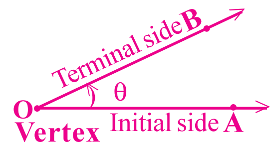
3 TRIGONOMETRIC FUNCTIONS

1 OMQ + 1 VSAQ + 1 LAQ [1 M + 2M +8 M = 11 M]

CONCEPTS & FORMULAS

1.1) ANGLE: Angle is the measurement of rotation of a given ray about its initial point (vertex). In the adjacent diagram, for the angle $\angle AOB = \theta$,

O is the vertex, $\overrightarrow{OA}$ is the initial side & $\overrightarrow{OB}$ is the terminal side .



- Angles are denoted by θ, α, β, γ.....; x, y, z;.....A, B, C, D; ....
- Angle measured in **anti clock** orientation is taken as **positive** and **clockwise** is **negative**.

1.2) MEASUREMENT OF ANGLES:

i) A basic unit of measurement of angle is one **complete rotation**.

ii) **One half** of a complete rotation is called a **straight angle**.

iii) **One fourth** of a complete rotation is called a **right angle**.

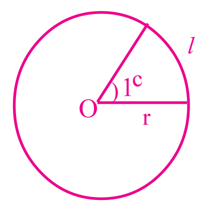
iv) **Degree Measure:** A right angle is divided into 90 equal parts called **degrees;** each degree is divided into 60 equal parts called **minutes** and each minute is divided into 60 equal parts called **seconds**.

Notation: One degree = 1°; one minute = 1' ; one second = 1"

Ex : 1 Right angle = 90°; 1 Straight angle = 180°; 1 Rotation = 360°

v) **Radian Measure (Circular Measure):**

Radian: The angle subtended by an '**arc of length**' equal to **radius** of the circle at its centre, is called a radian.



Notation: One radian is denoted by **1^c** or **1 rad**

Conventionally, x radians is denoted simply by x.

- 1 right angle $= \frac{\pi^c}{2} = 90°$; 1 straight angle $= \pi^c = 180°$; 1 revolution $= 2\pi^c = 360°$
- As an irrational $\pi \cong \frac{22}{7} \cong 3.14$; as an angle in radians we write $\pi \cong 180°$
- 1 Radian= $1^c \cong 57° 17' 45'' = 57.2958° = 206265$ seconds. Also $1° = 0.01745^c$

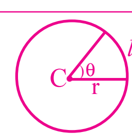
Conversion of Radians into Degrees and vice-versa:

- π radians $= \pi^c = 180°$. Hence, $1^c = \left(\frac{180}{\pi}\right)^o$; $1° = \left(\frac{\pi}{180}\right)^c$
- To convert x° into equivalent radian measure, multiply x by π/180
- To convert radian measure into degree measure, replace π by 180°

Ex1: $\frac{\pi}{2}\text{rad} = \frac{\cancel{180}°}{\cancel{2}} = 90°$; $\frac{\pi^c}{3} = \frac{\cancel{180}°}{\cancel{3}} = 60°$; $\frac{\pi}{6} = \frac{\cancel{180}°}{\cancel{6}} = 30°$; $\frac{2\pi}{3} = \frac{2\times\cancel{180}°}{\cancel{3}} = 120°$

Ex2: $60° = \left(\cancel{60}\times\frac{\pi}{\cancel{180}}\right)^c = \left(\frac{\pi}{3}\right)^c = \frac{\pi}{3}$; $45° = \left(\cancel{45}\times\frac{\pi}{\cancel{180}}\right) = \frac{\pi}{4}$; $720° = \left(\cancel{720}\times\frac{\pi}{\cancel{180}}\right) = 4\pi$

☞ If l is the length of an arc of a circle, r is radius and θ rad is the angle made by l at the centre then $\boldsymbol{l = r\theta}$.



1.3) TYPES OF ANGLES:

i) Acute angle: If $0^\circ < \theta < 90^\circ$ then θ is an acute angle. **Ex:** 30º, 45º,60º.........

ii) Obtuse angle: If $90^\circ < \theta < 180^\circ$ then θ is an obtuse angle. **Ex:** 120º, 135º,150º.........

iii) Complementary angles: If the sum of 2 angles is a right angle then the two angles are called complementary angles to each other. θ and $90^\circ - \theta$ are complementary angles.

Ex: 30º & 60º, 45º & 45º.........

iv) Supplementary angles: If the sum of 2 angles is a straight angle then the 2 angles are called supplementary angles to each other. θ and $180^\circ - \theta$ are supplementary angles.

Ex: 30º & 150º, 60º & 150º.........

v) Quadrantal angles: The angles 0º, 90º, 180º, 270º or integral multiples of these angles are called quadrantal angles. **Ex:** 90º, 180º, 270º , 360º, 450º, 540º, 720º, 900º,1080º, 1440º, ...

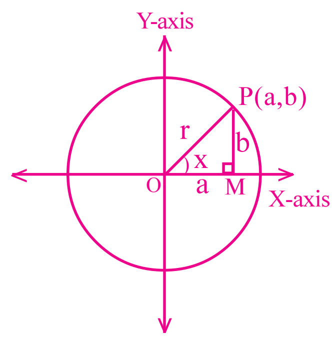
vi) Allied angles: Two angles are said to be allied angles to each other if their sum or difference is a multiple of a right angle.

Ex: $90^\circ \pm \theta, 180^\circ \pm \theta, 270^\circ \pm \theta, 360^\circ \pm \theta$.

2) TRIGONOMETRIC FUNCTIONS of angle x (T-Ratios):

Consider a circle with centre O and radius r in the xy plane.

Let P(a,b) be point on the circle such that $\angle POX = x$



The 6 trigonometric functions of x are defined as follows:

(1) $\sin x = \dfrac{\text{Opposite side}}{\text{Hypotenuse}} = \dfrac{b}{r}$ (2) $\csc x = \dfrac{\text{Hypotenuse}}{\text{Opposite side}} = \dfrac{r}{b}$

(3) $\cos x = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}} = \dfrac{a}{r}$ (4) $\sec x = \dfrac{\text{Hypotenuse}}{\text{Adjacent side}} = \dfrac{r}{a}$

(5) $\tan x = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{b}{a}$ (6) $\cot x = \dfrac{\text{Adjacent side}}{\text{Opposite side}} = \dfrac{a}{b}$

3) RECIPROCAL PAIRS OF T-RATIOS:

(i) $\sin x = \dfrac{1}{\csc x}$ (ii) $\cos x = \dfrac{1}{\sec x}$ (iii) $\tan x = \dfrac{1}{\cot x}$

(iv) $\sin x.\csc x = 1$ (v) $\cos x.\sec x = 1$ (vi) $\tan x.\cot x = 1$ (vii) $\tan x = \dfrac{\sin x}{\cos x} \Rightarrow \sin x = \tan x.\cos x$

4) TRIGONOMETRIC IDENTITIES:

1) $\sin^2 x + \cos^2 x \equiv 1$; $\sin^2 x \equiv 1 - \cos^2 x$; $\cos^2 x \equiv 1 - \sin^2 x, (\forall x \in R)$

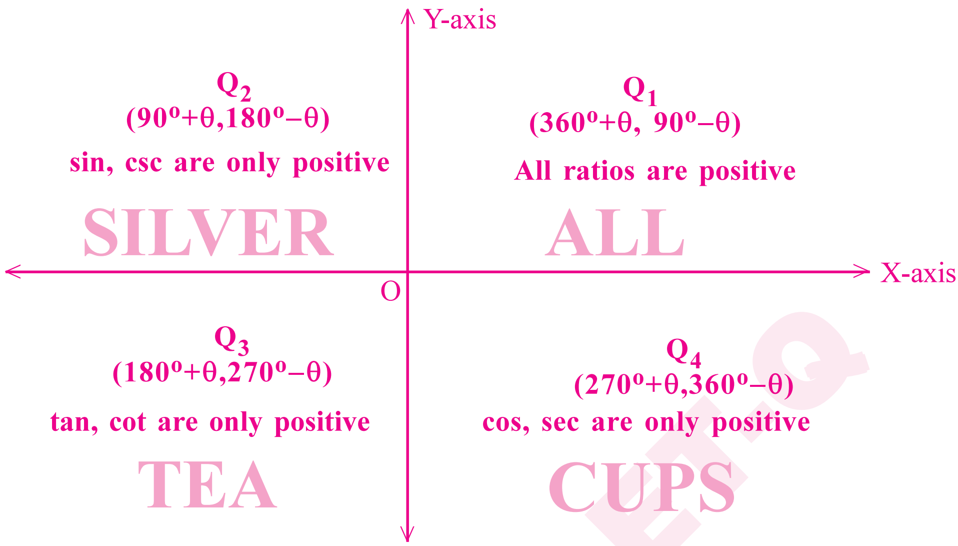
2) $\sec^2 x - \tan^2 x = 1$; $\sec^2 x = 1 + \tan^2 x$; $\tan^2 x = \sec^2 x - 1, [\forall x \in R - \{(2n+1)\pi/2, n \in Z\}]$

3) $\csc^2 x - \cot^2 x = 1$; $\csc^2 x = 1 + \cot^2 x$; $\cot^2 x = \csc^2 x - 1$, $[\ \forall x \in R - n\pi, n \in Z\ ]$

☞ $(\sin x)^2 = \sin^2 x$ (read as sine square x). But $(\sin x)^2$ should not be written as $\sin^2 x^2$ or $\sin x^2$

☞ Generally, in degree measure of angle we use θ and in radian measure we use x.

5) Determination of Trigonometric Ratios for Allied angles of θ (acute):



3 STEPS:

STEP 1: Locate the **quadrant** in which the given angle lies.

STEP 2: Use the Tip "**A**ll **S**ilver **T**ea **C**ups" to attach the **correct sign(±)** for the given T'ratio.

STEP 3: Apply the following conversions:

For **180º** ± θ (or) 360º ± θ the Trigonometric ratios **do not change** .

For **90º**± θ (or) 270º ± θ the Trigonometric ratios **change** as: sin⇔cos; tan ⇔ cot; csc ⇔ sec

Ex: **Simplify (i) sin(180º – θ), cos(180º + θ), tan(360º + θ), cot(360º – θ).**

(ii) sin(90º – θ), cos(90º + θ), tan(270º + θ), csc(270º – θ),

(iii) sin(– θ), cos(–θ), tan(–θ), sec(– θ)

Sol : **i)** sin(180º – θ) = sin θ, cos (180º + θ)= –cos θ

tan(360º + θ)= tan θ, cot(360º – θ)= – cot θ

ii) sin(90º – θ) = cosθ , cos(90º + θ)= –sin θ

tan(270º + θ)= –cot θ, csc(270º – θ)= –secθ

iii) sin(– θ) = –sin θ, cos(–θ) = cos θ, tan(–θ) = –tanθ, sec(– θ) = secθ [∵ $-\theta \in Q_4$]

Angle→ / Ratio↓	90º–θ Q_1	90º+θ Q_2	180º–θ Q_2	180º+θ Q_3	270º–θ Q_3	270º+θ Q_4	360º–θ Q_4	n.360º+θ =360º+θ Q_1	n.360º–θ =(–θ) Q_4
sin	cosθ	cosθ	sinθ	–sinθ	–cosθ	–cosθ	–sinθ	sinθ	–sinθ
cos	sinθ	–sinθ	–cosθ	–cosθ	–sinθ	sinθ	cosθ	cosθ	cosθ
tan	cotθ	–cotθ	–tanθ	tanθ	cotθ	–cotθ	–tanθ	tanθ	–tanθ

6) Values of Trigonometric ratios for certain STANDARD ANGLES:

Angle→ Ratio↓	0^o 0	30^o $\pi/6$	45^o $\pi/4$	60^o $\pi/3$	90^o $\pi/2$	120^o $2\pi/3$	135^o $3\pi/4$	150^o $5\pi/6$	180^o π	210^o $7\pi/6$	270^o $3\pi/2$	300^o $5\pi/3$	360^o 2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\frac{-\sqrt{3}}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	0	$\frac{1}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\to\pm\infty$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	$\to\pm\infty$	$-\sqrt{3}$	0
cot	$\to\pm\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$\to\pm\infty$	$\sqrt{3}$	0	$-\frac{1}{\sqrt{3}}$	$\to\pm\infty$
csc	$\to\pm\infty$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	$\to\pm\infty$	-2	-1	$\frac{-2}{\sqrt{3}}$	$\to\pm\infty$
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	$\to\pm\infty$	-2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	-1	$\frac{-2}{\sqrt{3}}$	$\to\pm\infty$	2	1

Note1 : Algebraically tan 90^o is undefined. But graphically tanx $\to\pm\infty$ when $x\to 90^o$

Note2 : To find the value of the given T'C ratio first write the given angle as a sum or difference of the **nearby Quadrant angle and apply the 3 steps given in the previous page.**

Ex: Find the values of all the 6 Trigonometric ratios for 120^o.

Sol : $120^o \in Q_2$ (Quadrant II)

$$\sin 120^o = \sin(180^o - 60^o) = \sin 60^o = \frac{\sqrt{3}}{2}$$

$$\cos 120^o = \cos(180^o - 60^o) = -\cos 60^o = -\frac{1}{2}$$

$$\tan 120^o = \tan(180^o - 60^o) = -\tan 60^o = -\sqrt{3}$$

$$\cot 120^o = \cot(180^o - 60^o) = -\cot 60^o = -\frac{1}{\sqrt{3}}$$

$$\csc 120^o = \csc(180^o - 60^o) = \csc 60^o = \frac{2}{\sqrt{3}}$$

$$\sec 120^o = \sec(180^o - 60^o) = -\sec 60^o = -2$$

$$\sin 120^o = \sin(90^o + 30^o) = \cos 30^o = \frac{\sqrt{3}}{2}$$

$$\cos 120^o = \cos(90^o + 30^o) = -\sin 30^o = -\frac{1}{2}$$

$$\tan 120^o = \tan(90^o + 30^o) = -\cot 30^o = -\sqrt{3}$$

$$\cot 120^o = \cot(90^o + 30^o) = -\tan 30^o = -\frac{1}{\sqrt{3}}$$

$$\csc 120^o = \csc(90^o + 30^o) = \sec 30^o = \frac{2}{\sqrt{3}}$$

$$\sec 120^o = \sec(90^o + 30^o) = -\csc 30^o = -2$$

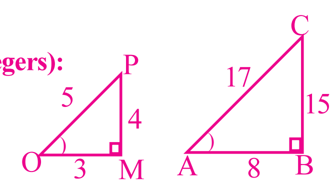
Practice Problems: Find the values of all the 6 trigonometric ratios of the following angles

30^o, 135^o, 240^o, 330^o, 450^o, 540^o, $2\pi/3$, $-\pi$

7) PYTHAGOREAN TRIPLETS (3 sides of a Right angled Triangle in Integers):

(3, 4, **5**); (5, 12, **13**); (7, 24, **25**); ...

(6, 8, **10**); (8, 15, **17**); (10, 24, **26**); ...



Tit Bits

- Length of Arc $l = r\theta$
- $1^c = \left(\frac{180}{\pi}\right)^o$; $1^o = \left(\frac{\pi}{180}\right)^c$
- $\cos(-x) = \cos x$; $\sin(-x) = -\sin x$

$\pi^c = 180^o$, $2\pi^c = 360^o$, $3\pi^c = 540^o$, $4\pi^c = 720^o$, $5\pi^c = 900^o$, $6\pi^c = 1080^o$

FORMULAE PAGE

1) TRIGONOMETRIC IDENTITIES

1. $\sin^2 x + \cos^2 x \equiv 1$; $\sin^2 x \equiv 1 - \cos^2 x$; $\cos^2 x \equiv 1 - \sin^2 x$
2. $\sec^2 x - \tan^2 x = 1$; $\sec^2 x = 1 + \tan^2 x$; $\tan^2 x = \sec^2 x - 1$
3. $\csc^2 x - \cot^2 x = 1$, $\csc^2 x = 1 + \cot^2 x$; $\cot^2 x = \csc^2 x - 1$

- Length of Arc $l = r\theta$
- $1^c = \left(\frac{180}{\pi}\right)^\circ$; $1^\circ = \left(\frac{\pi}{180}\right)^c$
- $\sin(-x) = -\sin x$; $\cos(-x) = \cos x$

2) COMPOUND ANGLES

4. (i) $\sin(x+y) = \sin x\cos y + \cos x\sin y$ (ii) $\sin(x-y) = \sin x\cos y - \cos x\sin y$
5. (i) $\cos(x+y) = \cos x\cos y - \sin x\sin y$ (ii) $\cos(x-y) = \cos x\cos y + \sin x\sin y$
6. (i) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ (ii) $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
7. (i) $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$ (ii) $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$
8. (i) $\sin(x+y) + \sin(x-y) = 2\sin x\cos y$ (ii) $\sin(x+y) - \sin(x-y) = 2\cos x\sin y$
9. (i) $\cos(x+y) + \cos(x-y) = 2\cos x\cos y$ (ii) $\cos(x+y) - \cos(x-y) = -2\sin x\sin y$
10. (i) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$ (ii) $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

3) MULTIPLE & SUB MULTIPLE ANGLES

11.1. $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$ 11.2. $\cos 2x = \cos^2 x - \sin^2 x$ 11.3. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

12. $\cos 2x = 1 - 2\sin^2 x$; $\cos 2x = 2\cos^2 x - 1$ 13. $1 - \cos 2x = 2\sin^2 x$; $1 + \cos 2x = 2\cos^2 x$

14. $\sin^2 x = \dfrac{1-\cos 2x}{2}$; $\cos^2 x = \dfrac{1+\cos 2x}{2}$ 15. $\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}$; $\cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}$

16. (i) $\sin 3A = 3\sin A - 4\sin^3 A$ (ii) $\cos 3A = 4\cos^3 A - 3\cos A$ (iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

4) TRANSFORMATIONS

17.1 $\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$ 17.2. $\sin x - \sin y = 2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

18.1 $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$ 18.2. $\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$