Previous IPE

SOLVED PAPERS

MARCH -2025(AP)

PREVIOUS PAPERS

IPE: MARCH-2025(AP)

Time : 3 Hours

MATHS-2B

Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **$10 \times 2 = 20$**

1. If the length of the tangent from (2, 5) to the circle $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$ then find k.
2. Find the equation of the circle whose end points of a diameter are (1, 2), (4, 6).
3. Find the equation of the common chord of circles $x^2 + y^2 - 4x - 4y + 3 = 0$, $x^2 + y^2 - 5x - 6y + 4 = 0$
4. Find the value of k if the line $2y = 5x + k$ is a tangent to the parabola $y^2 = 6x$.
5. If the eccentricity of a hyperbola is 5/4, then find the eccentricity of its conjugate hyperbola.
6. Evaluate $\int \frac{1}{7x+3} dx$
7. Evaluate $\int \sqrt{1 - \sin 2x}\, dx$
8. Evaluate $\int_0^4 |2 - x|\, dx$
9. Evaluate $\int_0^{\pi/2} \sin^{10} x\, dx$
10. Find the order and degree of the differential equation $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/3}$

SECTION-B

II. Answer any FIVE of the following SAQs: **$5 \times 4 = 20$**

11. Find the length of the chord intercepted by the circle $x^2 + y^2 - 8x - 2y - 8 = 0$ on the line $x + y + 1 = 0$
12. Find the radical centre of the circles $x^2 + y^2 - 4x - 6y + 5 = 0$, $x^2 + y^2 - 2x - 4y - 1 = 0$, $x^2 + y^2 - 6x - 2y = 0$
13. Find the eccentricity, length of latus rectum, centre, foci, length of major axis, minor axis, equation of the directrices of the ellipse $9x^2 + 16y^2 = 144$
14. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentricity e, passes through one end of the minor axis, then show that $e^4 + e^2 = 1$
15. Find the equation of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are (i) parallel to and (ii) perpendicular to the line $x + 2y = 0$
16. Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$
17. Solve $\frac{dy}{dx} - x\tan(y - x) = 1$

SECTION-C

III. Answer any FIVE of the following LAQs: **$5 \times 7 = 35$**

18. Show that the circles $x^2 + y^2 - 6x - 2y + 1 = 0$ and $x^2 + y^2 + 2x - 8y + 13 = 0$ touch each other. Find the point of contact and the equation of the common tangent at their point of contact.

19. If (2,0), (0,1), (4,5) and (0,c) are concyclic, then find c.

20. Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ sq.units

21. Evaluate the reduction formula for $I_n = \int \sin^n x\,dx$ and hence find $\int \sin^4 x\,dx$

22. Evaluate $\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5}dx$

23. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

24. Solve $\sqrt{1+x^2}\sqrt{1+y^2}dx + xydy = 0$

IPE AP MARCH-2025
SOLUTIONS

SECTION-A

1. If the length of the tangent from (2, 5) to the circle $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$ then find k.

Sol: Length of the tangent from (2, 5) to

$S = x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{S_{11}} = \sqrt{37}$;

On squaring both sides, we get $S_{11} = 37$

$\Rightarrow (2)^2 + 5^2 - 5(2) + 4(5) + k = 37$

$\Rightarrow 4 + 25 - 10 + 20 + k = 37$

$\Rightarrow 39 + k = 37 \Rightarrow k = -2$

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2. Find the equation of the circle whose end points of a diameter are (1, 2), (4, 6).

Sol: Given points $A(x_1, y_1) = (1, 2)$ and $B(x_2, y_2) = (4, 6)$

The equation of the circle with A(1, 2), B(4, 6) as ends of a diameter is given by

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \Rightarrow (x - 1)(x - 4) + (y - 2)(y - 6) = 0$

$\Rightarrow (x^2 - x - 4x + 4) + (y^2 - 6y - 2y + 12) = 0 \Rightarrow x^2 + y^2 - 5x - 8y + 16 = 0$

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3. Find the equation of the common chord of the circles:

$x^2 + y^2 - 4x - 4y + 3 = 0,\ x^2 + y^2 - 5x - 6y + 4 = 0$

A: Given Cirlces $S = x^2 + y^2 - 4x - 4y + 3 = 0,\ S' = x^2 + y^2 - 5x - 6y + 4 = 0$

Equation of common of chord (radical axis) is $S - S' = 0 \Rightarrow -4x + 5x - 4y + 6y + 3 - 4 = 0$

$\Rightarrow x + 2y - 1 = 0$

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4. Find the value of k if the line 2y = 5x + k is a tangent to the parabola $y^2 = 6x$.

Sol: Given parabola is $y^2 = 6x$

$\Rightarrow 4a = 6 \Rightarrow a = \frac{6}{4} = \frac{3}{2}$

Given line is $2y = 5x + k \Rightarrow y = \frac{5}{2}x + \frac{k}{2}$

Comparing with y = mx + c we get, $m = \frac{5}{2}, c = \frac{k}{2}$

Tangential condition is c = a/m

$\Rightarrow \frac{k}{2} = \frac{\frac{3}{\cancel{2}}}{\frac{5}{\cancel{2}}} \Rightarrow \frac{k}{2} = \frac{3}{5} \Rightarrow k = 2\left(\frac{3}{5}\right) = \frac{6}{5}$

5. If eccentricity of a hyperbola is 5/4, then find eccentricity of its conjugate hyperbola.

Sol: Let e = 5/4 and the eccentricity of the conjugate hyperbola be e_1

Then, $\frac{1}{e^2}+\frac{1}{e_1^2}=1 \Rightarrow \frac{1}{(5/4)^2}+\frac{1}{e_1^2}=1 \Rightarrow \frac{1}{e_1^2}=1-\frac{16}{25}=\frac{9}{25} \Rightarrow e_1^2=\frac{25}{9} \Rightarrow e_1=\frac{5}{3}$

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6. Evaluate $\int \frac{1}{7x+3}\,dx$.

Sol: Put 7x + 3 = t $\Rightarrow$ 7dx = dt $\Rightarrow$ dx = $\frac{1}{7}$ dt

$$\therefore I=\int \frac{1}{7x+3}dx=\int \frac{1}{t}\left(\frac{1}{7}dt\right)=\frac{1}{7}\int \frac{1}{t}dt=\frac{1}{7}\log|t|+c=\frac{1}{7}\log|7x+3|+c$$

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7. Evaluate $\int \sqrt{1-\sin 2x}\,dx$

Sol: $I=\int\sqrt{1-\sin 2x}\,dx \quad =\int\sqrt{(\sin^2 x+\cos^2 x)-2\sin x\cos x}\,dx \quad =\int\sqrt{(\sin x-\cos x)^2}\,dx$

$=\pm\int(\sin x-\cos x)\,dx = \pm(-\cos x-\sin x)+c.$ ($\pm$ As per the domain restriction.)

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8. Evaluate $\int_0^4 |2-x|\,dx$

Sol: $|2-x| = 2-x$ when $2-x \ge 0 \Rightarrow x \le 2$

$=-(2-x)$ when $2-x<0 \Rightarrow x>2$

$$\therefore \int_0^4 |2-x|\,dx=\int_0^2 |2-x|dx+\int_2^4 |2-x|dx$$

$$=\int_0^2 (2-x)dx+\int_2^4 (x-2)dx$$

$$=\left[2x-\frac{x^2}{2}\right]_0^2+\left[\frac{x^2}{2}-2x\right]_2^4$$

$$=\left(4-\frac{4}{2}\right)+[(8-8)-(2-4)]$$

$$=2+0+2=4$$

9. Evaluate $\int_0^{\pi/2} \sin^{10} x\, dx$

Sol: When n is even,

$$I_n = \int_0^{\pi/2} \sin^{10} x\, dx = \frac{(9)(7)(5)(3)(1)}{(10)(8)(6)(4)(2)} \cdot \frac{\pi}{2} = \frac{63\pi}{512}$$

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10. Find the order and degree of the differential equation $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3}$

Sol: Given D.E is $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{3}}$

Cubing on both sides, we get $\left(\frac{d^2y}{dx^2}\right)^3 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^5$. This is free from fractional powers.

Here, the highest derivative is $\left(\frac{d^2y}{dx^2}\right)^3$

$\therefore$ For the given D.E, order $=2$ and degree $=$exponent $=3$

SECTION-B

11. Find the length of the chord intercepted by the circle $x^2 + y^2 - 8x - 2y - 8 = 0$ on the line $x + y + 1 = 0$

Sol: Given circle is $x^2 + y^2 - 8x - 2y - 8 = 0$,

It's Centre C = (4, 1),

radius $r = \sqrt{16+1+8} = \sqrt{25} = 5$

Perpendicular distance from the centre (4, 1) to the line $x + y + 1 = 0$ is

$$p = \frac{|4+1+1|}{\sqrt{1^2+1^2}} = \frac{6}{\sqrt{2}} = \frac{3\times 2}{\sqrt{2}} = 3\sqrt{2}$$

∴ Length of the chord

$$= 2\sqrt{r^2 - p^2} = 2\sqrt{5^2 - (3\sqrt{2})^2}$$

$$= 2\sqrt{25 - 9(2)} = 2\sqrt{25-18} = 2\sqrt{7}$$

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12. Find the radical centre of the circles $x^2 + y^2 - 4x - 6y + 5 = 0$, $x^2 + y^2 - 2x - 4y - 1 = 0$, $x^2 + y^2 - 6x - 2y = 0$.

Sol: The given circles are

$S = x^2 + y^2 - 4x - 6y + 5 = 0$,

$S' = x^2 + y^2 - 2x - 4y - 1 = 0$,

$S'' = x^2 + y^2 - 6x - 2y = 0$

One radical axis is $S - S' = 0$

$\Rightarrow (-4x + 2x) + (-6y + 4y) + (5 + 1) = 0$

$\Rightarrow -2x - 2y + 6 = 0 \Rightarrow -2(x + y - 3) = 0$

$\Rightarrow x + y - 3 = 0$ ..... (1)

Another radical axis is $S - S'' = 0$

$\Rightarrow (-4x + 6x) + (-6y + 2y) + 5 = 0$

$\Rightarrow 2x - 4y + 5 = 0$ .... (2)

Now, $(1) \times 2 \Rightarrow 2x + 2y - 6 = 0$ ....(3)

$(3) - (2) \Rightarrow 6y - 11 = 0 \Rightarrow y = 11/6$

From(1), $x = 3 - y = 3 - \frac{11}{6} = \frac{18-11}{6} = \frac{7}{6}$

∴ the radical centre is (7/6, 11/6)

13. Find the eccentricity, coordinates of foci, length of latus rectum and equations of directrices of the ellipse $9x^2+16y^2=144$

Sol: Equation of ellipse is $9x^2+16y^2=144 \Rightarrow \frac{\cancel{9}x^2}{\cancel{144}}+\frac{\cancel{16}y^2}{\cancel{144}}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$.

Here, $a^2 = 16, b^2 = 9 \Rightarrow a > b$.

Hence, the ellipse is horizontal

(i) Eccentricity $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}$

(ii) Foci $= (\pm ae,0) = (\pm 4\left(\frac{\sqrt{7}}{4}\right),0) = (\pm\sqrt{7},0)$

(iii) Length of latus rectum $=\frac{2b^2}{a}=\frac{2(9)}{4}=\frac{9}{2}$

(iv) Equation of directrices is $x=\pm\frac{a}{e}=\pm 4\left(\frac{4}{\sqrt{7}}\right)=\frac{\pm 16}{\sqrt{7}} \Rightarrow \sqrt{7}x=\pm 16 \Rightarrow \sqrt{7}x\pm 16=0$

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14. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentricity e, passes through one end of the minor axis, then show that $e^4+e^2=1$

Sol: The equation of the normal at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

Let $L = (ae, b^2/a)$ be the one end of the latus rectum

Hence, equation of the normal at L is $\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2 \Rightarrow \frac{ax}{e}-ay=a^2-b^2$.....(1)

But, (1) passes through the one end $B'(0,-b)$ of minor axis

$\Rightarrow \frac{a(0)}{e}-a(-b)=a^2-b^2 \Rightarrow ab=a^2-a^2(1-e^2) \Rightarrow ab=a^2e^2 \Rightarrow e^2=\frac{b}{a}$

$\therefore e^4=\frac{b^2}{a^2}=\frac{a^2(1-e^2)}{a^2}=1-e^2 \Rightarrow e^4+e^2=1$

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15. Find the equation of the tangents to the hyperbola $x^2-4y^2=4$ which are (i) parallel (ii) perpendicular to the line $x+2y=0$

Sol: Given hyperbola is $x^2-4y^2=4$

$\Rightarrow \frac{x^2}{4}-\frac{y^2}{1}=1 \Rightarrow a^2=4, b^2=1$

Slope of the given line $x+2y=0$ is $m=-1/2 \Rightarrow$ Slope of its perpendicular is 2

Formula:

Tangent with slope m is $y=mx\pm\sqrt{a^2m^2-b^2}$

(i) Parallel tangent with slope $\frac{-1}{2}$ is

$y=-\frac{1}{2}x\pm\sqrt{4\left(\frac{1}{4}\right)-1}$

$\Rightarrow y=\frac{-x}{2} \Rightarrow x+2y=0$

(ii) Perpendicular tangent with slope 2 is

$$y = 2x \pm \sqrt{4(2^2)-1}$$

$$\Rightarrow y = 2x \pm \sqrt{15}$$

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16. Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Sol: We know $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. $\therefore I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ....(1)

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad ....(2)$$

From (1) and (2), $I + I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} \frac{\cancel{\sqrt{\sin x}+\sqrt{\cos x}}}{\cancel{\sqrt{\sin x}+\sqrt{\cos x}}}dx = \int_{\pi/6}^{\pi/3} 1dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6} \Rightarrow 2I = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

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17. Solve $\frac{dy}{dx} - x\tan(y-x) = 1$

Sol: Put, $y - x = t \Rightarrow \frac{dy}{dx} - 1 = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx}+1$

Hence, the given D.E becomes $\left(\frac{dt}{dx}+1\right) - x\tan t = 1 \Rightarrow \frac{dt}{dx} = x\tan t \Rightarrow \frac{1}{\tan t}dt = xdx$

$$\Rightarrow \int \cot t\,dt = \int x\,dx \Rightarrow \log(\sin t) = \frac{x^2}{2}+c$$

$$\Rightarrow 2\log\sin t = x^2 + c \Rightarrow 2\log\sin(y-x) = x^2 + c$$

SECTION-C

18. Show that the circles $x^2+y^2-6x-2y+1=0$ and $x^2+y^2+2x-8y+13=0$ touch each other. Find the point of contact and the equation of the common tangent at their point of contact.

Sol: First circle is $S \equiv x^2 + y^2 - 6x - 2y + 1 = 0$;

Centre $C_1 = (3, 1)$, radius $r_1 = \sqrt{9+1-1} = 3$

Second circle is $S' \equiv x^2 + y^2 + 2x - 8y + 13 = 0$

Centre $C_2 = (-1, 4)$, radius $r_2 = \sqrt{1+16-13} = 2$

$C_1C_2 = \sqrt{(3+1)^2 + (1-4)^2} = \sqrt{16+9} = \sqrt{25} = 5$

Also, $r_1 + r_2 = 3 + 2 = 5 = C_1C_2$.

$\therefore$ The circles touch each other **externally**.

Now, $r_1 : r_2 = 3 : 2$

So, the Point of contact P divides the join of $C_1(3, 1)$, $C_2(-1, 4)$ internally in the ratio 3 : 2

$\therefore P = \left(\frac{3(-1)+2(3)}{5}, \frac{3(4)+2(1)}{5}\right) = \left(\frac{3}{5}, \frac{14}{5}\right)$

The equation of the common tangent to the circles S=0 and S'=0 at the point of contact is given by $S - S' = 0$

$\Rightarrow (\cancel{x^2} + \cancel{y^2} - 6x - 2y + 1) - (\cancel{x^2} + \cancel{y^2} + 2x - 8y + 13) = 0$

$\Rightarrow (-6x - 2x) - 2y + 8y + 1 - 13 = 0$

$\Rightarrow -8x + 6y - 12 = 0 \Rightarrow -2(4x - 3y + 6) = 0$

$\Rightarrow 4x - 3y + 6 = 0$

19. If (2, 0), (0, 1), (4, 5), (0, c) are concyclic then find c.

Sol: Let A = (2, 0), B = (0, 1), C = (4, 5), D = (0, c)

We take $S(x_1,y_1)$ as the centre of the circle $\Rightarrow SA = SB = SC$

Now, $SA = SB \Rightarrow SA^2 = SB^2 \Rightarrow (x_1-2)^2+(y_1-0)^2=(x_1-0)^2+(y_1-1)^2$

$\Rightarrow (\cancel{x_1^2}-4x_1+4)+(\cancel{y_1^2})=(\cancel{x_1^2})+(\cancel{y_1^2}-2y_1+1)$

$\Rightarrow 4x_1-2y_1+1-4=0 \Rightarrow 4x_1-2y_1-3=0$.........(1)

Also, $SB = SC \Rightarrow SB^2 = SC^2 \Rightarrow (x_1-0)^2+(y_1-1)^2=(x_1-4)^2+(y_1-5)^2$

$\Rightarrow (\cancel{x_1^2})+(\cancel{y_1^2}-2y_1+1)=(\cancel{x_1^2}-8x_1+16)+(\cancel{y_1^2}-10y_1+25)$

$\Rightarrow 8x_1-2y_1+10y_1+1-16-25=0 \Rightarrow 8x_1+8y_1-40=0 \Rightarrow 8(x_1+y_1-5)=0 \Rightarrow x_1+y_1-5=0$......(2)

Solving (1) & (2) we get the centre $S(x_1,y_1)$

$2\times(2) \Rightarrow 2x_1+2y_1-10=0$..........(3)

$(1)+(3) \Rightarrow 6x_1-13=0 \Rightarrow 6x_1=13 \Rightarrow x_1=13/6$

$(2) \Rightarrow y_1=5-x_1=5-\frac{13}{6}=\frac{30-13}{6}=\frac{17}{6} \Rightarrow y_1=\frac{17}{6}$

$\therefore$ Centre of the circle is $S(x_1,y_1)=\left(\frac{13}{6},\frac{17}{6}\right)$

Also, we have A = (2, 0)

Hence, radius $r = SA \Rightarrow r^2 = SA^2$

$\therefore r^2 = SA^2 = \left(2-\frac{13}{6}\right)^2+\left(0-\frac{17}{6}\right)^2=\left(\frac{12-13}{6}\right)^2+\left(\frac{17}{6}\right)^2=\left(\frac{1}{36}\right)+\left(\frac{289}{36}\right)=\frac{290}{36}$

$\therefore$ Circle with Centre $\left(\frac{13}{6},\frac{17}{6}\right)$ and $r^2=\frac{290}{36}$ is $\left(x-\frac{13}{6}\right)^2+\left(y-\frac{17}{6}\right)^2=\frac{290}{36}$

Put D(0,c) in the above equation $\Rightarrow \left(0-\frac{13}{6}\right)^2+\left(c-\frac{17}{6}\right)^2=\frac{290}{36} \Rightarrow \left(c-\frac{17}{6}\right)^2=\frac{290}{36}-\frac{169}{36}=\frac{121}{36}$

$\Rightarrow \left(\frac{6c-17}{6}\right)^2=\frac{121}{36} \Rightarrow \frac{(6c-17)^2}{\cancel{36}}=\frac{11^2}{\cancel{36}} \Rightarrow 6c-17=\pm 11$

$\Rightarrow 6c=\pm 11+17 \Rightarrow 6c=28 \Rightarrow c=\frac{\cancel{28}}{\cancel{6}}=\frac{14}{3}$ (or) $\cancel{6}c=\cancel{6} \Rightarrow c=1$

$\therefore$ c = 14/3 (or) 1

20. If y_1, y_2, y_3 are the y-coordinates of the vertices of the triangle inscribed in the parabola $y^2 = 4ax$ then show that the area of the triangle is $\frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$ sq. units.

Sol: We take vertices as $P(x_1, y_1) = (at_1^2, 2at_1)$, $Q(x_2, y_2) = (at_2^2, 2at_2)$, $R(x_3, y_3) = (at_3^2, 2at_3)$

$$\text{Area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} x_1-x_2 & x_1-x_3 \\ y_1-y_2 & y_1-y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} at_1^2-at_2^2 & at_1^2-at_3^2 \\ 2at_1-2at_2 & 2at_1-2at_3 \end{vmatrix} = \frac{a.2a}{2}\begin{vmatrix} t_1^2-t_2^2 & t_1^2-t_3^2 \\ t_1-t_2 & t_1-t_3 \end{vmatrix}$$

$$= a^2\begin{vmatrix} (t_1-t_2)(t_1+t_2) & (t_1-t_3)(t_1+t_3) \\ t_1-t_2 & t_1-t_3 \end{vmatrix} = a^2(t_1-t_2)(t_1-t_3)\begin{vmatrix} t_1+t_2 & t_1+t_3 \\ 1 & 1 \end{vmatrix}$$

$$= a^2(t_1-t_2)(t_1-t_3)|(\cancel{t_1}+t_2-\cancel{t_1}-t_3)| = a^2\,|(t_1-t_2)(t_2-t_3)(t_3-t_1)|$$

$$= a^2\left|\left(\frac{y_1}{2a}-\frac{y_2}{2a}\right)\left(\frac{y_2}{2a}-\frac{y_3}{2a}\right)\left(\frac{y_3}{2a}-\frac{y_1}{2a}\right)\right| \quad \left(\because 2at_1 = y_1 \Rightarrow t_1 = \frac{y_1}{2a}, \ldots\ldots\right)$$

$$= \frac{a^2}{2a.2a.2a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)| = \frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)| \text{ Sq.Units}$$

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21. Evaluate the reduction formula for $I_n = \int\sin^n x\,dx$ and hence find $\int\sin^4 x\,dx$

Sol: Given $I_n = \int\sin^n x\,dx = \int\sin^{n-1}x(\sin x)dx$.

We take First function $u = \sin^{n-1}x$ and

Second function $v = \sin x \Rightarrow \int v = -\cos x$

From By parts rule, we have

$I_n = \sin^{n-1}x\,(-\cos x) - \int(n-1)\sin^{n-2}x\cos x\,(-\cos x)dx$

$= -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x\cos^2 x\,dx$

$= -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x(1-\sin^2 x)dx$

$= -\sin^{n-1}x\cos x + (n-1)[\int\sin^{n-2}x\,dx - \int\sin^n x\,dx]$

$= -\sin^{n-1}x\cos x + (n-1)[I_{n-2} - I_n]$

$I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$

$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - nI_n + I_n$

$\Rightarrow nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} + \cancel{I_n} - \cancel{I_n}$

$$\Rightarrow I_n = \frac{-\sin^{n-1}x\cos x}{n} + \left(\frac{n-1}{n}\right)I_{n-2} \quad \ldots.(1)$$

Put n = 4, 2, 0 successively in (1), we get $I_4 = -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}I_2$

$$= -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\left[-\frac{\sin x\cos x}{2} + \frac{1}{2}I_0\right]$$

$$= \frac{-\sin^3 x\cos x}{4} - \frac{3}{8}\sin x\cos x + \frac{3}{8}I_0$$

$$= -\frac{\sin^3 x\cos x}{4} - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + c \qquad [\because I_0 = x]$$

22. Evaluate $\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5}dx$

Sol: Let $(2\sin x + 3\cos x + 4) = A(3\sin x + 4\cos x + 5) + B\frac{d}{dx}(3\sin x + 4\cos x + 5) + C$ .....(i)

$\therefore (2\sin x + 3\cos x + 4) = A(3\sin x + 4\cos x + 5) + B(3\cos x - 4\sin x) + C$

$\Rightarrow (2\sin x + 3\cos x + 4) = \sin x(3A - 4B) + \cos x(4A + 3B) + (5A + C)$

Equating the coefficients of sinx , we get 3A – 4B = 2 ......(1)

Equating the coefficients of cosx , we get 4A + 3B = 3.......(2)

Equating the constant terms, we get 5A + C = 4 ....(3)

$(1) \times 4 \Rightarrow 12A - 16B = 8$ .....(4)

$(2) \times 3 \Rightarrow 12A + 9B = 9$ ....(5)

$(5) - (4) \Rightarrow 25B = 1 \Rightarrow B = \frac{1}{25}$

From(1), $3A = 2 + 4B = 2 + 4\left(\frac{1}{25}\right) = \frac{50+4}{25} = \frac{54}{25} \Rightarrow A = \frac{18}{25}$

From (3), $C = 4 - 5A = 4 - \cancel{5}\left(\frac{18}{\cancel{25}}\right) = 4 - \frac{18}{5} = \frac{20-18}{5} = \frac{2}{5} \Rightarrow C = \frac{2}{5}$

$\therefore$ Putting $A = \frac{18}{25}, B = \frac{1}{25}, C = \frac{2}{5}$ in (i) , we get Nr.

$(2\sin x + 3\cos x + 4) = \left(\frac{18}{25}\right)(3\sin x + 4\cos x + 5) + \left(\frac{1}{25}\right)\frac{d}{dx}(3\sin x + 4\cos x + 5) + \left(\frac{2}{5}\right)$

$\therefore\ I = \int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5}dx = \frac{18}{25}\int 1dx + \frac{1}{25}\int \frac{\frac{d}{dx}(3\sin x + 4\cos x + 5)}{3\sin x + 4\cos x + 5} + \frac{2}{5}\int \frac{dx}{3\sin x + 4\cos x + 5}$

$= \frac{18}{25}.x + \frac{1}{25}\log|3\sin x + 4\cos x + 5| + \frac{2}{5}\int \frac{dx}{3\sin x + 4\cos x + 5}$ ......(ii)

Now, we find $\int \frac{dx}{3\sin x + 4\cos x + 5}$

Put $\tan\frac{x}{2} = t \Rightarrow \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2dt}{1+t^2}$

$\therefore \int \frac{dx}{3\sin x + 4\cos x + 5} = \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) + 5}\left(\frac{2dt}{1+t^2}\right) = \int \frac{1}{\frac{6t + 4(1-t^2) + 5(1+t^2)}{\cancel{1+t^2}}}\left(\frac{2dt}{\cancel{1+t^2}}\right)$

$= 2\int \frac{dt}{6t + 4 - 4t^2 + 5 + 5t^2} = 2\int \frac{dt}{t^2 + 6t + 9} = 2\int \frac{dt}{(t+3)^2} = -\frac{2}{t+3} = -\frac{2}{\tan\frac{x}{2} + 3}$

From (ii), $I = \frac{18}{25}.x + \frac{1}{25}\log|3\sin x + 4\cos x + 5| - \frac{4}{5\left(\tan\frac{x}{2} + 3\right)} + c$

23. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Sol: Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

Also, $x=0\Rightarrow\theta=0; x=1\Rightarrow\theta=\frac{\pi}{4}$ and $1+x^2=1+\tan^2\theta=\sec^2\theta$

$$\therefore \int_0^1 \frac{\log(1+x)}{1+x^2}dx=\int_0^{\pi/4}\frac{\log(1+\tan\theta)}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}d\theta$$

$$=\int_0^{\pi/4}\log(1+\tan\theta)d\theta$$

$$\therefore I=\int_0^{\pi/4}\log[1+\tan\theta]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[\frac{(1+\tan\theta)+(1-\tan\theta)}{1+\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$=\int_0^{\pi/4}[\log 2-\log(1+\tan\theta)]\,d\theta$$

$$=\log 2\int_0^{\pi/4}1\,d\theta-\int_0^{\pi/4}\log(1+\tan\theta)d\theta=\log 2\int_0^{\pi/4}1\,d\theta-I$$

$$=\log 2[\theta]_0^{\pi/4}-I$$

$$\Rightarrow I+I=(\log 2)\left(\frac{\pi}{4}\right)\Rightarrow 2I=\left(\frac{\pi}{4}\right)(\log 2)$$

$$\Rightarrow I=\left(\frac{\pi}{8}\right)(\log 2)$$

24. Solve $\sqrt{1+x^2}\sqrt{1+y^2}dx + xydy = 0$

Sol: Given D.E is $\sqrt{1+x^2}\sqrt{1+y^2}dx + xydy = 0$ ....(1)

$\Rightarrow xydy = -\sqrt{1+x^2}\sqrt{1+y^2}dx$

$$\Rightarrow \frac{ydy}{\sqrt{1+y^2}} = \frac{-\sqrt{1+x^2}}{x}dx \Rightarrow \int\frac{ydy}{\sqrt{1+y^2}} = -\int\frac{\sqrt{1+x^2}}{x^2}xdx \text{.....................(2)}$$

Put $\sqrt{1+y^2} = t \Rightarrow 1+y^2 = t^2 \Rightarrow 2ydy = 2tdt \Rightarrow ydy = tdt$

Again, put $\sqrt{1+x^2} = s \Rightarrow 1+x^2 = s^2 \Rightarrow 2xdx = 2sds \Rightarrow xdx = sds$.

Also $1 + x^2 = s^2 \Rightarrow x^2 = s^2 - 1$

From (2), $\int\frac{\cancel{t}dt}{\cancel{t}} = -\int\frac{s(sds)}{s^2-1} \Rightarrow \int dt = -\int\frac{(s^2-1+1)}{s^2-1}ds \Rightarrow t = -\int\left[1+\frac{1}{s^2-1}\right]ds$

$$\Rightarrow t = -s - \frac{1}{2}\log\left(\frac{s-1}{s+1}\right) + c \Rightarrow t + s + \frac{1}{2}\log\left(\frac{s-1}{s+1}\right) = c$$

$$\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) = c$$