Previous IPE

SOLVED PAPERS

MARCH -2024 (TS)

PREVIOUS PAPERS

IPE: MARCH-2024(TS)

Time : 3 Hours **MATHS-2B** Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **$10 \times 2 = 20$**

1. If $x^2 + y^2 - 4x + 6y + c = 0$ represents a circle with radius 6, find the value of c.
2. Find the chord of contact of (1, 1) with respect to the circle $x^2 + y^2 = 9$
3. Find *k* if the pairs of circles $x^2 + y^2 + 4x + 8 = 0$, $x^2 + y^2 - 16y + k = 0$ are orthogonal.
4. Show that the line $2x - y + 2 = 0$ is a tangent to the parabola $y^2 = 16x$. Find the point of contact.
5. If the eccentricity of a hyperbola is 5/4, then find the eccentricity of its conjugate hyperbola.
6. Evaluate $\int \frac{1}{(x+3)\sqrt{x+2}}dx$
7. Evaluate $\int e^x\left(\frac{1+x\log x}{x}\right)dx$
8. Evaluate $\int_0^a \left(\sqrt{a}-\sqrt{x}\right)^2 dx$
9. Evaluate $\int_0^1 \frac{x^2}{1+x^2}dx$
10. Find the order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^2 - e^x = 4$

SECTION-B

II. Answer any FIVE of the following SAQs: **$5 \times 4 = 20$**

11. Find the value of k if the length of the tangent from (2, 5) to $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$.
12. Find the radical centre of the circles $x^2+y^2-4x-6y+5=0$, $x^2+y^2-2x-4y-1=0$, $x^2+y^2-6x-2y=0$.
13. Find the eccentricity, length of latusrectum, centre, foci, equation of the directrices of the ellipse $9x^2+16y^2=144$
14. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentricity e, passes through one end of the minor axis, then show that $e^4 + e^2 = 1$
15. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$, which are (i) parallel and (ii) perpendicular to the line $y = x - 7$
16. Find the area enclosed by $y = e^x$, $y = x$, $x = 0$, $x = 1$.
17. Solve $(e^x + 1)ydy + (y + 1)\,dx = 0$

SECTION-C

III. Answer any FIVE of the following LAQs: **$5 \times 7 = 35$**

18. Find the values of c if the points (2, 0), (0, 1), (4, 5), (0, c) are concyclic.
19. Show that the circles $x^2 + y^2 - 6x - 9y + 13 = 0$ and $x^2 + y^2 - 2x - 16y = 0$ touch each other. Find the point of contact.
20. Find the equation of the parabola whose axis is parallel to the x–axis and passing through the points (–2, 1), (1, 2) and (–1, 3)
21. Evaluate $\int \frac{dx}{4+5\sin x}$
22. Evaluate the reduction formula for $I_n = \int \sin^n x\,dx$ and hence find $\int \sin^4 x\,dx$
23. Evaluate $\int_0^\pi \frac{x}{1+\sin x}dx$
24. Solve $(1 + y^2)dx = (\mathrm{Tan}^{-1}y - x)dy$

IPE TS MARCH-2024

SOLUTIONS

SECTION-A

1. If $x^2 + y^2 - 4x + 6y + c = 0$ represents a circle with radius 6, find the value of c.

Sol: Given circle is $x^2 + y^2 - 4x + 6y + c = 0 \Rightarrow g = -2, f = 3, c = c$

But, radius = 6

$\therefore \sqrt{g^2 + f^2 - c} = 6 \quad \Rightarrow \sqrt{(-2)^2 + 3^2 - c} = 6 \quad \Rightarrow \sqrt{13 - c} = 6$.

Squaring on both sides,

$13 - c = 36 \Rightarrow c = 13 - 36 = -23$

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2. Find the chord of contact of (1, 1) with respect to the circle $x^2 + y^2 = 9$

Sol: Given point $P(x_1, y_1) = (1, 1)$ and circle $S = x^2 + y^2 - 9 = 0$

The equation of chord of contact of P(1, 1) w.r.to S = 0 is is $S_1 = 0 \quad \Rightarrow x_1x + y_1y - r^2 = 0$

$\Rightarrow 1(x) + 1(y) - 9 = 0 \quad \Rightarrow x + y - 9 = 0$

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3. Find *k* if the pairs of circles $x^2 + y^2 + 4x + 8 = 0$, $x^2 + y^2 - 16y + k = 0$ are orthogonal.

Sol: From the given circles, we get

$g = 2, \quad f = 0, \quad c = 8$ and $\quad g' = 0, \; f' = -8, \; c' = k$

Orthogonal condition: $2gg' + 2ff' = c + c' \quad \Rightarrow 2(2)(0) + 2(0)(-8) = 8 + k \quad \Rightarrow k = -8$

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4. S.T the line $2x - y + 2 = 0$ is a tangent to $y^2 = 16x$. Find the point of contact.

Sol: Given parabola is $y^2 = 16x \;\Rightarrow 4a = 16 \;\Rightarrow a = 4$

Given line is $2x - y + 2 = 0 \quad \Rightarrow y = 2x + 2$

Comparing with y = mx + c we get $\qquad m = 2, c = 2$

Tangential conditon is c = a/m $\qquad \therefore \frac{a}{m} = \frac{4}{2} = 2 = c$. So, the line is a tangent.

The point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{4}, \frac{8}{2}\right) = (1, 4)$

5. If eccentricity of a hyperbola is 5/4, then find eccentricity of its conjugate hyperbola.

Sol: Let $e=\frac{5}{4}$ and the eccentricity of the conjugate hyperbola be e_1

Then, $\frac{1}{e^2}+\frac{1}{e_1^2}=1 \Rightarrow \frac{1}{(5/4)^2}+\frac{1}{e_1^2}=1 \Rightarrow \frac{1}{e_1^2}=1-\frac{16}{25}=\frac{9}{25} \Rightarrow e_1^2=\frac{25}{9} \Rightarrow e_1=\frac{5}{3}$

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6. Evaluate $\int \frac{1}{(x+3)\sqrt{x+2}}dx$

Sol: Put, $\sqrt{x+2}=t \quad \Rightarrow x+2=t^2 \Rightarrow x=t^2-2 \Rightarrow dx=2tdt$

Also, $x + 3 = (x + 2) + 1= t^2 + 1$

$\therefore I=\int \frac{dx}{(x+3)\sqrt{x+2}}=\int \frac{2\cancel{t}dt}{(t^2+1)\cancel{t}}=2\int \frac{dt}{t^2+1}=2\mathrm{Tan}^{-1}(t)+c \Rightarrow 2\mathrm{Tan}^{-1}(\sqrt{x+2})+c$

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7. Evaluate $\int e^x\left(\frac{1+x\log x}{x}\right)dx$

Sol: Here, $f(x)=\log x \Rightarrow f'(x)=\frac{1}{x}$

$\therefore \int e^x\left(\frac{1+x\log x}{x}\right)dx=\int e^x\left(\frac{1}{x}+\log x\right)dx=e^x\log x+c$

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8. Evaluate $\int_0^a \sqrt{a^2-x^2}dx$

Sol: **Formula:** $\int \sqrt{a^2-x^2}dx$

$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\mathrm{Sin}^{-1}\frac{x}{a}+c$

$\therefore \int_0^a \sqrt{a^2-x^2}dx=\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\mathrm{Sin}^{-1}\frac{x}{a}\right]_0^a \quad =0+\frac{a^2}{2}\mathrm{Sin}^{-1}(1)=\frac{a^2}{2}\left(\frac{\pi}{2}\right)=\frac{\pi a^2}{4}$

9. Evaluate $\int_0^1 \frac{x^2}{1+x^2}\,dx$

Sol: $\int_0^1 \frac{x^2}{1+x^2}\,dx = \int_0^1 \frac{1+x^2-1}{1+x^2}dx = \int_0^1 \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)dx$

$= \int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx = \left[x - \mathrm{Tan}^{-1}x\right]_0^1 \quad = (1-\mathrm{Tan}^{-1}1) - (0 - \mathrm{Tan}^{-1}0) = 1 - \frac{\pi}{4}$

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10. Find the order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^2 - e^x = 4$

Sol: In the given D.E, highest derivative is $\left(\frac{d^3y}{dx^3}\right)^2$

$\therefore$ For the given D.E, order = 3 and degree = exponent = 2

SECTION-B

11. Find the value of k if the length of the tangent from (2, 5) to $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$.

Sol: Length of the tangent from (2, 5) to

$S = x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{S_{11}} = \sqrt{37}$;

On squaring both sides, we get $S_{11} = 37$

$\Rightarrow (2)^2 + 5^2 - 5(2) + 4(5) + k = 37$

$\Rightarrow 4 + 25 - 10 + 20 + k = 37 \quad \Rightarrow 39 + k = 37 \Rightarrow k = -2$

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12. Find the radical centre of the circles $x^2 + y^2 - 4x - 6y + 5 = 0$, $x^2 + y^2 - 2x - 4y - 1 = 0$, $x^2+y^2-6x-2y = 0$.

Sol: The given circles are $S = x^2+y^2-4x-6y+5=0$, $S' = x^2+y^2-2x-4y-1=0$, $S'' = x^2+y^2-6x-2y = 0$

One radical axis is $S - S' = 0$

$\Rightarrow (-4x + 2x) + (-6y + 4y) + (5 + 1) = 0$

$\Rightarrow -2x - 2y + 6 = 0 \quad \Rightarrow -2(x+y-3) = 0$

$\Rightarrow x + y - 3 = 0$ ..... (1)

Another radical axis is $S-S'' = 0 \Rightarrow (-4x + 6x) + (-6y + 2y) + 5 = 0$

$\Rightarrow 2x - 4y + 5 = 0$ .... (2)

Now, (1) × 2 $\Rightarrow 2x + 2y - 6 = 0$ ....(3)

(3) – (2) $\Rightarrow 6y - 11 = 0 \Rightarrow y = 11/6$

From(1), $x = 3-y = 3 - \frac{11}{6} = \frac{18-11}{6} = \frac{7}{6}$

∴ the radical centre is (7/6, 11/6)

13. Find the eccentricity, length of latusrectum, centre, foci, equation of the directrices of the ellipse $9x^2 + 16y^2 = 144$

Sol: Equation of ellipse is $9x^2 + 16y^2 = 144$ $\Rightarrow \frac{9x^2}{144}+\frac{16y^2}{144}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$.

Here, $a^2 = 16$, $b^2 = 9$ $\Rightarrow a > b$. Hence, the ellipse is horizontal

(i) Eccentricity $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}$

(ii) Foci = $(\pm ae, 0)$ = $\left(\pm 4\left(\frac{\sqrt{7}}{4}\right), 0\right) = (\pm\sqrt{7}, 0)$

(iii) Length of latus rectum = $\frac{2b^2}{a}=\frac{2(9)}{4}=\frac{9}{2}$

(iv) Equation of directrices is $x=\pm\frac{a}{e}=\pm 4\left(\frac{4}{\sqrt{7}}\right)=\frac{\pm 16}{\sqrt{7}} \Rightarrow \sqrt{7}x=\pm 16 \Rightarrow \sqrt{7}x \pm 16=0$

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14. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentricity e, passes through one end of the minor axis, then show that $e^4 + e^2 = 1$

Sol: The equation of the normal at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

Let $L = (ae, b^2/a)$ be the one end of the latus rectum

Hence equation of the normal at L is $\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2 \Rightarrow \frac{ax}{e}-ay=a^2-b^2$.....(1)

But, (1) passes through the one end B'(0, –b) of minor axis

$$\Rightarrow \frac{a(0)}{e}-a(-b)=a^2-b^2 \Rightarrow ab=a^2-a^2(1-e^2) \Rightarrow ab=a^2e^2 \Rightarrow e^2=\frac{b}{a}$$

$$\therefore e^4=\frac{b^2}{a^2}=\frac{a^2(1-e^2)}{a^2}=1-e^2 \Rightarrow e^4+e^2=1$$

15. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are (a) Parallel (b) Perpendicular to the line $y = x - 7$

Sol: Given hyperbola is $3x^2 - 4y^2 = 12 \Rightarrow \frac{x^2}{4} - \frac{y^2}{3} = 1 \Rightarrow a^2 = 4, b^2 = 3$

Slope of the given line $y = x - 7$ is $m = 1$

$\Rightarrow$ Slope of its perpendicular is -1

Formula:

Tangent with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

(i) Parallel tangent with slope $m = 1$ is $y = 1.x \pm \sqrt{4(1)^2 - 3} = x \pm 1$

$\Rightarrow x - y \pm 1 = 0$

(ii) Perpendicular tangent with slope $m = -1$ is $y = (-1)x \pm \sqrt{4(1)^2 - 3} = -x \pm 1$

$\Rightarrow x + y \pm 1 = 0$

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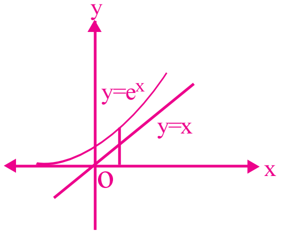
16. Find the area enclosed by $y = e^x$, $y = x$, $x = 0$, $x = 1$.

Sol: Here the upper boundary curve is $y = e^x$ and lower boundary line is $y = x$

$\therefore$ Required area $A = \int_0^1 (e^x - x)dx = \left[e^x - \frac{x^2}{2}\right]_0^1$



$= e - \frac{1}{2} - 1 = e - \frac{3}{2} = \frac{2e-3}{2}$ Sq.units

17. Solve $(e^x + 1)ydy + (y + 1)\,dx = 0$

Sol: Given D.E is $(e^x+1)ydy+(y+1)dx=0 \Rightarrow (e^x+1)ydy=-(y+1)dx=0$

$$\Rightarrow \frac{ydy}{y+1}=-\frac{dx}{e^x+1} \Rightarrow \int\frac{ydy}{y+1}=-\int\frac{dx}{e^x+1}$$

$$\Rightarrow \int\left(\frac{y+1-1}{y+1}\right)dy=-\int\frac{e^{-x}dx}{1+e^{-x}} \Rightarrow \int\left(\frac{y+1}{y+1}-\frac{1}{y+1}\right)dy=-\int\frac{e^{-x}}{1+e^{-x}}dx$$

$$\Rightarrow y-\log(y+1)=-\log(1+e^{-x})+\log c \Rightarrow y-\log(y+1)=-\log\left(1+\frac{1}{e^x}\right)+\log c$$

$$\Rightarrow y=\log(y+1)+\log\left(\frac{e^x+1}{e^x}\right)+\log c$$

$$\Rightarrow y=\log_e\left[(y+1)\frac{(e^x+1)}{e^x}c\right] \Rightarrow e^y=(y+1)\left(\frac{e^x+1}{e^x}\right)c \Rightarrow e^x.e^y=(y+1)(e^x+1)c$$

$\therefore$ The solution is $e^{x+y}=(y+1)(e^x+1)c$

SECTION-C

18. Find the values of c if the points (2, 0), (0, 1), (4, 5), (0, c) are concyclic.

Sol: Let A = (2, 0), B = (0, 1), C = (4, 5), D = (0, c)

We take $S(x_1,y_1)$ as the centre of the circle $\Rightarrow SA = SB = SC$

Now, $SA = SB \Rightarrow SA^2 = SB^2 \Rightarrow (x_1-2)^2+(y_1-0)^2=(x_1-0)^2+(y_1-1)^2$

$\Rightarrow (\cancel{x_1^2}-4x_1+4)+(\cancel{y_1^2})=(\cancel{x_1^2})+(\cancel{y_1^2}-2y_1+1)$

$\Rightarrow 4x_1-2y_1+1-4=0 \Rightarrow 4x_1-2y_1-3=0$.........(1)

Also, $SB = SC \Rightarrow SB^2 = SC^2 \Rightarrow (x_1-0)^2+(y_1-1)^2=(x_1-4)^2+(y_1-5)^2$

$\Rightarrow (\cancel{x_1^2})+(\cancel{y_1^2}-2y_1+1)=(\cancel{x_1^2}-8x_1+16)+(\cancel{y_1^2}-10y_1+25)$

$\Rightarrow 8x_1-2y_1+10y_1+1-16-25=0 \Rightarrow 8x_1+8y_1-40=0 \Rightarrow 8(x_1+y_1-5)=0 \Rightarrow x_1+y_1-5=0$......(2)

Solving (1) & (2) we get the centre $S(x_1,y_1)$

$2\times(2) \quad \Rightarrow 2x_1+2y_1-10 = 0$...........(3)

$(1)+(3) \quad \Rightarrow 6x_1-13 = 0 \Rightarrow 6x_1 = 13 \Rightarrow x_1=13/6$

$(2) \Rightarrow y_1 = 5-x_1 = 5-\frac{13}{6} = \frac{30-13}{6} = \frac{17}{6} \Rightarrow y_1 = \frac{17}{6}$

$\therefore$ Centre of the circle is $S(x_1,y_1) = \left(\frac{13}{6},\frac{17}{6}\right)$

Also, we have A = (2,0)

Hence, radius $r = SA \Rightarrow r^2 = SA^2$

$\therefore r^2 = SA^2 = \left(2-\frac{13}{6}\right)^2+\left(0-\frac{17}{6}\right)^2 = \left(\frac{12-13}{6}\right)^2+\left(\frac{17}{6}\right)^2 = \left(\frac{1}{36}\right)+\left(\frac{289}{36}\right) = \frac{290}{36}$

$\therefore$ Circle with Centre $\left(\frac{13}{6},\frac{17}{6}\right)$ and $r^2 = \frac{290}{36}$ is $\left(x-\frac{13}{6}\right)^2+\left(y-\frac{17}{6}\right)^2 = \frac{290}{36}$

Put, D(0, c) in the above equation $\Rightarrow \left(0-\frac{13}{6}\right)^2+\left(c-\frac{17}{6}\right)^2 = \frac{290}{36} \Rightarrow \left(c-\frac{17}{6}\right)^2 = \frac{290}{36}-\frac{169}{36} = \frac{121}{36}$

$\Rightarrow \left(\frac{6c-17}{6}\right)^2 = \frac{121}{36} \Rightarrow \frac{(6c-17)^2}{\cancel{36}} = \frac{11^2}{\cancel{36}} \Rightarrow 6c-17 = \pm 11$

$\Rightarrow 6c = \pm 11+17 \Rightarrow 6c = 28 \Rightarrow c = \frac{\cancel{28}}{\cancel{6}} = \frac{14}{3}$ (or) $\cancel{6}c = \cancel{6} \Rightarrow c = 1 \quad \therefore c = 14/3$ (or) 1

19. Show that the circles $x^2 + y^2 - 6x - 9y + 13 = 0$, $x^2 + y^2 - 2x - 16y = 0$ touch each other. Find the point of contact and the equation of the common tangent at that point.

Sol: First circle is $S \equiv x^2 + y^2 - 6x - 9y + 13 = 0$

Centre, $C_1 = (3, 9/2)$,

Radius, $r_1 = \sqrt{3^2 + \left(\frac{9}{2}\right)^2 - 13} = \sqrt{9 + \frac{81}{4} - 13} = \sqrt{\frac{36+81-52}{4}} = \sqrt{\frac{65}{4}} = \frac{\sqrt{65}}{2}$

Second circle is $S' \equiv x^2 + y^2 - 2x - 16y = 0$

Centre, $C_2 = (1, 8)$,

Radius, $r_2 = \sqrt{1^2 + 8^2 - 0} = \sqrt{1+64} = \sqrt{65}$

$C_1C_2 = \sqrt{(3-1)^2 + \left(\frac{9}{2} - 8\right)^2} = \sqrt{2^2 + \left(\frac{9-16}{2}\right)^2} = \sqrt{4 + \frac{49}{4}} = \sqrt{\frac{16+49}{4}} = \sqrt{\frac{65}{4}} = \frac{\sqrt{65}}{2}$

Also, $r_2 - r_1 = \sqrt{65} - \frac{\sqrt{65}}{2} = \frac{\sqrt{65}}{2} = C_1C_2$

$\therefore$ The circles touch each other **internally**.

Now, $r_1 : r_2 = \frac{\sqrt{65}}{2} : \sqrt{65} = \frac{1}{2} : 1 = 1 : 2$

So, the point of contact P divides the join of $C_1(3, 9/2)$, $C_2(1, 8)$ externally in the ratio 1 : 2

$$\therefore P = \left(\frac{1(1) - 2(3)}{1-2}, \frac{1(8) - 2(\frac{9}{2})}{1-2}\right) = (5, 1)$$

The equation of the common tangent to the circles S = 0 and S' = 0 at the point of contact is given by $S - S' = 0$

$\Rightarrow (x^2 + y^2 - 6x - 9y + 13) - (x^2 + y^2 - 2x - 16y) = 0$

$\Rightarrow -4x + 7y + 13 = 0 \quad \Rightarrow 4x - 7y - 13 = 0$

20. Find the equation of the parabola whose axis is parallel to the x–axis and passing through the points (–2, 1), (1, 2) and (–1, 3)

Sol: The equation of the parabola whose axis is parallel to the x-axis is $x = ly^2 + my + n$

The parabola passes through (–2, 1) $\Rightarrow -2 = l(1^2) + m(1) + n \Rightarrow l + m + n = -2$ ....(1)

The parabola passes through (1, 2) $\Rightarrow 1 = l(2)^2 + m(2) + n \Rightarrow 4l + 2m + n = 1$ ....(2)

The parabola passes through (–1, 3) $\Rightarrow -1= l(3)^2+m(3)+n \Rightarrow 9l+3m+n=-1$ ....(3)

(2) –(1) $\Rightarrow 4l + 2m + n - l - m - n = 1 + 2 \Rightarrow 3l+m = 3$ .....(4)

(3) –(2) $\Rightarrow 9l + 3m + n - 4l - 2m - n = -1 - 1 \Rightarrow 5l + m = -2$ .....(5)

(5) –(4) $\Rightarrow 5l + m - 3l - m = -2 - 3 \Rightarrow 2l = -5 \Rightarrow l= -5/2$

From (4), $m = 3 - 3l = 3 - 3(-5/2) = 21/2 \Rightarrow m = 21/2$

From (1), $l + m + n = -2 \Rightarrow -5/2 + 21/2 + n = -2 \Rightarrow n = -10$

Substituting the values $l = -5/2$, $m = 21/2$ and $n= -10$ in $x = ly^2 + my + n$ we get the equation of

the required parabola as $x = -\frac{-5}{2}y^2 + \frac{21}{2}y - 10 \Rightarrow 5y^2 + 2x - 21y + 20 = 0$

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21. Evaluate $\int \frac{dx}{4 + 5\sin x}$

Sol: Put, $\tan\frac{x}{2} = t \Rightarrow \sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$

$$\therefore I = \int \frac{dx}{4+5\sin x} = \int \frac{\frac{2dt}{1+t^2}}{4+5\frac{(2t)}{1+t^2}} = 2\int \frac{dt}{4(1+t^2)+10t} = \frac{2}{2}\int \frac{dt}{2t^2+5t+2}$$

$$= \int \frac{dt}{2\left(t^2+\frac{5}{2}t+1\right)} = \frac{1}{2}\int \frac{dt}{\left(t^2+2(t)\frac{5}{4}+\left(\frac{5}{4}\right)^2 - \frac{25}{16}+1\right)}$$

$$= \frac{1}{2}\int \frac{dt}{\left(t+\frac{5}{4}\right)^2 - \frac{9}{16}} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

$$\left[\because \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2}\frac{1}{2\left(\frac{3}{4}\right)}\log\left[\frac{t+\frac{5}{4}-\frac{3}{4}}{t+\frac{5}{4}+\frac{3}{4}}\right] = \frac{1}{3}\log\left[\frac{4t+2}{4t+8}\right] = \frac{1}{3}\log\left[\frac{2t+1}{2t+4}\right] + c = \frac{1}{3}\log\left[\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right] + c$$

22. Evaluate the reduction formula for $I_n=\int \sin^n x dx$ and hence find $\int \sin^4 x dx$

Sol: Given $I_n = \int \sin^n x dx = \int \sin^{n-1} x(\sin x)dx.$

We take First function $u = \sin^{n-1} x$ and

Second function $v = \sin x \quad \Rightarrow \int v = -\cos x$

From By parts rule, we have

$$I_n = \sin^{n-1} x\,(-\cos x) - \int (n-1)\sin^{n-2} x \cos x\,(-\cos x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x dx$$

$$= -\sin^{n-1} x\cos x + (n\text{-}1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1)[\int \sin^{n-2} xdx - \int \sin^n xdx]$$

$$= -\sin^{n-1} x\cos x + (n-1)[I_{n-2} - I_n]$$

$$I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$= -\sin^{n-1} x\cos x + (n-1)I_{n-2} - nI_n + I_n$$

$$\Rightarrow nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} + \cancel{I_n} - \cancel{I_n}$$

$$\Rightarrow I_n = \frac{-\sin^{n-1} x\cos x}{n} + \left(\frac{n-1}{n}\right)I_{n-2} \quad ....(1)$$

Put, $n = 4, 2, 0$ successively in (1), we get

$$I_4 = -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}I_2$$

$$= -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\left[-\frac{\sin x\cos x}{2} + \frac{1}{2}I_0\right]$$

$$= \frac{-\sin^3 x\cos x}{4} - \frac{3}{8}\sin x\cos x + \frac{3}{8}I_0$$

$$= -\frac{\sin^3 x\cos x}{4} - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + c \qquad [\because I_0 = x]$$

23. Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}dx$

Sol: We know $\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad \therefore I = \int_0^{\pi} \frac{x}{1+\sin x}dx = \int_0^{\pi} \frac{(\pi - x)dx}{1+\sin(\pi - x)}$

$$I = \int_0^{\pi} \frac{\pi\, dx}{1+\sin x} - \int_0^{\pi} \frac{x\, dx}{1+\sin x} = \int_0^{\pi} \frac{\pi\, dx}{1+\sin x} - I \Rightarrow I + I = 2I = \pi\int_0^{\pi} \frac{dx}{1+\sin x}$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{1+\sin x} = \frac{\pi}{2}\int_0^{\pi} \frac{(1-\sin x)}{1-\sin^2 x}dx = \frac{\pi}{2}\int_0^{\pi}\left(\frac{1-\sin x}{\cos^2 x}\right)dx$$

$$= \frac{\pi}{2}\left(\int_0^{\pi} \frac{1}{\cos^2 x}dx - \int_0^{\pi} \frac{\sin x}{\cos x}.\frac{1}{\cos x}dx\right) = \frac{\pi}{2}\int_0^{\pi} \sec^2 x dx - \int_0^{\pi} \sec x.\tan x\, dx$$

$$= \frac{\pi}{2}\left([\tan x]_0^{\pi} - [\sec x]_0^{\pi}\right) = \frac{\pi}{2}[(0-0)-(-1-1)] = \frac{\pi}{2}.2 = \pi$$

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24. Solve $(1+y^2)dx = (\mathrm{Tan}^{-1}y - x)dy$

Sol: Given D.E is $(1+y^2)dx = (\mathrm{Tan}^{-1}y - x)dy$

$\Rightarrow \frac{dx}{dy} = \frac{\mathrm{Tan}^{-1}y - x}{(1+y^2)} = \frac{\mathrm{Tan}^{-1}y}{1+y^2} - \frac{x}{1+y^2} \Rightarrow \frac{dx}{dy} + x\left(\frac{1}{1+y^2}\right) = \frac{\mathrm{Tan}^{-1}y}{1+y^2}$. This is a linear D.E in x

It is in the form $\frac{dx}{dy} + xP(y) = Q(y)$ where $P(y) = \frac{1}{1+y^2}$ and $Q(y) = \frac{\mathrm{Tan}^{-1}y}{1+y^2}$

Now, $\mathrm{I.F} = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\mathrm{Tan}^{-1}y}$

$\therefore$ The solution is $x.(\mathrm{I.F}) = \int(\mathrm{I.F})Q(y)dy \quad \Rightarrow x\left(e^{\mathrm{Tan}^{-1}y}\right) = \int e^{\mathrm{Tan}^{-1}y}\left(\frac{\mathrm{Tan}^{-1}y}{1+y^2}\right)dy$ ......(1)

Put $\mathrm{Tan}^{-1}y = t \Rightarrow \frac{dy}{1+y^2} = dt$

From (1), $xe^t = \int (e^t)t\,dt = e^t(t-1) + c = (\mathrm{Tan}^{-1}y - 1)e^{\mathrm{Tan}^{-1}y} + c$

$$\therefore xe^{\mathrm{Tan}^{-1}y} = (\mathrm{Tan}^{-1}y - 1)e^{\mathrm{Tan}^{-1}y} + c \quad \Rightarrow x = \mathrm{Tan}^{-1}y - 1 + ce^{-\mathrm{Tan}^{-1}y}$$