Previous IPE

SOLVED PAPERS

MARCH-2024 (AP)

PREVIOUS PAPERS

IPE: MARCH-2024(AP)

Time : 3 Hours **MATHS-1B** Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: $10 \times 2 = 20$

1. Find the value of y, if the line joining (3, y), (2, 7) is parallel to the line joining the points (–1, 4), (0, 6)
2. Find the image of the point (1,2) w.r.t. straight line 3x+4y–1=0
3. Find the distance between the mid point of the line segment $\overline{AB}$ and the point (3,–1,2) where A=(6,3,–4), B=(–2,–1,2).
4. Find the equation of the plane passing through the point (–2, 1, 3) and having (3,–5,4) as d.r.'s of its normal.
5. Compute $\underset{x\to 0}{\text{Lt}} \frac{\sin ax}{x\cos x}$
6. Compute $\underset{x\to\infty}{\text{Lt}} \frac{11x^3-3x+4}{13x^3-5x^2-7}$
7. If $f(x) = xe^x \sin x$ then find f(x).
8. If $y = ae^{nx} + be^{-nx}$, then P.T $y'' = n^2y$.
9. Find Δy and dy for the function $y = e^x + x$ when $x = 5,\ \Delta x = 0.02$
10. Verify Rolle's theorem for the function $f(x)=x(x+3)e^{-x/2}$ inn [–3,0]

SECTION-B

II. Answer any FIVE of the following SAQs: $5 \times 4 = 20$

11. Find the equation of locus P, if A = (4, 0), B = (–4, 0) and |PA–PB| = 4.
12. When the axes are rotated through an angle 45^o, the transformed equation of a curve is $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.
13. A straight line through P(3, 4) makes an angle of 60º with the positive direction of the X-axis. Find the coordinates of the points with the line which are 5 units away from P.
14. Find the real constants a, b so that the function f given by $f(x)=\begin{cases} \sin x & \text{if } x \le 0 \\ x^2+a & \text{if } 0<x<1 \\ bx+3 & \text{if } 1 \le x \le 3 \\ -3 & \text{if } x>3 \end{cases}$ is continuous on R.
15. Find the derivative of cosax from the first principle.
16. Find k, so that the length of the subnormal at any point on the curve $y=a^{1-k}x^k$ is a constant.
17. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?

SECTION-C

III. Answer any FIVE of the following LAQs: $5 \times 7 = 35$

18. Find the orthocentre of the triangle formed by the vertices (5, –2), (–1, 2), (1, 4)
19. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then Prove that (a) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ (b) $h^2 \ge ab,\ f^2 \ge bc,\ g^2 \ge ac$.
20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line x + 2y = k are mutually perpendicular.
21. S.T the lines whose direction cosines are $l + m + n = 0,\ 2mn + 3nl - 5lm = 0$ are perpendicular to each other.
22. Find derivative of $(\sin x)^{\log x} + x^{\sin x}$
23. If the tangent at a point on the curve $x^{2/3}+y^{2/3} = a^{2/3}$ intersects the coordinate axes in A,B then show that the length AB is a constant.
24. From a rectangular sheet of dimensions 30cm × 80cm, four equal squares of sides x cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. What is the value of x, so that volume of the box is the greatest?

IPE AP MARCH-2024
SOLUTIONS

SECTION-A

1. Find the value of y, if the line joining (3, y), (2, 7) is parallel to the line joining the points (–1, 4), (0, 6) is 2.

Sol: We take A = (3, y), B = (2, 7), then

Slope of $\overline{AB}$ is $m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-y}{2-3} = \frac{7-y}{-1} = y - 7$

Again, we take C= (–1, 4), D = (0, 6);

Slope of $\overline{CD}$ is $m_2 = \frac{6-4}{0+1} = \frac{2}{1} = 2$

Take, $m_1 = m_2$ [$\because$ Given lines are parallel]

$\Rightarrow y - 7 = 2 \Rightarrow y = 2 + 7 = 9 \quad \therefore y = 9$

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2. Find the image of (1, 2) in the straight line 3x + 4y – 1 = 0

Sol: Let (h, k) be the image of (1, 2) w.r.to 3x + 4y – 1 = 0

Here $(x_1, y_1) = (1, 2)$, a = 3, b = 4, c = –1.

$$\therefore \frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

$$\Rightarrow \frac{h-1}{3} = \frac{k-2}{4} = \frac{-2[3(1)+4(2)-1]}{3^2+4^2} = \frac{-2(10)}{25} = -2\left(\frac{2}{5}\right) = -\frac{4}{5}$$

Now, $\frac{h-1}{3} = -\frac{4}{5} \Rightarrow h - 1 = -\frac{12}{5} \Rightarrow h = 1 - \frac{12}{5} = \frac{5-12}{5} = -\frac{7}{5}$

Also $\frac{k-2}{4} = -\frac{4}{5} \Rightarrow k - 2 = -\frac{16}{5} \Rightarrow k = 2 - \frac{16}{5} = \frac{10-16}{5} = \frac{-6}{5}$

$\therefore$ The image is $(h,k) = \left(-\frac{7}{5}, -\frac{6}{5}\right)$

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3. Find the distance between the mid point of the line segment $\overline{AB}$ and the point (3,–1,2) where A = (6, 3, –4), B = (–2, –1, 2).

Sol: Let M be the mid point of AB $\Rightarrow M = \left(\frac{6-2}{2}, \frac{3-1}{2}, \frac{-4+2}{2}\right) = (2, 1, -1)$.

Now, P = (3, –1, 2), M = (2, 1, –1) $\Rightarrow PM = \sqrt{(3-2)^2 + (-1-1)^2 + (2+1)^2}$

$= \sqrt{1+4+9} = \sqrt{14}$ units.

4. Find the equation of the plane passing through the point (–2, 1, 3) and having (3, –5, 4) as d.r.'s of its normal.

Sol: The equation of the plane with a,b,c as normal d.r's passing through the point (x_1,y_1,z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$

Here, $(x_1, y_1, z_1) = (-2, 1, 3)$ and $a = 3, b = -5, c = 4$

$\therefore$ The required equation is $3(x + 2) - 5(y - 1) + 4(z - 3) = 0$

$\Rightarrow 3x + 6 - 5y + 5 + 4z - 12 = 0$

$\Rightarrow 3x - 5y + 4z - 1 = 0.$

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5. Compute $\underset{x\to 0}{\text{Lt}} \dfrac{\sin ax}{x\cos x}$

Sol: $$\underset{x\to 0}{\text{Lt}} \frac{\sin ax}{x\cos x} = \underset{x\to 0}{\text{Lt}} \left(\frac{\sin ax}{x}\right)\frac{1}{(\cos x)} = a\frac{1}{(\cos 0)} = a\frac{1}{1} = a \qquad \left(\because \lim_{x\to 0}\frac{\sin kx}{x} = k\right)$$

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6. Evaluate $\underset{x\to\infty}{\text{Lt}} \dfrac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$

Sol: Taking x^3, the highest power of x as common factor in the numerator and denominator.

$$\underset{x\to\infty}{\text{Lt}} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7} = \underset{x\to\infty}{\text{Lt}} \frac{\cancel{x^3}\left(11 - \frac{3}{x^2} + \frac{4}{x^3}\right)}{\cancel{x^3}\left(13 - \frac{5}{x} - \frac{7}{x^3}\right)} = \underset{\frac{1}{x}\to 0}{\text{Lt}} \frac{11 - \frac{3}{x^2} + \frac{4}{x^3}}{13 - \frac{5}{x} - \frac{7}{x^3}} = \frac{11 - 0 + 0}{13 - 0 - 0} = \frac{11}{13}$$

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7. If $f(x) = xe^x \sin x$ then find $f'(x)$.

Sol: $$\frac{d}{dx}(uvw) = uv\frac{d}{dx}w + uw\frac{d}{dx}v + vw\frac{d}{dx}u$$

$$\therefore f'(x) = \frac{d}{dx}(xe^x \sin x) = xe^x\frac{d}{dx}\sin x + x\sin x\frac{d}{dx}e^x + e^x\sin x\frac{d}{dx}(x)$$

$$= xe^x(\cos x) + x\sin x(e^x) + e^x \sin x(1)$$

$$= xe^x \cos x + x\sin x(e^x) + e^x \sin x$$

$$= e^x(x\cos x + x\sin x + \sin x)$$

8. If $y = ae^{nx} + be^{-nx}$, then prove that $y'' = n^2y$.

Sol: Given $y = ae^{nx} + be^{-nx}$; on diff. w.r.t x, we get

$y' = ae^{nx}(n) + be^{-nx}(-n)$. on diff. again w.r.t x, we get $y'' = ae^{nx}(n)(n) + be^{-nx}(-n)(-n)$

$\Rightarrow y'' = n^2ae^{nx} + n^2be^{-nx} = n^2(ae^{nx} + be^{-nx}) = n^2y$ $\therefore y'' = n^2y$.

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9. Find Δy and dy for the function $y = e^x + x$ when $x = 5$, $\Delta x = 0.02$

Sol: We take $y = f(x) = e^x + x$, $x = 5$, $\Delta x = 0.02$

(i) $\Delta y = f(x+\Delta x) - f(x)$

$= [e^{x+\Delta x} + (x+\Delta x)] - (e^x + x)$

$= [e^{5+0.02} + (\cancel{5} + 0.02)] - (e^5 + \cancel{5}) = e^{5.02} + 0.02 - e^5 = e^5(e^{0.02} - 1) + 0.02$

(ii) $dy = f'(x)\Delta x = (e^x+1)\,\Delta x = (e^5+1)(0.02)$

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10. Verify Rolle's theorem for the function $f(x) = x(x+3)\,e^{-x/2}$ on $[-3,0]$

Sol: Given function f(x) is (i) continuous on $[-3, 0]$ and (ii) differentiable in $(-3, 0)$

Now, $f(x) = x(x+3)e^{-x/2}$

$\Rightarrow f(-3) = (-3)(-3+3)e^{-3/2} = -3(0)e^{-3/2} = 0$. Also $f(0) = (0)(0+3)e^{-3/2} = 0$

$\therefore f(-3) = f(0)$

Hence, f(x) satisfies all the 3 conditions of Rolle's theorem.

$\therefore$ By Rolle's theorem, there exists $c \in (-3,0)$ such that $f'(c) = 0$

Now, $f(x) = x(x+3)e^{-x/2} = (x^2 + 3x)e^{-x/2}$.

$$f'(x) = (x^2+3x)e^{-x/2}\left(-\frac{1}{2}\right) + e^{-x/2}(2x+3)$$

$$= e^{-x/2}\left[\frac{-x^2-3x}{2} + 2x + 3\right] = e^{-x/2}\left[\frac{-x^2-3x+4x+6}{2}\right] = e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$$

Hence, $f'(c) = 0 \Rightarrow e^{-c/2}\left(\frac{-c^2+c+6}{2}\right) = 0 \Rightarrow -c^2 + c + 6 = 0$

$\Rightarrow c^2 - c - 6 = 0 \Rightarrow (c+2)(c-3) = 0 \Rightarrow c = -2$ or 3

Among these two values, $-2 \in (-3, 0)$. Hence, Rolle's theorem is verified.

SECTION-B

11. Find the equation of locus of P, if the line segment joining (4, 0) and (0, 4) subtends a right angle at P.

Sol:
- Let the locus point P = (x, y)
- Given points A = (4, 0), B = (0, 4)
- ★ **Given condition:** $\angle APB = 90^\circ$
- ★ $\Rightarrow PA^2 + PB^2 = AB^2$
- ★ $\Rightarrow [(x-4)^2 + (y-0)^2] + [(x-0)^2+(y-4)^2] = (4-0)^2+(0-4)^2$
- $\Rightarrow (x^2-8x+16)+y^2+x^2+(y^2-8y+16)=16+16$
- $\Rightarrow 2x^2+2y^2-8x-8y=0$
- $\Rightarrow 2(x^2+y^2-4x-4y)=0$
- $\Rightarrow x^2+y^2-4x-4y=0$
- Hence, Locus of P is $x^2+y^2-4x-4y=0$

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12. When the axes are rotated through an angle 45°, the transformed equation of a curve is $17x^2-16xy+17y^2=225$. Find the original equation of the curve.

Sol:
- ★ Given transformed(new) equation is taken as $17X^2 - 16XY + 17Y^2 = 225$ ....(1)
- Angle of rotation $\theta = 45^\circ$, then
- ★ $X = x\cos\theta + y\sin\theta = x\cos45^\circ + y\sin45^\circ = x\left(\frac{1}{\sqrt{2}}\right)+y\left(\frac{1}{\sqrt{2}}\right) \Rightarrow X=\frac{x+y}{\sqrt{2}}$

  $Y = y\cos\theta - x\sin\theta = y\cos45^\circ - x\sin45^\circ = y\left(\frac{1}{\sqrt{2}}\right)-x\left(\frac{1}{\sqrt{2}}\right) \Rightarrow Y=\frac{y-x}{\sqrt{2}}$
- From (1),original equation is
- $17\left(\frac{x+y}{\sqrt{2}}\right)^2-16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{y-x}{\sqrt{2}}\right)+17\left(\frac{y-x}{\sqrt{2}}\right)^2=225$
- ★ $\Rightarrow 17\left(\frac{x^2+y^2+2xy}{2}\right)-16\left(\frac{y^2-x^2}{2}\right)+17\left(\frac{y^2+x^2-2xy}{2}\right)=225$ $[\because (x+y)(y-x)=(y+x)(y-x)=y^2-x^2]$
- ★ $\Rightarrow \frac{17x^2+17y^2+\cancel{34xy}-16y^2+16x^2+17x^2+17y^2-\cancel{34xy}}{2}=225$
- ★ $\Rightarrow 50x^2+18y^2=2(225) \Rightarrow \cancel{2}(25x^2+9y^2)=\cancel{2}(225) \Rightarrow 25x^2+9y^2=225$

Hence, required original equation is $25x^2+9y^2=225$.

13. A straight line through P(3,4) makes an angle of 60° with the positive direction of the X-axis. Find the coordinates of the points with the line which are 5 units away from P.

Sol : Given point $P(x_1, y_1) = (3, 4)$, $r = 5, \theta = 60^\circ$

Required points are $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

$$= (3 \pm 5\cos 60^\circ, 4 \pm 5\sin 60^\circ) = \left(3 \pm 5\left(\frac{1}{2}\right), 4 \pm 5\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \left(3+5\left(\frac{1}{2}\right), 4+5\left(\frac{\sqrt{3}}{2}\right)\right) = \left(\frac{11}{2}, \frac{8+5\sqrt{3}}{2}\right) \text{ and } \left(3-5\left(\frac{1}{2}\right), 4-5\left(\frac{\sqrt{3}}{2}\right)\right) = \left(\frac{1}{2}, \frac{8-5\sqrt{3}}{2}\right)$$

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14. Find the real constants a, b so that the function f given by $f(x) = \begin{cases} \sin x & \text{if } x \le 0 \\ x^2 + a & \text{if } 0 < x < 1 \\ bx + 3 & \text{if } 1 \le x \le 3 \\ -3 & \text{if } x > 3 \end{cases}$ **is continuous on R.**

Sol: Given that f(x) is continuous on R $\Rightarrow$ f(x) is continuous at x = 0, 3

(i) f(x) is continuous at x = 0

$$\text{L.H.L} = \underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} \sin x = \sin 0 = 0$$

$$\text{R.H.L} = \underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} x^2 + a = 0^2 + a = a$$

But f(x) is continuous at x = 0 $\Rightarrow$ L.H.L = R.H.L $\Rightarrow$ a = 0

(ii) f(x) is continuous at x = 3

$$\text{L.H.L} = \underset{x \to 3-}{\text{Lt}} f(x) = \underset{x \to 3-}{\text{Lt}} (bx+3) = 3b+3$$

$$\text{R.H.L} = \underset{x \to 3+}{\text{Lt}} f(x) = \underset{x \to 3+}{\text{Lt}} (-3) = -3$$

But f(x) is continuous at x = 3 $\Rightarrow$ L.H.L = R.H.L $\Rightarrow$ 3b+3 = –3

$\Rightarrow$ 3b = –6 $\Rightarrow$ b = –2

15. Find the derivative of cosax from the first Principle.

Sol: We take f(x) = cosax, then f(x + h) = cosa(x + h) = cos(ax + ah)

From the first principle, $f'(x) = \underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h}$

$$= \underset{h\to 0}{Lt}\frac{\cos(ax+ah)-\cos(ax)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[-2\sin\left(\frac{(ax+ah)+ax}{2}\right)\sin\left(\frac{(\cancel{ax}+ah)-\cancel{ax}}{2}\right)\right] \quad \left[\because \cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -2\underset{h\to 0}{Lt}\frac{1}{h}\left[\sin\left(\frac{2ax+ah}{2}\right)\sin\left(\frac{ah}{2}\right)\right] = -2\underset{h\to 0}{Lt}\frac{1}{h}\left[\sin\left(ax+\frac{ah}{2}\right)\sin\frac{ah}{2}\right]$$

$$= -2\underset{h\to 0}{Lt}\sin\left(ax+\frac{ah}{2}\right)\underset{h\to 0}{Lt}\frac{\sin\left(\frac{ah}{2}\right)}{h} = -2\sin(ax+0)\left(\frac{a}{2}\right)$$

$$= -\cancel{2}\sin ax\left(\frac{a}{\cancel{2}}\right) = -a\sin ax$$

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16. Find the value of k, so that the length of the subnormal at any point on the curve $y = a^{1-k}x^k$ is a constant

Sol:
- Let P(x, y) be point on the curve $y = a^{1-k}x^k$

★ On diff. w.r.to x, we get $m = \dfrac{dy}{dx} = a^{1-k}(kx^{k-1})$

★ ∴ The length of the subnormal at point P(x,y) is $|y.m| = |y(a^{1-k}kx^{k-1})|$

★ $= |(a^{1-k}x^k)a^{1-k}(k)x^{k-1}| = |ka^{2-2k}.x^{2k-1}|$

★ The above expression becomes a constant if $2k-1 = 0 \Rightarrow k = 1/2$

17. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?

Sol:
- For the cube, we take
- length of the edge = x , Volume = V and

Surface area = S

★ Given $\frac{dV}{dt} = 9$ and x = 10 cm

★ Volume of the cube $V = x^3$

★ On diff. w.r.t 't', we get $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

★ $\Rightarrow 9 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$

★ Surface area $S=6x^2$

★ On diff. w.r.t 't', we get $\frac{dS}{dt} = 12x \frac{dx}{dt}$

★ $= 12x\left(\frac{3}{x^2}\right) = \frac{36}{x} = \frac{36}{10} = 3.6\, cm^2/\sec$

SECTION-C

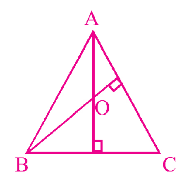
18. Find the orthocentre of the triangle whose vertices are (5, –2), (–1, 2), (1, 4)

Sol:
- Take O(x, y) as Orthocentre
- Vertices A = (5,–2), B = (–1,2), C= (1,4)

Step-1: Finding altitude through A(5, –2):

Slope of $\overline{BC}$ is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{1+1} = \frac{2}{2} = 1$

Its perpendicular slope is $\frac{-1}{m} = \frac{-1}{1} = -1$



Eq. of line through A(5, –2) with slope –1 is $y - y_1 = \frac{-1}{m}(x - x_1)$

$\Rightarrow y + 2 = -1(x - 5) \Rightarrow y + 2 = -x + 5 \quad \Rightarrow x + y - 3 = 0$ .... (1)

Step-2: Finding altitude through B(–1, 2):

Slope of $\overline{AC}$ is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-2)}{1-5} = \frac{6}{-4} = -\frac{3}{2}$

Its perpendicular slope is $\frac{-1}{m} = \frac{-1}{-\frac{3}{2}} = \frac{2}{3}$

Eq. of line through B(–1,2) with slope $\frac{2}{3}$ is $y - y_1 = \frac{-1}{m}(x - x_1)$

$\Rightarrow y-2 = \frac{2}{3}(x+1) \Rightarrow 3y - 6 = 2(x + 1)$

$\Rightarrow 3y - 6 = 2x + 2 \Rightarrow 2x-3y+2+6 = 0 \Rightarrow 2x - 3y + 8 = 0$ .... (2)

Step-3: Solving (1), (2), we get O;

(1) $\Rightarrow x + y - 3 = 0$

(2) $\Rightarrow 2x - 3y + 8 = 0$

$\therefore \frac{x}{1(8) - (-3)(-3)} = \frac{y}{(-3)(2) - (8)(1)} = \frac{1}{1(-3) - (2)(1)}$

$\Rightarrow \frac{x}{8-9} = \frac{y}{-6-8} = \frac{1}{-3-2}$

$\Rightarrow \frac{x}{-1} = \frac{y}{-14} = \frac{1}{-5} \Rightarrow x = \frac{-1}{-5} = \frac{1}{5};\ y = \frac{-14}{-5} = \frac{14}{5} \qquad \Rightarrow x = \frac{1}{5},\ y = \frac{14}{5}$

$\therefore$ Orthocentre $O(x,y) = \left(\frac{1}{5}, \frac{14}{5}\right)$

19. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then Prove that (a) $\Delta = abc+2fgh-af^2-bg^2-ch^2 = 0$ (b) $h^2 \geq ab,\ f^2 \geq bc,\ g^2 \geq ac$.

Sol:

★ Let $ax^2+2hxy+by^2+2gx+2fy+c \equiv (l_1x+m_1y+n_1)(l_2x+m_2y+n_2)$

- On equating like term coeff., we get

★ $a = l_1l_2,\ b = m_1m_2,\ c = n_1n_2,\ 2h = l_1m_2+l_2m_1,\ 2g = l_1n_2+l_2n_1,\ 2f = m_1n_2+m_2n_1$

- **(a)** $(2h)(2g)(2f) = (l_1m_2+l_2m_1)(l_1n_2+l_2n_1)(m_1n_2+m_2n_1)$

- On multiplying and regrouping, we get

★ $8fgh = l_1l_2(m_1^2n_2^2+m_2^2n_1^2)+m_1m_2(n_1^2l_2^2+n_2^2l_1^2)+n_1n_2(l_1^2m_2^2+l_2^2m_1^2)+2l_1l_2m_1m_2n_1n_2$

★ $= l_1l_2[(m_1n_2+m_2n_1)^2-2m_1m_2n_1n_2]+m_1m_2[(n_1l_2+n_2l_1)^2-2n_1n_2l_1l_2)]+$
$+n_1n_2[(l_1m_2+l_2m_1)^2-2l_1l_2m_1m_2)]+2l_1l_2m_1m_2n_1n_2$

★ $\Rightarrow 8fgh = a((2f)^2-2bc)+b((2g)^2-2ac)+c((2h)^2-2ab)+2abc = 4af^2+4bg^2+4ch^2-4abc$

- $\Rightarrow 8fgh = 4(af^2+bg^2+ch^2-abc) \Rightarrow 2fgh = af^2+bg^2+ch^2-abc$
- $\Rightarrow abc+2fgh-af^2-bg^2-ch^2 = 0$

★ **(b)** $h^2-ab = \left(\dfrac{l_1m_2+l_2m_1}{2}\right)^2-l_1l_2m_1m_2$

★ $= \dfrac{(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2}{4} = \dfrac{(l_1m_2-l_2m_1)^2}{4} \geq 0$

- $\therefore h^2-ab \geq 0 \Rightarrow h^2 \geq ab$

- Similarly, we can prove that $g^2 \geq ac, f^2 \geq bc$

20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = k$ are mutually perpendicular.

A:

- The given line is $x + 2y = k \Rightarrow \frac{x+2y}{k} = 1$ ...(1)
- Given curve is $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$............(2)
- Homogenising (1) & (2), we get

★ $2x^2 - 2xy + 3y^2 + 2x(1) - y(1) - (1)^2 = 0$

★ $\Rightarrow 2x^2 - 2xy + 3y^2 + 2x\left(\frac{x+2y}{k}\right) - y\left(\frac{x+2y}{k}\right) - \frac{(x+2y)^2}{k^2} = 0$

★ $\Rightarrow \frac{k^2(2x^2 - 2xy + 3y^2) + k(2x^2 + 4xy) - k(xy + 2y^2) - (x^2 + 4y^2 + 4xy)}{k^2} = 0$

★ $\Rightarrow k^2(2x^2 - 2xy + 3y^2) + k(2x^2 + 4xy) - k(xy + 2y^2) - (x^2 + 4y^2 + 4xy) = 0$

★ $\Rightarrow x^2(2k^2 + 2k - 1) + y^2(3k^2 - 2k - 4) + xy(-2k^2 + 3k - 4) = 0$

- If this pair or lines are perpendicular then

★ Coeff. x^2 + Coeff. y^2 =0

- $\Rightarrow (2k^2 \cancel{+2k} - 1) + (3k^2 \cancel{-2k} - 4) = 0 \Rightarrow 5k^2 - 5 = 0$
- $\Rightarrow \cancel{5}(k^2 - 1) = 0 \Rightarrow k^2 - 1 = 0 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

Hence, value of $k = \pm 1$

21. Show that the lines whose direction cosines are given by $l+m+n=0$, $2mn+3nl-5lm=0$ are perpendicular to each other.

Sol: Given $l + m + n = 0 \Rightarrow l = -(m + n)$ ....(1);

$2mn + 3nl - 5lm = 0$ .....(2)

Solving (1) & (2), we get, $2mn - 3n(m + n) + 5m(m + n) = 0$

$\Rightarrow 2mn-3mn-3n^2+5m^2+5mn=0 \Rightarrow 5m^2+4mn-3n^2=0$

$$\Rightarrow \frac{5m^2}{n^2}+\frac{4mn}{n^2}-\frac{3n^2}{n^2}=0 \Rightarrow 5\left(\frac{m}{n}\right)^2+4\left(\frac{m}{n}\right)-3=0$$

This is a quadratic equation in $\frac{m}{n}$, its roots are taken as $\frac{m_1}{n_1}, \frac{m_2}{n_2}$,

Product of roots $\frac{m_1}{n_1}.\frac{m_2}{n_2}=\frac{c}{a}=\frac{-3}{5}$ [$\because$ Product of roots of $ax^2+bx+c=0$ is $\alpha\beta=\frac{c}{a}$]

$$\Rightarrow \frac{m_1m_2}{n_1n_2}=-\frac{3}{5} \Rightarrow \frac{m_1m_2}{3}=\frac{n_1n_2}{-5} \text{ ....(3)}$$

From (1), $n=-(l+m)$

Substituting this value of n in (2), we get $-2m(l+m)-3l(l+m)-5lm=0$

$\Rightarrow -2lm-2m^2-3l^2-3lm-5lm=0 \Rightarrow 3l^2+10lm+2m^2=0$

$$\Rightarrow \frac{3l^2}{m^2}+\frac{10lm}{m^2}+\frac{2m^2}{m^2}=0 \Rightarrow 3\left(\frac{l}{m}\right)^2+10\left(\frac{l}{m}\right)+2=0$$

This is a quadratic equation in $\frac{l}{m}$, its roots are taken as $\frac{l_1}{m_1}, \frac{l_2}{m_2}$

Product of roots $\frac{l_1}{m_1}.\frac{l_2}{m_2}=\frac{c}{a}=\frac{2}{3}$

$$\Rightarrow \frac{l_1l_2}{m_1m_2}=\frac{2}{3} \Rightarrow \frac{l_1l_2}{2}=\frac{m_1m_2}{3} \text{ .....(4)}$$

From (3) and (4), we get $\frac{l_1l_2}{2}=\frac{m_1m_2}{3}=\frac{n_1n_2}{-5}=k$ (say)

$\Rightarrow l_1l_2 = 2k,\ m_1m_2 = 3k,\ n_1n_2 = -5k$

$\therefore\ l_1l_2 + m_1m_2 + n_1n_2 = 2k + 3k-5k = 5k-5k = 0.$

Hence, proved that the two lines are perpendicular.

22. Find the derivative of $(\sin x)^{\log x} + x^{\sin x}$

Sol: Let $y = (\sin x)^{\log x} + x^{\sin x}$..........(1)

Take $u = (\sin x)^{\log x}$

Taking log on both sides, we get

$\log u = \log(\sin x)^{\log x}$

$\Rightarrow \log u = \log x\,[\log \sin x]$

On diff. using uv formula, we get

$$\frac{1}{u}\frac{du}{dx} = \log x\left(\frac{1}{\sin x}(\cos x)\right) + \log \sin x\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \left(\log x.\cot x + \frac{\log \sin x}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = u\left(\log x.\cot x + \frac{\log \sin x}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = \sin x^{\log x}\left(\log x \cot x + \frac{\log \sin x}{x}\right)$$

Take $v = x^{\sin x}$

Taking log on both sides, we get

$\log v = \log x^{\sin x}$

$\Rightarrow \log v = \sin x\,[\log x]$

On diff. using uv formula, we get

$$\frac{1}{v}\frac{dv}{dx} = \sin x\left(\frac{1}{x}\right) + \log x.\cos x$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left(\frac{\sin x}{x} + \log x.\cos x\right)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\frac{\sin x}{x} + \log x.\cos x\right)$$

$$\Rightarrow \frac{dv}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x.\cos x\right)$$

From (1), $y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = \sin x^{\log x}\left(\log x.\cot x + \frac{\log \sin x}{x}\right) + x^{\sin x}\left(\frac{\sin x}{x} + \log x.\cos x\right)$$

23. If the tangent at a point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the coordinate axes in A,B then show that the length AB is a constant.

Sol: ★ The point on the curve taken as $P(a\cos^3\theta, a\sin^3\theta)$

- $x = a\cos^3\theta$ and $y = a\sin^3\theta$

★ $$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(a\sin^3\theta)}{\frac{d}{d\theta}(a\cos^3\theta)} = \frac{\cancel{a}.3\cancel{\sin^2}\theta(\cancel{\cos\theta})}{\cancel{a}.3\cancel{\cos^2}\theta(\cancel{-\sin\theta})} = -\frac{\sin\theta}{\cos\theta}$$

- So, slope of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ is $m = -\frac{\sin\theta}{\cos\theta}$

★ $\therefore$ Equation of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ having slope $-\frac{\sin\theta}{\cos\theta}$ is $y - y_1 = m(x - x_1)$

★ $$\Rightarrow y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$$

- $$\Rightarrow \frac{y - a\sin^3\theta}{\sin\theta} = -\frac{(x - a\cos^3\theta)}{\cos\theta}$$

- $$\Rightarrow \frac{y}{\sin\theta} - \frac{a\sin^3\theta}{\sin\theta} = -\frac{x}{\cos\theta} + \frac{a\cos^3\theta}{\cos\theta}$$

- $$\Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a\cos^2\theta + a\sin^2\theta = a(\cos^2\theta + \sin^2\theta) = a(1)$$

- $$\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$$

★ $\therefore A = (a\cos\theta, 0), B = (0, a\sin\theta)$

- $$\therefore AB = \sqrt{(a\cos\theta - 0)^2 + (0 - a\sin\theta)^2}$$

$$= \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = \sqrt{a^2(\cos^2\theta + \sin^2\theta)} = \sqrt{a^2(1)} = a$$

$\therefore$ Hence, proved that AB is a constant.

24. From a rectangular sheet of dimensions 30cm x 80cm, four equal squares of sides x cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. What is the value of x, so that the volume of the box is the greatest?

Sol: ★ For the open box, we take

height $h = x$
length $l = 80 - 2x$
breadth $b = 30 - 2x$ .....(1)

★ Volume $V = lbh = (80-2x)(30-2x)(x)$

- $= 2(40-x)\,2(15-x)(x)$
- $= 4(40-x)(15-x)(x) = 4(600-40x-15x+x^2)x$
- $= 4(600-55x+x^2)x = 4(x^3-55x^2+600x)$

★ $V(x) = 4(x^3-55x^2+600x)$ .....(2)

- On diff. (2) w.r.to x, we get,
- $V'(x) = 4(3x^2-110x+600)$ .....(3)
- At max. or min., we have $V'(x)=0 \Rightarrow \not{4}(3x^2-110x+600)=0$

★ $\Rightarrow 3x^2-90x-20x+600=0$

- $\Rightarrow 3x(x-30)-20(x-30)=0 \Rightarrow (3x-20)(x-30)=0$
- $\Rightarrow 3x = 20$ (or) $x = 30 \Rightarrow x = 20/3$ (or) $x = 30$
- Now, on diff. (3), w.r.to x, we get

★ $V''(x) = 4(6x-110)$ ....(4)

★ At $x = \frac{20}{3}$, from (4), we get

★ $V''\left(\frac{20}{3}\right) = 4\left(\not{6}\left(\frac{20}{\not{3}}\right)-110\right) = 4(40-110) = 4(-70) = -280$

- Thus, $V''\left(\frac{20}{3}\right) < 0$
- $\therefore$ V(x) has maximum value at $x = \frac{20}{3}$ cm