sol: Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1-\cos^2 x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x} dx$$

$$= \frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x}{4-3\cos^2 x} dx = \frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x - 4}{4-3\cos^2 x} dx$$

$$= \frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{4-3\cos^2 x} dx + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4}{4-3\cos^2 x} dx$$

$$= \frac{-1}{3}\int_0^{\frac{\pi}{2}} 1 dx + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3} dx = -\frac{1}{3}[x]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1+\tan^2 x) - 3} dx$$

$$= -\frac{\pi}{6} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x} dx \text{.....(1)}$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x} dx$

Put, $2\tan x = t \Rightarrow 2\sec^2 x dx = dt$

When $x = 0$ we have $t = 0$ and $x = \frac{\pi}{2}$ we have $t = \infty$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x} dx = \int_0^{\infty} \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_0^{\infty} = \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right] = \frac{\pi}{2}$$

$$\therefore \text{from (1), we get } I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

11. Evaluate $\int \frac{1}{1+\sin x+\cos x}dx.$

Sol: Put $\tan\frac{x}{2}=t$ then $\sin x=\frac{2t}{1+t^2}$; $\cos x=\frac{1-t^2}{1+t^2}$ and $dx=\frac{2dt}{1+t^2}$

$$\therefore\ I=\int\frac{1}{1+\sin x+\cos x}dx=\int\frac{\left(\frac{2dt}{1+t^2}\right)}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int\frac{2dt}{(1+t^2)+2t+(1-t^2)}$$

$$=2\int\frac{1}{2+2t}dt=\int\frac{1}{1+t}dt=\log|1+t|+C=\log\left|1+\tan\left(\frac{x}{2}\right)\right|+C$$

12. Find $\int\left[\sqrt{\cot x}+\sqrt{\tan x}\right]dx$

Sol: We have $I=\int[\sqrt{\cot x}+\sqrt{\tan x}]\,dx=\int\sqrt{\tan x}(1+\cot x)\,dx$

Put $\tan x=t^2\Rightarrow \sec^2x\,dx=2t\,dt$ or $dx=\frac{2tdt}{1+t^4}$

Then $I=\int t\left(1+\frac{1}{t^2}\right)\frac{2t}{(1+t^4)}dt=2\int\frac{(t^2+1)}{t^4+1}dt=2\int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t^2+\frac{1}{t^2}\right)}=2\int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2}$

Put $t-\frac{1}{t}=y$, so that $\left(1+\frac{1}{t^2}\right)dt=dy.$

Then $\mathbf{I}=2\int\frac{dy}{y^2+\left(\sqrt{2}\right)^2}=\sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}}+\mathbf{C}=\sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}}+\mathbf{C}$

$$=\sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)+\mathbf{C}=\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+\mathbf{C}$$

13. Find the integral of $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Sol: We have $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=\dfrac{x^4+3x^2+2}{x^4+7x^2+12}=\dfrac{(x^4+7x^2+12)-4x^2+10}{x^4+7x^2+12}=1-\dfrac{(4x^2+10)}{(x^2+3)(x^2+4)}$ ...(1)

Put x^2=y then

$$\text{Let}\ \frac{(4y+10)}{(y+3)(y+4)}=\frac{A}{(y+3)}+\frac{B}{(y+4)}=\frac{A(y+4)+B(y+3)}{(y+3)(y+4)}$$

$\Rightarrow A(y+4)+B(y+3)=(4y+10)$ ....(1)

Put y= –3 in (1) $\Rightarrow A(-3+4)+B(0)=-12+10=-2\Rightarrow A=-2$

Put y= –4 in (1) $\Rightarrow A(0)+B(-4+3)=-16+10=-6\Rightarrow B=6$

From (1),

$$\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=\int 1-\left[\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}\right]dx=\int\left[1+\frac{2}{x^2+\left(\sqrt{3}\right)^2}-\frac{6}{x^2+2^2}\right]dx$$

$$=x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C=x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

14. Find the integral of $\dfrac{1}{x^4-1}$

Sol: $G.E = \dfrac{1}{(x^4-1)} = \dfrac{1}{(x^2-1)(x^2+1)} = \dfrac{1}{(x+1)(x-1)(x^2+1)}$

Let $\dfrac{1}{(x+1)(x-1)(x^2+1)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)} + \dfrac{Cx+D}{(x^2+1)}$

$$= \frac{A(x-1)(1+x^2)+B(x+1)(1+x^2)+(Cx+D)(x^2-1)}{(x+1)(x-1)(x^2+1)}$$

$A(x-1)(1+x^2)+B(x+1)(1+x^2)+(Cx+D)(x^2-1)=1 ......(1)$

$\Rightarrow A(x^3+x-x^2-1)+B(x^3+x+x^2+1)+Cx^3+Dx^2-Cx-D=1$

$\Rightarrow (A+B+C)x^3+(-A+B+D)x^2+(A+B-C)x+(-A+B-D)=1$

Put $x=-1$ in (1) $\Rightarrow$ $A(-1-1)(1+1)+0+0=1 \Rightarrow -4A=1 \Rightarrow A=-1/4$

Put $x=1$ in (1) $\Rightarrow$ $0+B(1+1)(1+1)+0=1 \Rightarrow 4B=1 \Rightarrow B=1/4$

Equating the coefficients of x^3 we get

$A+B+C=0 \quad \Rightarrow \quad C=-A-B=\dfrac{1}{4}-\dfrac{1}{4}=0$

Equating the coefficients of x^2 we get

$-A+B-D=1 \Rightarrow D=A-B=-\dfrac{1}{4}-\dfrac{1}{4}=\dfrac{-2}{4}=-\dfrac{1}{2}$

$\therefore A=-\dfrac{1}{4}, B=\dfrac{1}{4}, C=0, D=-\dfrac{1}{2}$

$\therefore \dfrac{1}{(x^4-1)} = \dfrac{-1}{4(x+1)} + \dfrac{1}{4(x-1)} - \dfrac{1}{2(1+x^2)}$

$\Rightarrow \displaystyle\int \frac{1}{(x^4-1)}dx = \int \frac{-1}{4(x+1)}dx + \int \frac{1}{4(x-1)}dx - \int \frac{1}{2(1+x^2)}dx$

$\Rightarrow \displaystyle\int \frac{1}{(x^4-1)}dx = -\frac{1}{4}\log|x+1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{-1}x + C$

$= \dfrac{1}{4}\log\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\tan^{-1}x + C \qquad \left[\because \log a - \log b = \log\left(\dfrac{a}{b}\right)\right]$