LAQ PQ QUESTIONS

1. Solve the system of equations using matrix method
$x - y + z = 4,\ 2x + y - 3z = 0,\ x + y + z = 2$

2. Find the shortest distance between lines
$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

3. Find the distance between the lines l_1 and l_2 given by
$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}),\ \vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$

4. Find the distance between the pair of parallel lines
$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{-3}$ and $\frac{x+2}{-1} = \frac{y+1}{-2} = \frac{z+2}{3}$

5. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

6. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

7. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$ What is the probability that the student knows the answer given that he answered it correctly?

8. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

9. An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

10. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

11. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

12. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

13. Evaluate $\int_0^{\pi} \log(1 + \cos x)dx$

14. Show that the differential equation $(x - y)\,dy - (x + y)\,dx = 0$ is homogeneous and solve it.

LAQ PQ SOLUTIONS

DETERMINANTS

1. **Solve the system of equations using matrix method**
$x - y + z = 4,\ 2x + y - 3z = 0,\ x + y + z = 2$

Sol: Given system of equations are written in the form AX = B, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Now $|A| = 1(1+3)+1(2+3)+1(2-1) = 4+5+1 = 10 \neq 0$

So, A is non-singular. Hence A^{-1} exists.

Cofactor matrix of A:

$A_{11} = 4;\ A_{12} = -5;\ A_{13} = 1$

$A_{21} = 2;\ A_{22} = 0;\ A_{23} = -2$

$A_{31} = 2;\ A_{32} = 5;\ A_{33} = 3$

$$\text{Hence,}\quad \operatorname{adj} A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\operatorname{adj} A = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\text{Now } AX=B \Rightarrow X=A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ x=2, y= –1 and z=1

3D-GEOMETRY

2. Find the shortest distance between lines
$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ **and** $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

Sol: Comparing the given skew lines with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ we get

$\overline{a_1} = 6\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{a}_2 = -4\hat{i} - \hat{k}$ and $\vec{b}_1 = \hat{i} - 2\hat{j} + 2\hat{k}$, $\vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$

Now $\overrightarrow{a_1} - \overrightarrow{a_2} = (6\hat{i} + 2\hat{j} + 2\hat{k}) - (-4\hat{i} - \hat{k}) = 10\hat{i} + 2\hat{j} + 3\hat{k}$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k} = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$$|\overrightarrow{b_1} \times \overrightarrow{b_2}| = \sqrt{(8)^2 + (8)^2 + (4)^2} = \sqrt{64+64+16} = \sqrt{144} = 12$$

Shortest distance between the lines is $d = \dfrac{|(\overrightarrow{a_1} - \overrightarrow{a_2}).(\overrightarrow{b_1} \times \overrightarrow{b_2})|}{|\vec{b}_1 \times \vec{b}_2|}$

$$d = \frac{|(10\hat{i} + 2\hat{j} + 3\hat{k}).(8\hat{i} + 8\hat{j} + 4\hat{k})|}{12} = \frac{|80+16+12|}{12} = \frac{108}{12} = 9 \text{ units.}$$

---

3. Find the distance between the lines l_1 and l_2 given by
$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$, $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$ **(Parallel lines)**

Sol: Given parallel lines $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ .....(1)

$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$ .....(2)

Comparing (1) and (2) with $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$

$\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$ and $\bar{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

Here, $\vec{a}_1 - \vec{a}_2 = (\hat{i} + 2\hat{j} - 4\hat{k}) - (3\hat{i} + 3\hat{j} - 5\hat{k}) = -2\hat{i} - \hat{j} + \hat{k}$

$$\therefore (\vec{a}_1 - \vec{a}_2) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 1 \\ 2 & 3 & 6 \end{vmatrix} = (-6-3)\hat{i} - (-12-2)\hat{j} + (-6+2)\hat{k} = -9\hat{i} + 14\hat{j} - 4\hat{k}$$

Also $|\vec{b}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$

$\therefore$ the distance between the given parallel lines is

$$d = \frac{|(\vec{a}_1 - \vec{a}_2)\vec{b}|}{|\vec{b}|} = \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{7} = \frac{\sqrt{293}}{7}$$

4. Find the distance between the pair of parallel lines

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{-3} \text{ and } \frac{x+2}{-1}=\frac{y+1}{-2}=\frac{z+2}{3}$$

Sol: From given lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{-3}$ and $\frac{x+2}{-1}=\frac{y+1}{-2}=\frac{z+2}{3}$ we have

Hence $\vec{a}_1=(1,2,3)=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{a}_2=(-2,-1,-2)=-2\hat{i}-\hat{j}-2\hat{k}$

$\Rightarrow \vec{a}_1-\vec{a}_2=(\hat{i}+2\hat{j}+3\hat{k})-(-2\hat{i}-\hat{j}-2\hat{k})=3\hat{i}+3\hat{j}+5\hat{k}$

We can take $\vec{b}=(1,2,-3)=\hat{i}+2\hat{j}-3\hat{k}$

$$|\vec{b}|=\sqrt{1^2+2^2+(-3)^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$\text{Now } (\vec{a}_1-\vec{a}_2)\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 3 & 5\\ 1 & 2 & -3\end{vmatrix}=(-9-10)\hat{i}-(-9-5)\hat{j}+(6-3)\hat{k}=-19\hat{i}+14\hat{j}+3\hat{k}$$

$$|(\vec{a}_1-\vec{a}_2)\times\vec{b}|=\sqrt{(-19^2)+(14)^2+3^2}=\sqrt{381+256+9}=\sqrt{566}$$

Distance between given parallel lines $d=\frac{|(\vec{a}_1-\vec{a}_2)\times\vec{b}|}{|\vec{b}|}=\frac{\sqrt{566}}{\sqrt{14}}=\sqrt{\frac{283}{7}}$

PROBABILITY

5. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Sol: Let E_1 and E_2 respectively denote the event that a red ball is transferred from bag I to II and a black ball is transferred from bag I to II.

$P(E_1)=\frac{3}{7}$ and $P(E_2)=\frac{4}{7}$

Let A be the event that the ball drawn is red.

When a red ball is transferred from bag I to II, $P(A|E_1)=\frac{5}{10}=\frac{1}{2}$

When a black ball is transferred from bag I to II, $P(A|E_2)=\frac{4}{10}=\frac{2}{5}$

$$P(E_2|A)=\frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)}=\frac{\frac{4}{7}\times\frac{2}{5}}{\frac{3}{7}\times\frac{1}{2}+\frac{4}{7}\times\frac{2}{5}}=\frac{16}{31}$$

6. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Sol: Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur.

Then $P(S_1)$ = Probability that six occurs $=\frac{1}{6}$

$P(S_2)$ = Probability that six does not occur $=\frac{5}{6}$

$P(E|S_1)$ = Probability that the man reports that six occurs when six has actually occurred on the die = Probability that the man speaks the truth $=\frac{3}{4}$

$P(E|S_2)$ = Probability that the man reports that six occurs when six has not actually occurred on the die

= Probability that the man does not speak the truth $=1-\frac{3}{4}=\frac{1}{4}$

Thus, by Bayes' theorem, we have

Probability that the report of the man that six has occurred is actually a six = $P(S_1|E)$

$$=\frac{P(S_1)P(E|S_1)}{P(S_1)P(E|S_1)+P(S_2)P(E|S_2)}=\frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}}=\frac{1}{8}\times\frac{24}{8}=\frac{3}{8}$$

Hence, the required probability is $\frac{3}{8}$

7. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Sol: Let E_1 and E_2 be the respective events that the student knows the answer and he guesses the answer.

Let A be the event that the answer is correct. $P(E_1)=\frac{3}{4}, P(E_2)=\frac{1}{4}$

The probability that the students answered correctly, given that he knows the answer, is 1.

$\therefore P(A|E_1)=1$

Probability that the student answered correctly, given that he guessed, is 1/4.

$\therefore P(A|E_2)=1/4$

The probability that the student knows the answer, given that he answered it correctly, is given by $P(E_1|A)$..

By using Bayes' theorem, we have

$$P(E_1|A)=\frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}=\frac{\frac{3}{4}\times 1}{\frac{3}{4}\times 1+\frac{1}{4}\times\frac{1}{4}}=\frac{\frac{3}{4}}{\frac{3}{4}+\frac{1}{16}}=\frac{\frac{3}{4}}{\frac{13}{16}}=\frac{12}{13}$$

APPLICATION OF DERIVATIVES

8. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Sol: Let a number be x .
Then, the other number be (16–x).
Let the sum of the cubes of these numbers be denoted by S(x).

Then, $S(x) = x^3 + (16-x)^3$.

$\therefore S'(x) = 3x^2 + 3(16-x)^2(-1) = 3x^2 - 3(16-x)^2 \Rightarrow S''(x) = 6x + 6(16-x)$

Now, $S'(x) = 0 \Rightarrow 3x^2 - 3(16-x)^2 = 0 \Rightarrow x^2 - (16-x)^2 = 0$

$\Rightarrow x^2 - 256 - x^2 + 32x = 0 \Rightarrow x = \frac{256}{32} \Rightarrow x = 8$

Also, $S''(8) = 6(8) + 6(16-8) = 48 + 48 = 96 > 0$

By second derivative test, x=8 is the point of local minima of S.

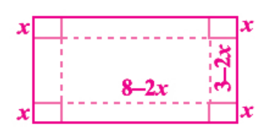
Thus, the numbers are 8 and (16–8)=8.

Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 each.

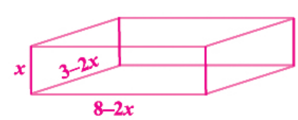
9. An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

Sol: Let x metre be the length of a side of the removed squares.

Then, the height of the box is x, length is 8 – 2x and breadth is 3 – 2x.



(a)



(b)

If V(x) is the volume of the box, then

$V(x) = x(3-2x)(8-2x) = 4x^3 - 22x^2 + 24x$

$V'(x) = 12x^2 - 44x + 24 = 4(x-3)(3x-2) \Rightarrow V''(x) = 24x - 44$

Now $V'(x) = 0 \Rightarrow x = 3, \frac{2}{3}$. But $x \neq 3$

Thus, we have $x = \frac{2}{3}$. Now $V''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 44 = -28 < 0$

$\Rightarrow x = \frac{2}{3}$ is the point of maxima, i.e., if we remove a square of side 2/3 metre from each corner of the sheet and make a box from the remaining sheet, then the volume of the box such obtained will be the largest and it is given by

$$V\left(\frac{2}{3}\right) = 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right) = \frac{200}{27} m^3.$$

10. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

Sol: Let a number be x. Then, the other number is $y=(35-x)$.

Let $P(x)=x^2y^5$. Then we have, $P(x)=x^2(35-x)^5$

$$P'(x)=2x(35-x)^5+x^2 5(35-x)^4(-1)=2x(35-x)^5-5x^2(35-x)^4$$

$$=x(35-x)^4[2(35-x)-5x]=x(35-x)^4(70-7x)=7x(35-x)^4(10-x)$$

$$P''(x)=7(35-x)^4(10-x)+7x[-(35-x)^4-4(35-x)^3(10-x)]$$

$$=7(35-x)^4(10-x)-7x(35-x)^4-28x(35-x)^3(10-x)$$

$$=7(35-x)^3[(35-x)(10-x)-x(35-x)-4x(10-x)]$$

$$=7(35-x)^3[350-45x+x^2-35x+x^2-40x+4x^2]$$

$$=7(35-x)^3(6x^2-120x+350)$$

Now $P'(x)=0 \Rightarrow x=0, x=35, x=10$

When, x=35 then, $P'(x)=P(x)=0 \Rightarrow y=35-35=0$

This will make the product x^2y^5 equal to 0.

When, x=0 then y=35–0=35. This will make the product x^2y^5 equal to 0.

Hence, x=0 and x=35 cannot be the possible values of x.

When x=10. Then, $P''(x)=7(35-10)^3(6\times100-120\times10+350)=7(25)^3(-250)<0$

By second derivative test, P(x) will be the maximum when x=10 and y=35–10=25

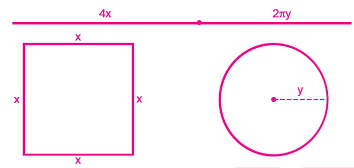
Hence, the required numbers are 10 and 25.

11. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Sol: Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28-l)$.

Now, side of square is 1/4.



Let r be the radius of the circle.

Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28-l)$

The combined areas of the square and the circle A, is given by,

$$A = (\text{side of the square})^2 + \pi r^2 = \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2 = \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$

Hence, $\frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28-l)$

$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

Now, $\frac{d^2A}{dl^2} = 0$

$$\Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28-l) = 0 \Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0 \Rightarrow (\pi+4)l - 112 = 0 \Rightarrow l = \frac{112}{\pi+4}$$

When, $l = \frac{112}{\pi+4}$ we have $\frac{d^2A}{dl^2} > 0$

By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi+4}$ cm.

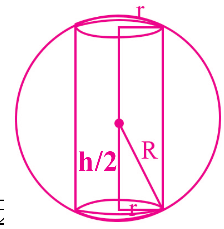
Hence, the combined area is the minimum when the length of the wire in making the square is $l = \frac{112}{\pi+4}$ cm while the length of the wire in making the circle is

$$\left(28 - \frac{112}{\pi+4}\right) = \frac{28\pi}{\pi+4} \text{cm}$$

12. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Sol: A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$

The volume (V) of the cylinder is given by, $V = \pi r^2 h = 2\pi r^2\sqrt{R^2 - r^2}$

$\therefore\ V = 2\pi r^2\sqrt{R^2 - r^2}$

$\Rightarrow \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$

$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$

$= \frac{4\pi r(R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}} = \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}} = 0 \Rightarrow 4\pi r R^2 = 6\pi r^3 \Rightarrow r^2 = \frac{2R^2}{3}$

Also, $\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}(4\pi R^2 - 18\pi r^2) - (4\pi r R^2 - 6\pi r^3)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)}$

$= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + r(4\pi r R^2 - 6\pi r^3)}{(R^2 - r^2)^{\frac{3}{2}}} = \frac{(4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2)}{(R^2 - r^2)^{\frac{3}{2}}}$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}$ we have $\frac{d^2V}{dr^2} < 0$

Thus, the volume is the maximum when $r^2 = \frac{2R^2}{3}$

When, $r^2 = \frac{2R^2}{3}$. Then, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

$\therefore$ Maximum volume of cylinder $= \pi r^2 h$

$$= 2\pi r^2\sqrt{R^2 - r^2} = 2\pi\left(\frac{2R^2}{3}\right)\sqrt{R^2 - \left(\frac{2R^2}{3}\right)} = \frac{4\pi}{3\sqrt{3}}R^3$$

☺ HINT BOX ☺

$R^2 = r^2 + \frac{h^2}{4} \Rightarrow r^2 = R^2 - \frac{h^2}{4}$

$\therefore\ V = \pi r^2 h = \pi h\left(R^2 - \frac{h^2}{4}\right)$

$= \pi\left(R^2 h - \frac{h^3}{4}\right)$

$V' = \pi\left(R^2 - \frac{3h^2}{4}\right)$

$V'' = \pi\left(0 - \frac{6h}{4}\right) < 0$

For max. and min. $V' = 0$

$R^2 = \frac{3h^2}{4} \Rightarrow h^2 = \frac{4R^2}{3}$

$\Rightarrow h = \frac{2R}{\sqrt{3}}$

INTEGRATION

13. Evaluate $\int_0^{\pi} \log(1+\cos x)dx$

Sol: Consider $I=\int_0^{\pi}\log(1+\cos x)dx$.....(1) $\left(\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$\Rightarrow I=\int_0^{\pi}\log[1+\cos(\pi-x)]dx \Rightarrow I=\int_0^{\pi}\log(1-\cos x)dx$.....(2)

Adding (1) and (2), we get $2I=\int_0^{\pi}[\log(1+\cos x)+\log(1-\cos x)]dx$

$2I=\int_0^{\pi}\log(1-\cos^2 x)dx \Rightarrow 2I=\int_0^{\pi}\log(\sin^2 x)dx$ $[\because \log x^n = n\log x]$

$\Rightarrow 2I=2\int_0^{\pi}\log(\sin x)dx \Rightarrow I=\int_0^{\pi}\log(\sin x)dx \quad \ldots(3)$

$\therefore \sin(\pi-x)=\sin x$

We know that $\int_0^{2a} f(x)dx=2\int_0^a f(x)dx$ if $f(2a-x)=f(x)$

$\therefore I=2\int_0^{\frac{\pi}{2}}\log\sin x dx \quad \ldots(4) \quad \Rightarrow I=2\int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2}-x\right)dx=2\int_0^{\frac{\pi}{2}}\log\cos x dx \quad \ldots(5)$

Adding (4) and (5), we get

$2I=2\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos x dx) \Rightarrow I=\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos x+\log 2-\log 2)dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log 2\sin x\cos x-\log 2)dx \Rightarrow I=\int_0^{\frac{\pi}{2}}\log\sin 2x dx-\int_0^{\frac{\pi}{2}}\log 2 dx$

Put $2x=t \Rightarrow 2dx=dt$ When $x=0\Rightarrow t=0; x=\frac{\pi}{2}\Rightarrow t=\pi$

$\therefore I=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt-\frac{\pi}{2}\log 2=\frac{1}{\cancel{2}}\cancel{2}\int_0^{\pi/2}\log(\sin t)dt-\frac{\pi}{2}\log 2=-\frac{\pi}{2}\log 2-\frac{\pi}{2}\log 2=-\pi\log 2$

$\left[\because \int_0^{\pi/2}\log(\sin x)dx=\frac{\pi}{2}\log 2\right]$

DIFFERENTIAL EQUATIONS

14. Show that the differential equation $(x-y)\,dy-(x+y)\,dx=0$ is homogeneous and solve it.

Sol: Given D.E is $(x-y)dy-(x+y)dx=0 \Rightarrow \dfrac{dy}{dx}=\dfrac{x+y}{x-y}$......(1)

Let $F(x,y)=\dfrac{x+y}{x-y}$

Then $F(\lambda x,\lambda y)=\dfrac{\lambda x+\lambda y}{\lambda x-\lambda y}=\dfrac{\lambda(x+y)}{\lambda(x-y)}=\lambda^0 F(x,y)$

$\therefore$ Given Differential Equation is a homogeneous equation.

Let $y=vx \Rightarrow \dfrac{dy}{dx}=\dfrac{d}{dx}(vx) \Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$(1) \Rightarrow v+x\dfrac{dv}{dx}=\dfrac{x+vx}{x-vx}=\dfrac{1+v}{1-v} \Rightarrow x\dfrac{dv}{dx}=\dfrac{1+v}{1-v}-v=\dfrac{1+v-v(1-v)}{1-v}$

$\Rightarrow x\dfrac{dv}{dx}=\dfrac{1+v^2}{1-v}=\dfrac{1-v}{(1+v^2)}dv=\dfrac{dx}{x}$

$\Rightarrow \left(\dfrac{1}{1+v^2}-\dfrac{v}{1+v^2}\right)dv=\dfrac{dx}{x}$

$\tan^{-1}v-\dfrac{1}{2}\log(1+v^2)=\log x+C \Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right)-\dfrac{1}{2}\log\left[1+\left(\dfrac{y}{x}\right)^2\right]=\log x+C$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right)-\dfrac{1}{2}\log\left[\dfrac{x^2+y^2}{x^2}\right]=\log x+C$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right)-\dfrac{1}{2}\left[\log(x^2+y^2)-\log x^2\right]=\log x+C$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right)=\dfrac{1}{2}\log(x^2+y^2)+C$

This is the required general soltuion to the given D.E.