Previous IPE

SOLVED PAPERS

MARCH -2024 (TS)

PREVIOUS PAPERS

IPE: MARCH-2024(TS)

MATHS-1A

Time : 3 Hours Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **10 × 2 = 20**

1. If $f : R - \{0\} \to R$ is defined by $f(x) = x^3 - \frac{1}{x^3}$, then show that f(x) + f(1/x) = 0.
2. Find the domain of the real function $f(x) = \frac{1}{(x^2-1)(x+3)}$
3. If $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$ then find the values of x, y, z and a.
4. If ω is a complex cube root of 1 then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$
5. Let $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$, $\bar{b} = 3\bar{i} + \bar{j}$. Find a unit vector in the direction of $\bar{a} + \bar{b}$
6. Find the vector equation (V.E) of the line passing through the point $2\bar{i} + 3\bar{j} + \bar{k}$ and parallel to $4\bar{i} - 2\bar{j} + 3\bar{k}$.
7. If the vectors $2\bar{i} + \lambda\bar{j} - \bar{k}$, $4\bar{i} - 2\bar{j} + 2\bar{k}$ are perpendicular to each other then find λ.
8. Prove that $\sin 78^\circ + \cos 132^\circ = \frac{\sqrt{5}-1}{4}$ **9.** Find the value of $\sin^2 82\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$
10. Show that $\operatorname{Tanh}^{-1}\frac{1}{2} = \frac{1}{2}\log_e 3$.

SECTION-B

II. Answer any FIVE of the following SAQs: **5 x 4=20**

11. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then show that $(aI+bE)^3 = a^3I + 3a^2bE$.
12. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then prove that the four points $3\vec{a} + 2\vec{b} - 5\vec{c}$, $-3\vec{a} + 8\vec{b} - 5\vec{c}$ and $-3\vec{a} + 2\vec{b} + \vec{c}$ are coplanar.
13. For any three vectors $\bar{a}, \bar{b}, \bar{c}$ prove that $[\bar{b}+\bar{c} \;\; \bar{c}+\bar{a} \;\; \bar{a}+\bar{b}] = 2[\bar{a}\,\bar{b}\,\bar{c}]$.
14. Prove that tan70°–tan20° = 2 tan 50° **15.** Solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$
16. Prove that $\operatorname{Tan}^{-1}\frac{1}{2} + \operatorname{Tan}^{-1}\frac{1}{5} + \operatorname{Tan}^{-1}\frac{1}{8} = \frac{\pi}{4}$. **17.** In a ΔABC if a:b:c= 7:8:9, then find cos A : cos B: cos C

SECTION-C

III. Answer any FIVE of the following LAQs: **5 x 7=35**

18. Let A = {1,2,3}, B={α,β,γ}, C={p,q,r} and f:A→B,g : B → C are defined by f ={(1,α), (2,γ),(3,β)}, g={(α,q),(β,r),(γ,p)} then show that f and g are bijective functions and $(gof)^{-1} = f^{-1}og^{-1}$
19. Using P.M.I, prove that $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ...n \text{ terms} = \frac{n}{3n+1}$ ∀n∈N.
20. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$
21. Solve x+y+z=1, 2x+2y+3z=6, x+4y+9z=3 by using Matrix inversion method.
22. If $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k}, \bar{b} = \bar{i} - 2\bar{j} + \bar{k}, \bar{c} = -\bar{i} + \bar{j} - 4\bar{k}$ and $\bar{d} = \bar{i} + \bar{j} + \bar{k}$ then compute $|(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})|$
23. If A+B+C=180°, then show that sin2A+sin2B+sin2C=4sinAsinBsinC.
24. If $r_1 = 2, r_2 = 3, r_3 = 6$ and r = 1, prove that a = 3, b = 4 and c = 5.

IPE TS MARCH-2024
SOLUTIONS

SECTION-A

1. If $f : R - \{0\} \to R$ is defined by $f(x) = x^3 - \frac{1}{x^3}$, then show that $f(x) + f(1/x) = 0$.

Sol: Given $f(x)=x^3-\frac{1}{x^3} \Rightarrow f\left(\frac{1}{x}\right)=\frac{1}{x^3}-x^3$.

$$\therefore f(x)+f\left(\frac{1}{x}\right)=\left(x^3-\frac{1}{x^3}\right)+\left(\frac{1}{x^3}-x^3\right)=0$$

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2. Find the domain of the real function $\frac{1}{(x^2-1)(x+3)}$

Sol: Given f(x) is defined when $(x^2-1)(x+3)\neq 0$

$\Rightarrow (x-1)(x+1)(x+3)\neq 0 \Rightarrow x\neq 1,-1,-3$

$\therefore$ Domain = $R-\{1,-1,-3\}$

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3. If $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix}=\begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$ then find the values of x, y, z and a.

Sol: Given $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix}=\begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$

On equating corresponding elements, we get $x-3=5 \Rightarrow x=5+3=8$; $2y-8=2$

$\Rightarrow 2y=2+8 \Rightarrow 2y=10 \Rightarrow y=5$

$z+2=-2 \Rightarrow z=-2-2=-4$; $a-4=6 \Rightarrow a=6+4 \Rightarrow a=10$

$\therefore x=8, y=5, z=-4, a=10$

4. If ω is a complex cube root of 1 then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$

Sol: $LHS = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$ $(\because R_1 \to R_1+R_2+R_3)$

$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 = \text{R.H.S} \ (\because 1+\omega+\omega^2 = 0)$

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5. Let $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$ **and** $\bar{b} = 3\bar{i} + \bar{j}$**. Find a unit vector in the direction of** $\bar{a} + \bar{b}$

Sol: Given $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$, $\bar{b} = 3\bar{i} + \bar{j}$, then

$\bar{a} + \bar{b} = (\bar{i} + 2\bar{j} + 3\bar{k}) + (3\bar{i} + \bar{j}) = 4\bar{i} + 3\bar{j} + 3\bar{k}$

$\Rightarrow |\bar{a} + \bar{b}| = \sqrt{4^2 + 3^2 + 3^2} = \sqrt{16+9+9} = \sqrt{34}$

$\therefore$ Required unit vector$= \dfrac{\bar{a} + \bar{b}}{|\bar{a} + \bar{b}|} = \dfrac{4\bar{i} + 3\bar{j} + 3\bar{k}}{\sqrt{34}}$

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6. Find the vector equation (V.E) of the line passing through the point $2\bar{i} + 3\bar{j} + \bar{k}$ **and parallel to** $4\bar{i} - 2\bar{j} + 3\bar{k}$**.**

Sol: Given point $A(\bar{a}) = 2\bar{i} + 3\bar{j} + \bar{k}$ and given vector $\bar{b} = 4\bar{i} - 2\bar{j} + 3\bar{k}$

V.E of the line passing through the point $A(\bar{a})$ and parallel to the vector $\bar{b}$ is $\bar{r} = \bar{a} + t\bar{b}, t \in R$

$\Rightarrow \bar{r} = (2\bar{i} + 3\bar{j} + \bar{k}) + t(4\bar{i} - 2\bar{j} + 3\bar{k}), t \in R$

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7. If the vectors $2\bar{i} + \lambda\bar{j} - \bar{k}$ **and** $4\bar{i} - 2\bar{j} + 2\bar{k}$ **are perpendicular to each other then find λ.**

Sol: Let $\bar{a} = 2\bar{i} + \lambda\bar{j} - \bar{k}$, $\bar{b} = 4\bar{i} - 2\bar{j} + 2\bar{k}$

Given $\bar{a} \perp \bar{b} \Rightarrow \bar{a}.\bar{b} = 0$

$\therefore (2\bar{i} + \lambda\bar{j} - \bar{k}).(4\bar{i} - 2\bar{j} + 2\bar{k}) = 0$

$\Rightarrow 2(4) - 2\lambda - 2 = 0 \Rightarrow 8 - 2\lambda - 2 = 0 \Rightarrow 6 - 2\lambda = 0 \Rightarrow 2\lambda = 6 \Rightarrow \lambda = 3$

8. Prove that $\sin 78^\circ + \cos 132^\circ = \frac{\sqrt{5}-1}{4}$

Sol: $L.H.S = \sin 78^\circ + \cos 132^\circ = \sin 78^\circ + \cos(90^\circ + 42^\circ)$

$$= \sin 78^\circ - \sin 42^\circ = 2\cos\left(\frac{78^\circ+42^\circ}{2}\right)\sin\left(\frac{78^\circ-42^\circ}{2}\right) \quad \left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= 2\cos 60^\circ \sin 18^\circ = \cancel{2}\left(\frac{1}{\cancel{2}}\right)\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{4} = R.H.S$$

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9. Find the value of $\sin^2 82\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$

Sol: We know $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$

$$\therefore \sin^2 82\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ = \sin\left(82\frac{1}{2}^\circ + 22\frac{1}{2}^\circ\right).\sin\left(82\frac{1}{2}^\circ - 22\frac{1}{2}^\circ\right)$$

$$= \sin(105^\circ).\sin 60^\circ = \sin(60^\circ + 45^\circ).\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}(\sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} + \frac{1}{2}.\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) = \frac{\sqrt{3}(\sqrt{3}+1)}{4\sqrt{2}}$$

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10. Show that $\mathrm{Tanh}^{-1}\frac{1}{2} = \frac{1}{2}\log_e 3$

Sol: We know $\mathrm{Tanh}^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

$$\therefore \mathrm{Tanh}^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\log_e\left(\frac{\frac{3}{\cancel{2}}}{\frac{1}{\cancel{2}}}\right) = \frac{1}{2}\log_e(3)$$

SECTION-B

11. **If $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, E=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ then S.T $(aI+bE)^3=a^3I+3a^2bE$, where I is unit matrix of order 2.**

Sol: Given $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, E=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$

$$aI+bE=a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}=\begin{bmatrix}a & 0\\0 & a\end{bmatrix}+\begin{bmatrix}0 & b\\0 & 0\end{bmatrix}=\begin{bmatrix}a & b\\0 & a\end{bmatrix}$$

$$L.H.S=(aI+bE)^3=\begin{bmatrix}a & b\\0 & a\end{bmatrix}^3=\begin{bmatrix}a & b\\0 & a\end{bmatrix}\begin{bmatrix}a & b\\0 & a\end{bmatrix}\begin{bmatrix}a & b\\0 & a\end{bmatrix}=\begin{bmatrix}a^2+0 & ab+ab\\0+0 & 0+a^2\end{bmatrix}\begin{bmatrix}a & b\\0 & a\end{bmatrix}$$

$$=\begin{bmatrix}a^2 & 2ab\\0 & a^2\end{bmatrix}\begin{bmatrix}a & b\\0 & a\end{bmatrix}=\begin{bmatrix}a^3 & 3a^2b\\0 & a^3\end{bmatrix}\text{........(1)}$$

$$R.H.S=a^3I+3a^2bE=a^3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+3a^2b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}=\begin{bmatrix}a^3 & 0\\0 & a^3\end{bmatrix}+\begin{bmatrix}0 & 3a^2b\\0 & 0\end{bmatrix}=\begin{bmatrix}a^3 & 3a^2b\\0 & a^3\end{bmatrix}\text{......(2)}$$

$\therefore$ from (1), (2) L.H.S= R.H.S Hence, proved.

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12. **Show that the four points $-\bar{a}+4\bar{b}-3\bar{c}, 3\bar{a}+2\bar{b}-5\bar{c}, -3\bar{a}+8\bar{b}-5\bar{c}, -3\bar{a}+2\bar{b}+\bar{c}$ are coplanar, where $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors.**

Sol: We take $\overline{OP}=-\bar{a}+4\bar{b}-3\bar{c}, \overline{OQ}=3\bar{a}+2\bar{b}-5\bar{c}$,

$\overline{OR}=-3\bar{a}+8\bar{b}-5\bar{c}, \overline{OS}=-3\bar{a}+2\bar{b}+\bar{c}$, where 'O' is the origin.

$\therefore \overline{PQ}=\overline{OQ}-\overline{OP}=(3\bar{a}+2\bar{b}-5\bar{c})-(-\bar{a}+4\bar{b}-3\bar{c})=4\bar{a}-2\bar{b}-2\bar{c}$

$\overline{PR}=\overline{OR}-\overline{OP}=(-3\bar{a}+8\bar{b}-5\bar{c})-(-\bar{a}+4\bar{b}-3\bar{c})=-2\bar{a}+4\bar{b}-2\bar{c}$

$\overline{PS}=\overline{OS}-\overline{OP}=(-3\bar{a}+2\bar{b}+\bar{c})-(-\bar{a}+4\bar{b}-3\bar{c})=-2\bar{a}-2\bar{b}+4\bar{c}$

Now, $[\overline{PQ}\ \overline{PR}\ \overline{PS}]=\begin{vmatrix}4 & -2 & -2\\-2 & 4 & -2\\-2 & -2 & 4\end{vmatrix}[\bar{a}\ \bar{b}\ \bar{c}]=[4(16-4)+2(-8-4)-2(4+8)][\bar{a}\ \bar{b}\ \bar{c}]$

$=[4(12)+2(-12)-2(12)]\ [\bar{a}\ \bar{b}\ \bar{c}]=[48-24-24][\bar{a}\ \bar{b}\ \bar{c}]=0\times[\bar{a}\ \bar{b}\ \bar{c}]=0$

So, $\overline{PQ}, \overline{PR}, \overline{PS}$ are coplanar.

Hence the four points

P,Q,R,S are coplanar.

13. For any three vectors $\bar{a},\bar{b},\bar{c}$ prove that $[\bar{b}+\bar{c}\ \ \bar{c}+\bar{a}\ \ \bar{a}+\bar{b}]=2[\bar{a}\bar{b}\bar{c}]$.

Sol: $$\left[\bar{b}+\bar{c}\ \ \bar{c}+\bar{a}\ \ \bar{a}+\bar{b}\right]=\begin{vmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{vmatrix}[\bar{a}\ \bar{b}\ \bar{c}]=[0(0-1)-1(0-1)+1(1-0)][\bar{a}\ \bar{b}\ \bar{c}]=2[\bar{a}\ \bar{b}\ \bar{c}]$$

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14. Prove that $\tan 70^\circ - \tan 20^\circ = 2\tan 50^\circ$

Sol: Consider $70^\circ - 20^\circ = 50^\circ \Rightarrow \tan(70^\circ - 20^\circ) = \tan 50^\circ$ Apply tan(A–B) formula

$$\Rightarrow \frac{\tan 70^\circ - \tan 20^\circ}{1+\tan 70^\circ \tan 20^\circ} = \tan 50^\circ \Rightarrow \frac{\tan 70^\circ - \tan 20^\circ}{1+\tan 70^\circ \cot 70^\circ} = \tan 50^\circ \qquad [\because \tan 20^\circ = \tan(90^\circ - 70^\circ) = \cot 70^\circ]$$

$$\Rightarrow \frac{\tan 70^\circ - \tan 20^\circ}{1+1} = \tan 50^\circ \Rightarrow \frac{\tan 70^\circ - \tan 20^\circ}{2} = \tan 50^\circ \Rightarrow \tan 70^\circ - \tan 20^\circ = 2\tan 50^\circ$$

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15. Solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$

Sol: Given equation is $\sin x + \sqrt{3}\cos x = \sqrt{2}$

On dividing by $\sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$, we get

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2} \Rightarrow \sin 30^\circ \sin x + \cos 30^\circ \cos x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6} = \cos\frac{\pi}{4} \Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}.$$ Here P.V is $\alpha = \frac{\pi}{4}$

$\therefore$ General solution is given by $\theta = 2n\pi \pm \alpha,\ n \in Z$

$$\Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4} \Rightarrow x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}, n \in Z$$

16. Prove that $\text{Tan}^{-1}\frac{1}{2}+\text{Tan}^{-1}\frac{1}{5}+\text{Tan}^{-1}\frac{1}{8}=\frac{\pi}{4}$

Sol: We know, $\text{Tan}^{-1}x+\text{Tan}^{-1}y=\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\therefore \text{Tan}^{-1}\frac{1}{2}+\text{Tan}^{-1}\frac{1}{5}=\text{Tan}^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}.\frac{1}{5}}\right)=\text{Tan}^{-1}\left(\frac{\frac{5+2}{10}}{\frac{10-1}{10}}\right)=\text{Tan}^{-1}\frac{7}{9}$$

$$\therefore \text{L.H.S}=\text{Tan}^{-1}\frac{7}{9}+\text{Tan}^{-1}\frac{1}{8}=\text{Tan}^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}.\frac{1}{8}}\right)=\text{Tan}^{-1}\left(\frac{\frac{56+9}{72}}{\frac{72-7}{72}}\right)$$

$$=\text{Tan}^{-1}\left(\frac{65}{65}\right)=\text{Tan}^{-1}1=\frac{\pi}{4}=\text{R.H.S}$$

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17. In a ΔABC if a:b:c = 7:8:9 then show that cosA:cosB:cosC = 14:11:6

Sol: Given that a:b:c = 7:8:9 $\Rightarrow$ a=7k, b=8k, c=9k

Now $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{(8k)^2+(9k)^2-(7k)^2}{2(8k)(9k)}=\frac{k^2(8^2+9^2-7^2)}{2(8)(9)(k^2)}$

$$=\frac{64+81-49}{(2)(8)(9)}=\frac{96}{(2)(8)(9)}=\frac{2}{3}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{(7k)^2+(9k)^2-(8k)^2}{(2)(7k)(9k)}=\frac{k^2(7^2+9^2-8^2)}{(2)(7)(9)(k^2)}$$

$$=\frac{49+81-64}{(2)(7)(9)}=\frac{66}{(2)(7)(9)}=\frac{11}{21}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{(7k)^2+(8k)^2-(9k)^2}{(2)(7k)(8k)}=\frac{k^2(7^2+8^2-9^2)}{(2)(7)(8)(k^2)}$$

$$=\frac{49+64-81}{(2)(7)(8)}=\frac{32}{(2)(7)(8)}=\frac{2}{7}$$

$$\therefore \cos A:\cos B:\cos C=\frac{2}{3}:\frac{11}{21}:\frac{2}{7}=21\left(\frac{2}{3}\right):21\left(\frac{11}{21}\right):21\left(\frac{2}{7}\right)=14:11:6$$

SECTION-C

18. Let $A = \{1,2,3\}$, $B=\{\alpha,\beta,\gamma\}$, $C=\{p,q,r\}$ and $f:A\rightarrow B, g : B \rightarrow C$ are defined by $f =\{(1,\alpha), (2,\gamma),(3,\beta)\}$, $g=\{(\alpha,q),(\beta,r),(\gamma,p)\}$ then show that f and g are bijective functions and $(gof)^{-1} = f^{-1}og^{-1}$

Sol: From the ordered pairs of f,g it is clear that both f & g are one-one and onto, hence bijective.

Given $f =\{(1,\alpha), (2,\gamma),(3,\beta)\}$, $g=\{(\alpha,q),(\beta,r),(\gamma,p)\}$

$\Rightarrow gof=\{(1,q),(2,p),(3,r)\} \Rightarrow (gof)^{-1}= \{(q,1),(p,2),(r,3)\}$

Also $g^{-1} =\{(q,\alpha), (r,\beta), (p,\gamma)\}$, $f^{-1} = \{(\alpha,1), (\gamma,2), (\beta,3)\}$

$\Rightarrow f^{-1}og^{-1} = \{(q,1), (p,2), (r,3)\}$.

19. Using P.M.I, prove that $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+n$ terms $=\frac{n}{3n+1}$

Sol: To find n^{th} term:

1,4,7... are in A.P with a=1, d=3

$\therefore T_n = a+(n-1)d \Rightarrow T_n=1+(n-1)3=1+3n-3=3n-2$

4,7,10... are in A.P with a=4, d=3

$\therefore T_n=4+(n-1)3=3n+1$

$\therefore n^{th}$ term is $T_n = \frac{1}{(3n-2)(3n+1)}$

Let $S(n) : \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

Step 1: L.H.S of $S(1) = \frac{1}{1.4}=\frac{1}{4}$; R.H.S of $S(1)=\frac{1}{3.1+1}=\frac{1}{4}$

$\therefore$ L.H.S = R.H.S.

So, S(1) is true

Step 2: Assume that S(k) is true ,for $k\in N$

$$S(k) : \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \quad .....(1)$$

Step 3: We show that S(k+1) is true

$(k+1)^{th}$ term $=\frac{1}{(3(k+1)-2)(3(k+1)+1)}=\frac{1}{(3k+1)(3k+4)}$

On adding $(k+1)^{th}$ term to both sides of (1), we get

$$L.H.S = \left[\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3k-2)(3k+1)}\right]+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)\cancel{(3k+1)}}{\cancel{(3k+1)}(3k+4)}=\frac{k+1}{3k+4}=\frac{k+1}{3k+3+1}=\frac{k+1}{3(k+1)+1}=R.H.S$$

$\therefore$ L.H.S =R.H.S.

So, S(k+1) is true whenever S(k) is true

Hence, by P.M.I the given statement is true, for all $n\in N$

20. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol: $$\text{L.H.S} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} \quad (\because R_2 \Rightarrow R_2 - R_1,\ R_3 \Rightarrow R_3 - R_1)$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & (b-a)(b+a) & (b-a)(b^2+ba+a^2) \\ 0 & (c-a)(c+a) & (c-a)(c^2+ca+a^2) \end{vmatrix} \quad \left(\begin{array}{l} \because a^2-b^2 = (a-b)(a-b) \\ a^3-b^3 = (a-b)(a^2+ab+b^2) \end{array} \right)$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & a+c & a^2+ac+c^2 \end{vmatrix}$$

(Taking $(b-a)$ common from R_2 and $(c-a)$ common form R_3)

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & ac-ab+c^2-b^2 \end{vmatrix} (\because R_3 \to R_3 - R_2)$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & a(c-b)+(c+b)(c-b) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & (c-b)(a+b+c) \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & 1 & (a+b+c) \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)1[(a+b)(a+b+c)-(a^2+ab+b^2)]$$

$$= (b-a)(c-a)(c-b)[(a+b)^2+c(a+b)-(a^2+ab+b^2)]$$

$$= (b-a)(c-a)(c-b)[(\cancel{a^2}+\cancel{b^2}+2ab)+(ca+cb)\cancel{-a^2}-ab\cancel{-b^2}]$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca) = \text{R.H.S}$$

21. By using Matrix inversion method, solve x+y+z=1, 2x+2y+3z=6, x+4y+9z=3 .

Sol: Matrix equation of the given system of equations is AX=D, where

$$A=\begin{bmatrix}1&1&1\\2&2&3\\1&4&9\end{bmatrix}; X=\begin{bmatrix}x\\y\\z\end{bmatrix}, D=\begin{bmatrix}1\\6\\3\end{bmatrix}$$

$\therefore$ The solution of AX=D is $X=A^{-1}D$

First we find A^{-1}

$$\det A=|A|=\begin{vmatrix}1&1&1\\2&2&3\\1&4&9\end{vmatrix}=1(18-12)-1(18-3)+1(8-2)=1(6)-1(15)+1(6)=6-15+6=-3\neq 0$$

The co-factor matrix of A is

$$\begin{bmatrix}+\begin{vmatrix}2&3\\4&9\end{vmatrix}&-\begin{vmatrix}2&3\\1&9\end{vmatrix}&+\begin{vmatrix}2&2\\1&4\end{vmatrix}\\-\begin{vmatrix}1&1\\4&9\end{vmatrix}&+\begin{vmatrix}1&1\\1&9\end{vmatrix}&-\begin{vmatrix}1&1\\1&4\end{vmatrix}\\+\begin{vmatrix}1&1\\2&3\end{vmatrix}&-\begin{vmatrix}1&1\\2&3\end{vmatrix}&+\begin{vmatrix}1&1\\2&2\end{vmatrix}\end{bmatrix}=\begin{bmatrix}(18-12)&-(18-3)&(4-2)\\-(9-4)&(9-1)&-(4-1)\\(3-2)&-(3-2)&(2-2)\end{bmatrix}=\begin{bmatrix}6&-15&6\\-5&8&-3\\1&-1&0\end{bmatrix}$$

$$\Rightarrow \text{Adj } A=\begin{bmatrix}6&-5&1\\-15&8&-1\\6&-3&0\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj}A)=\frac{1}{-3}\begin{bmatrix}6&-5&1\\-15&8&-1\\6&-3&0\end{bmatrix}$$

$$\text{Now, } X=A^{-1}D==\frac{1}{-3}\begin{bmatrix}6&-5&1\\-15&8&-1\\6&-3&0\end{bmatrix}\begin{bmatrix}1\\6\\3\end{bmatrix}=\frac{-1}{3}\begin{bmatrix}6-30+3\\-15+48-3\\6-18+0\end{bmatrix}=\frac{-1}{3}\begin{bmatrix}-21\\30\\-12\end{bmatrix}=\begin{bmatrix}7\\-10\\4\end{bmatrix}$$

$$\therefore X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}7\\-10\\4\end{bmatrix}$$

$\therefore$ The solution is x=7, y= –10, z=4

22. If $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k}$, $\bar{b} = \bar{i} - 2\bar{j} + \bar{k}$, $\bar{c} = -\bar{i} + \bar{j} - 4\bar{k}$ and $\bar{d} = \bar{i} + \bar{j} + \bar{k}$, then compute $\left|\left(\bar{a}x\bar{b}\right)x\left(\bar{c}x\bar{d}\right)\right|$.

Sol: Given $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k}$, $\bar{b} = \bar{i} - 2\bar{j} + \bar{k}$, $\bar{c} = -\bar{i} + \bar{j} - 4\bar{k}$, $\bar{d} = \bar{i} + \bar{j} + \bar{k}$,

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = \bar{i}(1-6) - \bar{j}(2+3) + \bar{k}(-4-1) = -5\bar{i} - 5\bar{j} - 5\bar{k} \quad \text{............(1)}$$

$$\bar{c}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 1 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \bar{i}(1+4) - \bar{j}(-1+4) + \bar{k}(-1-1) = 5\bar{i} - 3\bar{j} - 2\bar{k} \quad \text{............(2)}$$

From (1) & (2),

$$\therefore (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -5 & -5 & -5 \\ 5 & -3 & -2 \end{vmatrix} = \bar{i}(10-15) - \bar{j}(10+25) + \bar{k}(15+25) = -5\bar{i} - 35\bar{j} + 40\bar{k}$$

$$= 5(-\bar{i} - 7\bar{j} + 8\bar{k})$$

$$\therefore |(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})| = 5\sqrt{(-1)^2 + (-7)^2 + 8^2} = 5\sqrt{1+49+64} = 5\sqrt{114}$$

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23. If A+B+C=180°, then show that sin2A+sin2B+sin2C=4sinAsinBsinC.

Sol: L.H.S = sin2A+sin2B+sin2C

$= 2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \sin 2C$ $\quad \left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

= 2sin(A+B)cos(A–B)+2sinCcosC $\quad [\because \sin 2\theta = 2\sin\theta\cos\theta]$

= 2 sin (180°–C) cos(A–B)+2sinCcosC $\quad$ [∵ (A+B)+C=180°]

= 2sinC cos(A–B)+2sinCcosC $\quad [\because \sin(180^\circ - \theta) = \sin\theta]$

= 2sinC[cos(A–B)+cosC] $\quad$ [Taking 2sinC common]

= 2sinC[cos(A–B)+cos(180°–(A+B)] $\quad$ [∵ (A+B)+C=180°]

= 2sinC[cos(A–B)–cos(A+B)] $\quad [\because \cos(180^\circ - \theta) = -\cos\theta]$

= 2sinC (2sinA sinB) $\quad$ [∵ cos(A–B)–cos(A+B)=2sinAsinB]

= 4sinA sinB sinC = R.H.S

24. If $r_1 = 2$, $r_2 = 3$, $r_3 = 6$, $r = 1$, then prove that $a = 3$, $b = 4$, $c = 5$.

Sol: Given $r_1 = 2$, $r_2 = 3$, $r_3 = 6$ and $r = 1$, then

$$\Delta = \sqrt{rr_1r_2r_3} = \sqrt{1\times2\times3\times6} = \sqrt{36} = 6$$

Now, $r = \frac{\Delta}{s} \Rightarrow 1 = \frac{6}{s} \quad \therefore s = 6$

(i) $r_1 = \frac{\Delta}{s-a} \Rightarrow s - a = \frac{\Delta}{r_1} = \frac{6}{2} = 3$

$\therefore s - a = 3 \Rightarrow 6 - a = 3$

$\Rightarrow a = 6 - 3 = 3$

(ii) $r_2 = \frac{\Delta}{s-b} \Rightarrow s - b = \frac{\Delta}{r_2} = \frac{6}{3} = 2$

$\therefore s - b = 2 \Rightarrow 6 - b = 2$

$\Rightarrow b = 6 - 2 = 4$

(iii) $r_3 = \frac{\Delta}{s-c} \Rightarrow s - c = \frac{\Delta}{r_3} = \frac{6}{6} = 1$

$\therefore s - c = 1 \Rightarrow 6 - c = 1$

$\Rightarrow c = 6 - 1 = 5$