Previous IPE

SOLVED PAPERS

MARCH -2024 (TS)

PREVIOUS PAPERS

IPE: MARCH-2024(TS)

Time : 3 Hours **MATHS-1B** Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **$10 \times 2 = 20$**

1. Show that the points $(-5, 1), (5, 5), (10, 7)$ are collinear.
2. Find the distance between the parallel lines $5x - 3y - 4 = 0, 10x - 6y - 9 = 0$.
3. Find 4^{th} vertex of parallelogram whose consecutive vertices are $(2, 4, -1), (3, 6, -1)$ and $(4, 5, 1)$
4. Write the equation of the plane $4x - 4y + 2z + 5 = 0$ in the intercept form.
5. Compute $\underset{x \to 0}{Lt} \dfrac{e^x - 1}{\sqrt{1+x} - 1}$ 6. Compute $\underset{x \to \frac{\pi}{2}}{Lt} \dfrac{\cos x}{\left(x - \frac{\pi}{2}\right)}$
7. If $f(x) = 2x^2 + 3x - 5$ then prove that $f'(0) + 3f'(-1) = 0$.
8. Find the derivative of $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ w.r.to x.
9. Find the approximate value of $\sqrt[3]{65}$
10. Verify Rolle's theorem for the function $y = f(x) = x^2 + 4$ on $[-3, 3]$

SECTION-B

II. Answer any FIVE of the following SAQs: **$5 \times 4 = 20$**

11. If the distance from 'P' to the points (2,3) and (2,–3) are in the ratio 2:3, then find the equation of locus of P.
12. When the axes are rotated through an angle α, find the transformed equation of $x\cos\alpha + y\sin\alpha = p$
13. Find the value of k if the angle between the straight lines $4x - y + 7 = 0$, $kx - 5y - 9 = 0$ is 45°
14. Evaluate $\underset{x \to 0}{Lt} \dfrac{\cos ax - \cos bx}{x^2}$
15. Find the derivative of $\cot x$ from the first principle.
16. Find the length of subtangent and subnormal at a point on the curve $y = b\sin(x/a)$.
17. The volume of a cube is increasing at a rate of 8 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 12 cm?

SECTION-C

III. Answer any FIVE of the following LAQs: **$5 \times 7 = 35$**

18. Find the equation of the straight line parallel to the line $3x + 4y = 7$ and passing through the point of intersection of the lines $x - 2y - 3 = 0$, $x + 3y - 6 = 0$.
19. Prove that the line $lx + my + n = 0$ and the pair of lines $(lx + my)^2 - 3(mx - ly)^2 = 0$ form an equilateral triangle and its area is $\dfrac{n^2}{\sqrt{3}(l^2 + m^2)}$
20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = k$ are mutually perpendicular.
21. Find the angle between whose Dc's satisfy the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.
22. If $y = x\sqrt{a^2 + x^2} + a^2\log\left(x + \sqrt{a^2 + x^2}\right)$, then show that $\dfrac{dy}{dx} = 2\sqrt{a^2 + x^2}$
23. If the tangent at a point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the coordinate axes in A,B then show that the length AB is a constant.
24. From a rectangular sheet of dimensions $30\text{cm} \times 80\text{cm}$, four equal squares of sides x cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. What is the value of x, so that volume of the box is the greatest?

IPE TS MARCH-2024

SOLUTIONS

SECTION-A

1. Show that the points (–5, 1), (5, 5), (10, 7) are collinear.

Sol: Let A = (–5, 1), B = (5, 5), C = (10, 7)

Slope of the line joining A(–5, 1), B(5, 5) is $m = \frac{5-1}{5+5} = \frac{4}{10} = \frac{2}{5}$

Equation of the line passing through A(–5, 1) with slope 2/5 is

$y-1 = \frac{2}{5}(x+5) \Rightarrow 5(y-1) = 2(x+5) \Rightarrow 2x-5y+15=0$ ......(1)

Putting the coordinates of C(10, 7) in (1), we have 2(10) –5(7)+15 = 20–35+15=35–35 = 0

Thus C satisfies the equation of AB. Hence A, B, C are collinear.

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2. Find the distance between the parallel lines 5x – 3y – 4 = 0, 10x – 6y – 9 = 0.

Sol:
- We write the first line 5x–3y–4 = 0 as 10x–6y–8 = 0 ....(1)
- Second line is 10x – 6y – 9 = 0 ....(2)
- $\therefore$ The distance between (1) & (2) is

★ $\frac{|c_1 - c_2|}{\sqrt{a^2+b^2}} = \frac{|-8+9|}{\sqrt{10^2+6^2}} = \frac{|1|}{\sqrt{100+36}} = \frac{1}{\sqrt{136}}$

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3. Find the fourth vertex of the parallelogram whose consecutive vertices are (2, 4, –1), (3, 6, –1) and (4, 5, 1)

Sol:
- We take A = (2, 4, –1), B = (3, 6, –1), C = (4, 5, 1) and fourth vertex D = (a, b, c)
- In the parallelogram ABCD,

★ mid point of AC = mid point of BD

★ $\Rightarrow \left(\frac{2+4}{2}, \frac{4+5}{2}, \frac{-1+1}{2}\right) = \left(\frac{3+a}{2}, \frac{6+b}{2}, \frac{-1+c}{2}\right)$

- $\Rightarrow \frac{3+a}{\cancel{2}} = \frac{6}{\cancel{2}} \Rightarrow a+3=6 \Rightarrow a=6-3=3;$
- $\frac{6+b}{\cancel{2}} = \frac{9}{\cancel{2}} \Rightarrow b+6=9 \Rightarrow b=9-6=3;$
- $\frac{-1+c}{2} = 0 \Rightarrow c-1=0 \Rightarrow c=1.$ Hence, fourth vertex D = (3, 3, 1)

4. Write the equation of the plane $4x - 4y + 2z + 5 = 0$ in the intercept form.

Sol: The given equation of the plane is $4x - 4y + 2z + 5 = 0 \Rightarrow 4x - 4y + 2z = -5$

$$\Rightarrow \frac{4x}{-5} + \frac{-4y}{-5} + \frac{2z}{-5} = 1 \Rightarrow \frac{x}{\left(\frac{-5}{4}\right)} + \frac{y}{\left(\frac{5}{4}\right)} + \frac{z}{\left(\frac{-5}{2}\right)} = 1$$ which is in the intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

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5. Evaluate $\underset{x\to 0}{Lt} \frac{e^x - 1}{\sqrt{1+x} - 1}$

Sol: $$\underset{x\to 0}{Lt} \frac{e^x - 1}{\sqrt{1+x} - 1} = \underset{x\to 0}{Lt}\left(\frac{e^x - 1}{x}\right)\left(\frac{x}{\sqrt{1+x} - 1}\right)$$

$$= \underset{x\to 0}{Lt} \frac{e^x - 1}{x} . \underset{x\to 0}{Lt} \frac{x}{\sqrt{1+x} - 1} = 1. \underset{x\to 0}{Lt} \frac{x}{\sqrt{1+x} - 1}$$

$$= \underset{x\to 0}{Lt} \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} = \underset{x\to 0}{Lt} \frac{x(\sqrt{1+x}+1)}{1+x-1} = \underset{x\to 0}{Lt} \frac{x(\sqrt{1+x}+1)}{x}$$

$$= \underset{x\to 0}{Lt} (\sqrt{1+x}+1) = \sqrt{1+0}+1 = 1+1 = 2$$

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6. Evaluate $\underset{x\to \pi/2}{Lt} \frac{\cos x}{\left(x - \frac{\pi}{2}\right)}$

Sol: Put $x - \frac{\pi}{2} = y$ then $x = \frac{\pi}{2} + y$ and $x \to \frac{\pi}{2} \Rightarrow y \to 0$

$$\therefore \underset{x\to \pi/2}{Lt} \frac{\cos x}{x - \frac{\pi}{2}} = \underset{y\to 0}{Lt} \frac{\cos\left(\frac{\pi}{2} + y\right)}{y} = \underset{y\to 0}{Lt} \frac{-\sin y}{y} = -1$$

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7. If $f(x)=2x^2+3x-5$ then prove that $f'(0)+3f'(-1)=0$.

Sol: Given $f(x)=2x^2+3x-5 \Rightarrow f'(x) = 4x+3$

Then $f'(0) = 0+3=3$ and $f'(-1) = -4+3= -1$

$\therefore f'(0)+3f'(-1) = 3+3(-1)=3-3=0$

8. Find the derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol: We take $x=\tan\theta$, then $\theta=\tan^{-1}x$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=\sin^{-1}(\sin 2\theta)=2\theta=2(\tan^{-1}x) \quad [\because x=\tan\theta \Rightarrow \theta=\tan^{-1}x]$$

$$\therefore \frac{d}{dx}(2\tan^{-1}x)=2\frac{d}{dx}\tan^{-1}x=2\left(\frac{1}{1+x^2}\right)=\frac{2}{1+x^2}$$

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9. Find the approximate value of $\sqrt[3]{65}$

Sol: Given $\sqrt[3]{65}=\sqrt[3]{64+1}$

$\therefore$ known value x = 64 and Δx=1

$$f(x)=\sqrt[3]{x}=x^{\frac{1}{3}} \Rightarrow f'(x)=\frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3x^{2/3}}$$

Formula: $f(x+\Delta x)=[f(x)+f'(x)\Delta x]_{\text{at } x}$

$$\therefore \sqrt[3]{65}\cong\sqrt[3]{x}+\frac{1}{3x^{2/3}}\Delta x=\sqrt[3]{64}+\frac{1}{3(64)^{2/3}}(1) \quad =4+\frac{1}{3(4^3)^{\frac{2}{3}}}(1)=4+\frac{1}{3(4^2)}=4+\frac{1}{3(16)}$$

$$=4+\frac{1}{48}=\frac{192+1}{48}=\frac{193}{48}=4.0208$$

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10. Verify Rolle's theorem for the function $y=f(x)=x^2+4$ on [–3,3]

Sol :

- Given $f(x)=x^2+4 \Rightarrow f'(x)=2x$
- f(x) is (i) continuous on [–3,3]
- (ii) differentiable in (–3,3)

★ (iii) $f(-3)=(-3)^2+4=9+4=13$;

$f(3)=3^2+4=9+4=13$

- $\Rightarrow f(-3)=f(3)$

★ So, from Rolle's theorem, $f'(c)=0$

$\Rightarrow 2c=0 \quad \Rightarrow c=0$

★ $\therefore$ $c=0\in(-3,3)$.

- Hence, Rolle's theorem is verified.

SECTION-B

11. If the distance from 'P' to the points (2, 3) and (2, –3) are in the ratio 2 : 3, then find the equation of locus of P.

Sol:

- We take A = (2, 3), B = (2, –3) and P = (x, y) is a point on the locus.
- ★ **Given condition:** $\frac{PA}{PB}=\frac{2}{3} \Rightarrow 3PA = 2PB \Rightarrow 9PA^2 = 4PB^2$
- ★ $\Rightarrow 9[(x-2)^2+(y-3)^2] = 4[(x-2)^2+(y+3)^2]$
- $\Rightarrow 9[(x^2+4-4x)+(y^2+9-6y)] = 4[(x^2+4-4x)+(y^2+9+6y)]$
- $\Rightarrow 9x^2 \cancel{+36} - 36x + 9y^2 + 81 - 54y = 4x^2 + 16 - 16x + 4y^2 \cancel{+36} + 24y$
- $\Rightarrow 9x^2-4x^2+9y^2-4y^2-36x+16x-54y-24y+81-16 = 0 \Rightarrow 5x^2+5y^2-20x-78y+65 = 0$
- Hence, locus of P is $5x^2+5y^2-20x-78y+65 = 0$.

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12. When the axes are rotated through an angle α, find the transformed equation of xcosα + ysinα = p

Sol:

- Given original equation is $x\cos\alpha + y\sin\alpha = p$........(1)
- Angle of rotation $\theta = \alpha$, then
- ★ $x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos\alpha - Y\sin\alpha$

  $y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos\alpha + X\sin\alpha$
- From (1), transformed equation is
- $(X\cos\alpha - Y\sin\alpha)\cos\alpha + (Y\cos\alpha + X\sin\alpha)\sin\alpha = p$
- $\Rightarrow X\cos^2\alpha \cancel{-Y\sin\alpha\cos\alpha} \cancel{+Y\cos\alpha\sin\alpha} + X\sin^2\alpha = p$
- $\Rightarrow X(\cos^2\alpha + \sin^2\alpha) = p \Rightarrow X(1) = p \Rightarrow X = p$

13. Find the value of k if the angle between the straight lines

$4x - y + 7 = 0$, $kx - 5y - 9 = 0$ is 45°

Sol:
- Given line is 4x–y+7=0. It's slope $m_1 = \frac{-a}{b} = \frac{-4}{-1} = 4$
- Another line is kx–5y–9=0. It's slope is $m_2 = \frac{-a}{b} = \frac{-k}{-5} = \frac{k}{5}$

★ Angle between the lines is 45°, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \Rightarrow \tan 45^\circ = \left|\frac{4-(k/5)}{1+4(k/5)}\right|$

★ $\Rightarrow 1 = \left|\frac{20-k}{5+4k}\right| \Rightarrow |5+4k| = |20-k|$

$\Rightarrow 5+4k = \pm(20-k) \Rightarrow 5+4k = 20-k \Rightarrow 5k = 15 \Rightarrow k = 3$

- (or) $5+4k = -(20-k) = k-20 \Rightarrow 3k = -25 \Rightarrow k = -25/3$

$\therefore$ k = 3 or –25/3

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14. Evaluate $\lim\limits_{x\to 0} \frac{\cos ax - \cos bx}{x^2}$

Sol: $\lim\limits_{x\to 0} \frac{\cos ax - \cos bx}{x^2}$

$= \lim\limits_{x\to 0} \frac{2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{bx-ax}{2}\right)}{x^2}$ $\left(\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right)$

$= 2\lim\limits_{x\to 0}\left(\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\frac{\sin\left(\frac{b-a}{2}\right)x}{x}\right) = 2\left(\lim\limits_{x\to 0}\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\lim\limits_{x\to 0}\frac{\sin\left(\frac{b-a}{2}\right)x}{x}\right)$

$= 2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) = \frac{b^2 - a^2}{2}$ $\left(\because \lim\limits_{x\to 0}\frac{\sin kx}{x} = k\right)$

15. Find the derivative of cotx from the first principle.

Sol: We take f(x) = cotx, then

$f(x+h) = \cot(x+h)$

From the first principle, $f'(x) = \underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h}$

$$= \underset{h\to 0}{Lt}\frac{\cot(x+h)-\cot(x)}{h} = \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{\cos(x+h).\sin x-\sin(x+h).\cos x}{\sin(x+h).\sin x}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{-\sin((x+h)-x)}{\sin(x+h)\sin x}\right] \quad [\because \cos A\sin B-\sin A\cos B=-\sin(A-B)]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{-\sinh}{\sin(x+h)\sin x}\right]$$

$$= \underset{h\to 0}{Lt}\left(\frac{-\sinh}{h}\right).\underset{h\to 0}{Lt}\frac{1}{\sin(x+h).\sin x} = -1\left(\frac{1}{\sin x.\sin x}\right) = -\operatorname{cosec}^2 x$$

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16. Find the length of subtangent and subnormal at a point on the curve $y = b\sin\left(\frac{x}{a}\right)$.

Sol:
- Let P(x,y) be point on the curve $y = b\sin\left(\frac{x}{a}\right)$

★ On diff. w.r.to x, we get $\dfrac{dy}{dx} = b\left(\cos\dfrac{x}{a}\right)\dfrac{1}{a}$ $\quad\therefore$ Slope $m = \left(\dfrac{dy}{dx}\right)_P = \dfrac{b}{a}\cos\left(\dfrac{x}{a}\right)$

★ (i) Length of sub-tangent $= \left|\dfrac{y}{m}\right| = \left|\dfrac{y}{\frac{b}{a}\cos\frac{x}{a}}\right| = \left|\dfrac{b\sin\left(\frac{x}{a}\right)}{\frac{b}{a}\cos\left(\frac{x}{a}\right)}\right| = \left|\dfrac{b\sin\left(\frac{x}{a}\right)}{\frac{b}{a}\cos\left(\frac{x}{a}\right)}\right| = \left|a\tan\dfrac{x}{a}\right|$

★ (ii) Length of sub-normal $|ym| = \left|b\sin\left(\dfrac{x}{a}\right)\dfrac{b}{a}\cos\left(\dfrac{x}{a}\right)\right|$

★ $= \left|\dfrac{b^2}{a}\dfrac{1}{2}2\sin\left(\dfrac{x}{a}\right)\cos\left(\dfrac{x}{a}\right)\right| = \left|\dfrac{b^2}{2a}\sin\left(\dfrac{2x}{a}\right)\right|$ $\quad[\because \sin 2\theta = 2\sin\theta\cos\theta]$

17. The volume of a cube is increasing at a rate of 8 cubic centimeters per second. How fast is the surface area increasing when length of edge is 12cm?

Sol : For the cube, we take length of the edge = x , Volume = V and Surface area = S

Given $\frac{dV}{dt} = 8$ and x = 12 cm

Volume of the cube $V = x^3$

On diff. w.r.t 't', we get $\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow 8 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$

Surface area $S = 6x^2$

On diff. w.r.t 't', we get $\frac{dS}{dt} = 12x \frac{dx}{dt}$

$$= \cancel{12}\,\cancel{x}\left(\frac{8}{\cancel{3}\,\cancel{x^2}}\right) = \frac{32}{x} = \frac{\cancel{32}}{\cancel{12}} = \frac{8}{3}\ cm^2 / sec$$

SECTION-C

18. Find the equation of the straight line parallel to 3x + 4y = 7 and passing through the point of intersection of the lines x – 2y – 3 = 0 and x + 3y – 6 = 0.

Sol: First we solve the given two equations $x-2y-3=0,\ x+3y-6=0$.

$$\frac{x}{(-2)(-6)-3(-3)}=\frac{y}{(-3)(1)-1(-6)}=\frac{1}{1(3)-1(-2)}$$

$$\Rightarrow \frac{x}{12+9}=\frac{y}{-3+6}=\frac{1}{3+2}\Rightarrow \frac{x}{21}=\frac{y}{3}=\frac{1}{5}\Rightarrow x=\frac{21}{5},y=\frac{3}{5}\ \therefore \text{Point of Intersection}=\left(\frac{21}{5},\frac{3}{5}\right)$$

Now, the equation of any line parallel to $3x + 4y = 7$ is of the form $3x + 4y = k$.

If this line passes through $\left(\frac{21}{5},\frac{3}{5}\right)$ then $3\left(\frac{21}{5}\right)+4\left(\frac{3}{5}\right)=k$

$$\Rightarrow \frac{63}{5}+\frac{12}{5}=k\Rightarrow \frac{75}{5}=k\Rightarrow k=15$$

$\therefore$ The equation of the required line is $3x + 4y = 15 \Rightarrow 3x + 4y - 15 = 0$

19. Prove that the line $lx + my + n = 0$ and the pair of lines $(lx+my)^2 - 3(mx-ly)^2 = 0$ form an equilateral triangle and its area is $\dfrac{n^2}{\sqrt{3}(l^2+m^2)}$

Sol : The equation of the given pair of lines is $(lx+my)^2 - 3(mx-ly)^2 = 0$

$$\Rightarrow [lx + my + \sqrt{3}(mx - ly)][lx + my - \sqrt{3}(mx - ly)] = 0$$

$$\Rightarrow [(l + \sqrt{3}m)x + (-\sqrt{3}l + m)y][(l - \sqrt{3}m)x + (\sqrt{3}l + m)y] = 0$$

∴ the given pair of lines represent the two lines

$(l + \sqrt{3}m)x + (-\sqrt{3}l + m)y = 0$ .....(1); $(l - \sqrt{3}m)x + (\sqrt{3}l + m)y = 0$ ....(2)

Also the other given line is $lx+my+n=0$ ......(3)

Let A be the angle between (1) and (3), then $\cos A = \dfrac{l(l+\sqrt{3}m) + m(-\sqrt{3}l + m)}{\sqrt{\left((l+\sqrt{3}m)^2 + (-\sqrt{3}l+m)^2\right)(l^2+m^2)}}$

$$= \frac{l^2 + \sqrt{3}lm - \sqrt{3}lm + m^2}{\sqrt{\left(l^2 + 3m^2 + 2\sqrt{3}lm + 3l^2 + m^2 - 2\sqrt{3}lm\right)(l^2+m^2)}} = \frac{l^2+m^2}{\sqrt{(4l^2+4m^2)(l^2+m^2)}} = \frac{l^2+m^2}{2(l^2+m^2)} = \frac{1}{2} \Rightarrow A = 60^\circ$$

Let B be the angle between (2) and (3), then $\cos B = \dfrac{l(l-\sqrt{3}m) + m(\sqrt{3}l + m)}{\sqrt{((l-\sqrt{3}m)^2 + (\sqrt{3}l+m)^2)(l^2+m^2)}}$

$$= \frac{l^2 - \sqrt{3}lm + \sqrt{3}lm + m^2}{\sqrt{\left(l^2 + 3m^2 - 2\sqrt{3}lm + 3l^2 + m^2 + 2\sqrt{3}lm\right)(l^2+m^2)}} = \frac{l^2+m^2}{\sqrt{\left(4l^2+4m^2\right)(l^2+m^2)}} = \sqrt{\frac{l^2+m^2}{4(l^2+m^2)}} = \frac{1}{2} \Rightarrow B = 60^\circ$$

(1), (2), (3) intersect each other in pairs ⇒ a triangle will be formed by the intersection of these lines

∴ the angle between (1), (2) is 180º–(60º + 60º) = 60º

∴ the given line forms an equilateral triangle.

Let the length of the altitude of the triangle be p

⇒ the perpendicular distance from the origin O(0, 0) to the line $lx+my+n=0$ is $p = \dfrac{|n|}{\sqrt{l^2+m^2}}$

The area of equilateral triangle $= \dfrac{p^2}{\sqrt{3}} = \dfrac{n^2}{\sqrt{3}(l^2+m^2)}$ sq.units

20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line $x+2y=k$ are mutually perpendicular.

Sol:

- The given line is $x+2y=k \Rightarrow \frac{x+2y}{k}=1$ ...(1)
- Given curve is $2x^2-2xy+3y^2+2x-y-1=0$............(2)
- Homogenising (1)&(2), we get

★ $2x^2-2xy+3y^2+2x(1)-y(1)-(1)^2=0$

★ $\Rightarrow 2x^2-2xy+3y^2+2x\left(\frac{x+2y}{k}\right)-y\left(\frac{x+2y}{k}\right)-\frac{(x+2y)^2}{k^2}=0$

★ $\Rightarrow \frac{k^2(2x^2-2xy+3y^2)+k(2x^2+4xy)-k(xy+2y^2)-(x^2+4y^2+4xy)}{k^2}=0$

★ $\Rightarrow k^2(2x^2-2xy+3y^2)+k(2x^2+4xy)-k(xy+2y^2)-(x^2+4y^2+4xy)=0$

★ $\Rightarrow x^2(2k^2+2k-1)+y^2(3k^2-2k-4)+xy(-2k^2+3k-4)=0$

- If this pair or lines are perpendicular then

★ Coeff. x^2 + Coeff. y^2 =0

- $\Rightarrow (2k^2 \cancel{+2k}-1)+(3k^2 \cancel{-2k}-4)=0 \Rightarrow 5k^2-5=0$
- $\Rightarrow \cancel{5}(k^2-1)=0 \Rightarrow k^2-1=0 \Rightarrow k^2=1 \Rightarrow k=\pm 1$

Hence, value of $k=\pm 1$

21. Find the angle between the lines whose dc's are related by $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.

Sol: Given $3l + m + 5n = 0 \Rightarrow m = -3l - 5n$ .....(1) , $6mn - 2nl + 5lm = 0$ .....(2)

Solving (1) & (2) we get $6n(-3l-5n) - 2nl + 5l(-3l-5n) = 0$

$\Rightarrow -18ln - 30n^2 - 2ln - 15l^2 - 25ln = 0$

$\Rightarrow -15l^2 - 45ln - 30n^2 = 0$

$\Rightarrow -15(l^2 + 3ln + 2n^2) = 0 \Rightarrow l^2 + 3ln + 2n^2 = 0$

$\Rightarrow l^2 + ln + 2ln + 2n^2 = 0 \Rightarrow l(l+n) + 2n(l+n) = 0$

$\Rightarrow (l+n)(l+2n) = 0 \Rightarrow l = -n$ or $l = -2n$

Case (i): Put $l = -n$ in (1), then

$m = -3(-n) - 5n = 3n - 5n = -2n$

$\therefore m = -2n$

Now, $l : m : n = -\cancel{n} : -2\cancel{n} : \cancel{n} = -1 : -2 : 1 = 1 : 2 : -1$

So, d.r's of $L_1 = (a_1, b_1, c_1) = (1, 2, -1)$.....(3)

Case (ii): Put $l = -2n$ in (1), then

$m = -3(-2n) - 5n = 6n - 5n = n \therefore m = n$

Now, $l : m : n = -2\cancel{n} : \cancel{n} : \cancel{n} = -2 : 1 : 1 = 2 : -1 : -1$

So, d.r's of $L_2 = (a_2, b_2, c_2) = (2, -1, -1)$....(4)

If θ is the angle between the lines then from (3), (4), we get

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}} = \frac{|1(2) + 2(-1) + (-1)(-1)|}{\sqrt{(1^2 + 2^2 + (-1)^2)(2^2 + (-1)^2 + (-1)^2)}}$$

$$= \frac{|\cancel{2} - \cancel{2} + 1|}{\sqrt{(6)(6)}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

$$\therefore \cos\theta = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\frac{1}{6}$$

Hence angle between the lines is $\cos^{-1}\frac{1}{6}$

22. If $y = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)$, then show that $\frac{dy}{dx} = 2\sqrt{a^2+x^2}$

Sol:

- Given $y = x\sqrt{a^2+x^2} + a^2\log(x+\sqrt{a^2+x^2})$;
- On diff. w.r.to x, we get

★ $\frac{dy}{dx} = \left[x\frac{d}{dx}\sqrt{a^2+x^2} + \sqrt{a^2+x^2}\frac{d}{dx}(x)\right] + a^2\left[\frac{d}{dx}\log(x+\sqrt{a^2+x^2})\right]$

★ $= \left(x\frac{1}{2\sqrt{a^2+x^2}}\frac{d}{dx}(a^2+x^2) + (\sqrt{a^2+x^2})(1)\right) + a^2\left(\frac{1}{x+\sqrt{a^2+x^2}}\frac{d}{dx}(x+\sqrt{a^2+x^2})\right)$

- $= \left(\frac{x}{\cancel{2}\sqrt{a^2+x^2}}(\cancel{2}x) + \sqrt{a^2+x^2}\right) + \left(\frac{a^2}{x+\sqrt{a^2+x^2}}\right)\left(1+\frac{1}{\cancel{2}\sqrt{a^2+x^2}}(\cancel{2}x)\right)$

- $= \left(\frac{x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2}\right) + \left(\frac{a^2}{x+\sqrt{a^2+x^2}}\right)\left(1+\frac{x}{\sqrt{a^2+x^2}}\right)$

★ $= \left(\frac{x^2+(a^2+x^2)}{\sqrt{a^2+x^2}}\right) + \left(\frac{a^2}{\cancel{x+\sqrt{a^2+x^2}}}\right)\left(\frac{\cancel{\sqrt{a^2+x^2}+x}}{\sqrt{a^2+x^2}}\right)$

- $= \frac{a^2+2x^2}{\sqrt{a^2+x^2}} + \frac{a^2}{\sqrt{a^2+x^2}}$

- $= \frac{a^2+2x^2+a^2}{\sqrt{a^2+x^2}}$

- $= \frac{2a^2+2x^2}{\sqrt{a^2+x^2}}$

- $= \frac{2(a^2+x^2)}{\sqrt{a^2+x^2}}$

★ $= 2\sqrt{a^2+x^2}$ $[\because \frac{a}{\sqrt{a}} = \sqrt{a}]$

$\therefore \boxed{\frac{dy}{dx} = 2\sqrt{a^2+x^2}}$

23. If the tangent at a point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the coordinate axes in A,B then show that the length AB is a constant.

Sol: ★ The point on the curve taken as $P(a\cos^3\theta, a\sin^3\theta)$

- $x = a\cos^3\theta$ and $y = a\sin^3\theta$

★ $\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(a\sin^3\theta)}{\frac{d}{d\theta}(a\cos^3\theta)} = \frac{a.3\sin^2\theta(\cos\theta)}{a.3\cos^2\theta(-\sin\theta)} = -\frac{\sin\theta}{\cos\theta}$

- So, slope of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ is $m = -\frac{\sin\theta}{\cos\theta}$

★ $\therefore$ Equation of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ having slope $-\frac{\sin\theta}{\cos\theta}$ is $y-y_1=m(x-x_1)$

★ $\Rightarrow y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$

- $\Rightarrow \frac{y - a\sin^3\theta}{\sin\theta} = -\frac{(x - a\cos^3\theta)}{\cos\theta}$

- $\Rightarrow \frac{y}{\sin\theta} - \frac{a\sin^3\theta}{\sin\theta} = -\frac{x}{\cos\theta} + \frac{a\cos^3\theta}{\cos\theta}$

- $\Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a\cos^2\theta + a\sin^2\theta = a(\cos^2\theta + \sin^2\theta) = a(1)$

- $\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$

★ $\therefore A=(a\cos\theta, 0), B=(0, a\sin\theta)$

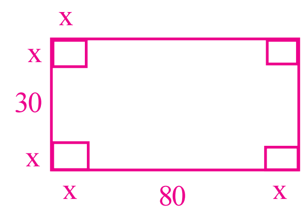
- $\therefore AB = \sqrt{(a\cos\theta - 0)^2 + (0 - a\sin\theta)^2}$

$$= \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = \sqrt{a^2(\cos^2\theta + \sin^2\theta)} = \sqrt{a^2(1)} = a$$

$\therefore$ Hence, proved that AB is a constant.

24. From a rectangular sheet of dimensions 30cm x 80cm, four equal squares of sides x cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. What is the value of x, so that the volume of the box is the greatest?

Sol:



- ★ For the open box, we take

  height $h = x$
  length $l = 80-2x$
  breadth $b = 30-2x$ .....(1)

- ★ Volume $V = lbh = (80-2x)(30-2x)(x)$
- $= 2(40-x)\,2(15-x)(x)$
- $= 4(40-x)(15-x)(x) = 4(600-40x-15x+x^2)x$
- $= 4(600-55x+x^2)x = 4(x^3-55x^2+600x)$
- ★ $V(x) = 4(x^3-55x^2+600x)$ .....(2)
- On diff. (2) w.r.to x, we get,
- $V'(x) = 4(3x^2-110x+600)$ .....(3)
- At max. or min., we have $V'(x) = 0 \Rightarrow \cancel{4}(3x^2-110x+600)=0$
- ★ $\Rightarrow 3x^2-90x-20x+600=0$
- $\Rightarrow 3x(x-30)-20(x-30)=0 \Rightarrow (3x-20)(x-30)=0$
- $\Rightarrow 3x=20$ (or) $x=30 \Rightarrow x=20/3$ (or) $x=30$
- Now, on diff. (3), w.r.to x, we get
- ★ $V''(x) = 4(6x-110)$ ....(4)
- ★ At $x = \frac{20}{3}$, from (4), we get
- ★ $V''\left(\frac{20}{3}\right) = 4\left(\cancel{6}\left(\frac{20}{\cancel{3}}\right)-110\right) = 4(40-110) = 4(-70) = -280$
- Thus, $V''\left(\frac{20}{3}\right) < 0$
- $\therefore$ V(x) has maximum value at $x = \frac{20}{3}$ cm