Previous IPE

SOLVED PAPERS

MARCH -2024 (AP)

PREVIOUS PAPERS

IPE: MARCH-2024(AP)

MATHS-2B

Time : 3 Hours Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: $10 \times 2 = 20$

1. Find the equation of the circle with $(4, 2), (1, 5)$ as ends of a diameter.
2. Find the value of k if the length of the tangent from (2,5) to $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$.
3. Find k if the pairs of circles $x^2+y^2+4x+8=0$, $x^2+y^2-16y+k=0$ are orthogonal.
4. Find the coordinates of the point on the parabola $y^2=8x$, whose focal distance is 10.
5. If the angle between the asymptotes is 30° then find its eccentricity.
6. Evaluate $\int \sec^2x.\csc^2x dx$ **7.** Find $\int e^{\log(1+\tan^2x)}dx$. **8.** Find $\int_0^{\pi/2} \cos^7x\sin^2x dx$
9. Find the area of the region enclosed by the given curves $x = 4 - y^2$, $x = 0$.
10. Find the order and degree of $\left(\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right)^{6/5} = 6y$

SECTION-B

II. Answer any FIVE of the following SAQs: $5 \times 4 = 20$

11. Find the length of the chord intercepted by the circle $x^2+y^2-x+3y-22=0$ onthe line $y = x - 3$
12. S.T the circles $x^2+y^2-8x-2y+8 = 0$, $x^2+y^2-2x+6y+6 =0$ touch each other and find the point of contact.
13. Find the equation of tangent and normal to the ellipse $9x^2+16y^2=144$ at the endof the latus rectum in the first quadrant.
14. If P(x, y) is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci S & S' then SP + S'P is a constant.
15. Find the centre, eccentricity, foci, length of latus rectum and equations of the directrices of $x^2 - 4y^2 = 4$
16. Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ **17.** Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{1+x^2}{1+x^3}$

SECTION-C

III. Answer any FIVE of the following LAQs: $5 \times 7 = 35$

18. Find the values of c if the points (2,0), (0,1), (4,5), (0,c) are concyclic.
19. Find the equation to the pair of transverse common tangents to the circles $x^2 + y^2 - 4x - 10y + 28 = 0$ and $x^2 + y^2 + 4x - 6y + 4 = 0$
20. Define parabola. Derive its equation in the standard form.
21. Evaluate the reduction formula for $I_n = \int \sin^n x dx$ and hence find $\int \sin^4 x dx$
22. Evaluate $\int \frac{x+1}{x^2+3x+12}dx$.
23. Evaluate $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx$ **24.** Solve $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

IPE AP MARCH-2024
SOLUTIONS

SECTION-A

1. Find the equation of the circle with (4, 2), (1, 5) as ends of a diameter.

Sol: **Formula:** The equation of the circle with $A(x_1,y_1)$, $B(x_2,y_2)$ as ends of a diameter is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\therefore$ Equation of the required circle is $(x-4)(x-1)+(y-2)(y-5)=0$

$\Rightarrow (x^2-x-4x+4)+(y^2-5y-2y+10)=0 \Rightarrow x^2+y^2-5x-7y+14=0$

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2. Find the value of k if the length of the tangent from (2,5) to $x^2+y^2-5x+4y+k=0$ is $\sqrt{37}$.

Sol: Length of the tangent from (2, 5) to

$S = x^2+y^2-5x+4y+k = 0$ is $\sqrt{S_{11}} = \sqrt{37}$;

On squaring both sides, we get $S_{11} = 37$

$\Rightarrow (2)^2 + 5^2 - 5(2) + 4(5) + k = 37 \quad \Rightarrow 4 + 25 - 10 + 20 + k = 37$

$\Rightarrow 39 + k = 37 \Rightarrow k = -2$

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3. Find *k* if the pairs of circles $x^2+y^2+4x+8 = 0$, $x^2+y^2-16y+k = 0$ are orthogonal.

Sol: From the given circles, we get

$g = 2,\ f = 0,\ c = 8$ and $g' = 0,\ f' = -8,\ c' = k$

Orthogonal condition: $2gg' + 2ff' = c+c' \quad \Rightarrow 2(2)(0) + 2(0)(-8) = 8 + k \Rightarrow k = -8$

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4. Find the coordinates of the point on parabola $y^2 = 8x$, whose focal distance is 10.

Sol: Given parabola is $y^2 = 8x \quad \Rightarrow 4a = 8 \quad \Rightarrow a = 2$

Given focal distance $SP = 10$

Formula: Focal distance $SP = x_1+a \quad \Rightarrow x_1 + 2 = 10 \quad \Rightarrow x_1 = 8$.

But, $y_1^2 = 8x_1 \Rightarrow y_1^2 = 8(8) \Rightarrow y_1 = \pm 8$

$\therefore\ P(x_1,y_1) = (8, \pm 8)$

5. If the angle between the asymptotes is 30° then find its eccentricity.

Sol: The angle between the asymptotes of the hyperbola S = 0 is $2\text{Sec}^{-1}e$

$\therefore\ 2\text{Sec}^{-1}e = 30^\circ \Rightarrow \text{Sec}^{-1}e = 15^\circ \quad \Rightarrow e = \sec 15^\circ = \sqrt{6} - \sqrt{2}$

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6. Evaluate $\int \sec^2 x.\csc^2 x dx$

Sol: $\int \sec^2 x.\csc^2 x dx = \int \frac{1}{\cos^2 x \sin^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x} dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx$

$= \int (\sec^2 x + \csc^2 x) dx$

$= \int \sec^2 x dx + \int \csc^2 x dx$

$= \tan x - \cot x + c$

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7. Find $\int e^{\log(1+\tan^2 x)} dx$.

Sol: $I = \int e^{\log_e (1+\tan^2 x)} dx = \int (1+\tan^2 x) dx = \int \sec^2 x\, dx = \tan x + c \qquad \left[\because \int e^{\log_e f(x)} = f(x) \right]$

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8. Find $\int_0^{\pi/2} \cos^7 x \sin^2 x dx$

Sol: $\int_0^{\pi/2} \cos^7 x \sin^2 x\, dx = \frac{[(6)(4)(2)][(1)]}{(9)(7)(5)(3)} = \frac{16}{315}$

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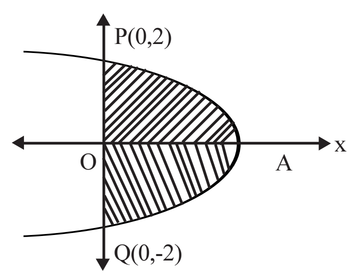
9. Find the area of the region enclosed by the given curves $x = 4 - y^2$, $x = 0$.

Sol : The given curve $x = 4 - y^2$ is a horizontal (left) parabola.

Put y = 0, then we get A = (4,0).

Put x = 0 then we get P(0, 2) and Q(0, –2).



From the diagram, Required area = 2 Area of OAP

$A = 2\int_0^2 x dy = 2\int_0^2 (4 - y^2) dy = 2\left(4y - \frac{y^3}{3} \right)_0^2 = 2.\frac{16}{3} = \frac{32}{3}$ sq. units

10. Find the order and degree of the differential equation $\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right]^{\frac{6}{5}}=6y$

Sol: Given D.E is $\left(\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right)^{6/5}=6y$

$$\Rightarrow \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=(6y)^{5/6}$$

Here, the highest order derivative is $\frac{d^2y}{dx^2}$ $\therefore$ order = 2

The exponent of $\frac{d^2y}{dx^2}$ is 1 $\therefore$ degree = 1

SECTION-B

11. Find the length of the chord intercepted by the circle $x^2 + y^2 - x + 3y - 22 = 0$ on the line $y = x - 3$.

Sol: Given circle $x^2 + y^2 - x + 3y - 22 = 0$

It's Centre, C = (1/2,–3/2)

$$\text{radius, } r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 22} = \sqrt{\frac{1}{4} + \frac{9}{4} + 22} = \sqrt{\frac{1+9+88}{4}} = \sqrt{\frac{98}{4}} = \sqrt{\frac{49}{2}} = \frac{7}{\sqrt{2}}$$

Perpendicular distance from the centre(1/2, –3/2) to the line $y = x-3 = 0 \Rightarrow x - y - 3 = 0$

$$\text{is } p = \frac{\left|\frac{1}{2} + \frac{3}{2} - 3\right|}{\sqrt{1^2 + 1^2}} = \frac{\left|\frac{1+3-6}{2}\right|}{\sqrt{2}} = \frac{\left|\frac{-2}{2}\right|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \text{ Length of the chord } = 2\sqrt{r^2 - p^2} = 2\sqrt{\left(\frac{49}{2}\right) - \frac{1}{2}} = 2\sqrt{\frac{48}{2}} = 2\sqrt{24}$$

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12. Show that the circles $x^2 + y^2 - 8x - 2y + 8 = 0$, $x^2 + y^2 - 2x + 6y + 6 = 0$ touch each other and find the point of contact.

Sol: Given circle is $S = x^2+y^2-8x-2y+8=0$, centre $C_1=(4,1)$, radius $r_1 = \sqrt{16+1-8} = \sqrt{9} = 3$

Other circle is $S'=x^2+y^2-2x+6y+6=0$, centre $C_2=(1,-3)$, radius $r_2 = \sqrt{1+9-6} = \sqrt{4} = 2$

$$C_1C_2 = \sqrt{(4-1)^2 + (1+3)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Also, $r_1 + r_2 = 3 + 2 = 5$. Here, $C_1C_2 = r_1 + r_2$

$\therefore$ the two circles touch each other externally.

Also the point of contact I divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 3:2$ internally.

$$\therefore \text{Point of contact } I = \left(\frac{3(1)+2(4)}{3+2}, \frac{3(-3)+2(1)}{3+2}\right) = \left(\frac{11}{5}, \frac{-7}{5}\right)$$

13. Find the equation of tangent and normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of the latus rectum in the first quadrant.

Sol: Given Ellipse $9x^2 + 16y^2 = 144 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow a^2 = 16, b^2 = 9 \Rightarrow a = 4, b = 3$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{16-9}{16}} = \sqrt{\frac{7}{16}} \Rightarrow e = \frac{\sqrt{7}}{4}$$

Positive end of latus rectum $L = \left(ae, \frac{b^2}{a}\right) = \left(4 . \frac{\sqrt{7}}{4}, \frac{9}{4}\right) = \left(\sqrt{7}, \frac{9}{4}\right)$

Equation of the tangent at L is $S_1 = 0$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \Rightarrow \frac{x\sqrt{7}}{16} + \frac{y}{9}\left(\frac{9}{4}\right) = 1 \Rightarrow \sqrt{7}x + 4y = 16$$

Equation of the normal at L is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \Rightarrow \frac{16x}{\sqrt{7}} - \frac{9y}{\frac{9}{4}} = 16 - 9$

$$\Rightarrow \frac{16x}{\sqrt{7}} - 4y = 7 \Rightarrow 16x - 4\sqrt{7}y = 7\sqrt{7}$$

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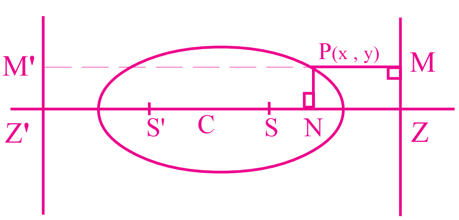
14. If P(x, y) is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci S & S' then $SP + S'P$ is a constant.

Proof: Let N be the foot of the perpendicular from P(x, y) on to the x-axis.

from the diagram, $PM = NZ = CZ - CN = \frac{a}{e} - x$

and $PM' = NZ' = CN + CZ' = x + \frac{a}{e}$



Now, $\frac{SP}{PM} = e \Rightarrow SP = ePM = e\left(\frac{a}{e} - x\right) = a - ex$

and $\frac{S'P}{PM'} = e \Rightarrow S'P = ePM' = e\left(x + \frac{a}{e}\right) = ex + a = a + ex$

$\therefore SP + S'P = (a - ex) + (a + ex) = 2a$ which is a constant

15. Find the centre, eccentricity, foci, length of latus rectum and equations of the directrices of the Hyperbola $x^2 - 4y^2 = 4$

Sol: Given hyperbola is $x^2 - 4y^2 = 4 \Rightarrow \frac{x^2}{4} - \frac{y^2}{1} = 1$. Here $a^2=4$, $b^2=1$

(i) Centre C= (0,0)

(ii) Eccentricity $e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{4+1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

(iii) Foci $= (\pm ae, 0) = \left(\pm 2\left(\frac{\sqrt{5}}{2}\right), 0\right) = \left(\pm\sqrt{5}, 0\right)$

(iv) Equation of the directrices is $x = \pm\frac{a}{e} \Rightarrow x = \pm\frac{2}{\frac{\sqrt{5}}{2}} \Rightarrow x = \pm\frac{4}{\sqrt{5}}$

(v) Length of latusrectum $= \frac{2b^2}{a} = \frac{2(1)}{2} = 1$

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16. Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Sol: We know $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

$$\therefore I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \text{ ....(1)} \quad = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)} + \sqrt{\cos\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)}} dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \text{ ....(2)}$$

From (1) and (2), $I + I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_{\pi/6}^{\pi/3} 1 dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \Rightarrow 2I = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

17. Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{1+x^2}{1+x^3}$

Sol: Given D.E is $\frac{dy}{dx}+y\left(\frac{3x^2}{1+x^3}\right)=\frac{1+x^2}{1+x^3}$. This is a linear D.E in y.

It is in the form $\frac{dy}{dx}+yP(x)=Q(x)$ where $P(x)=\frac{3x^2}{1+x^3}$ and $Q(x)=\frac{1+x^2}{1+x^3}$

Here, $P(x)=\frac{3x^2}{1+x^3} \Rightarrow \int P(x)\,dx=\int\frac{3x^2}{1+x^3}dx=\log(1+x^3)$ $\left[\because \int\frac{f'(x)}{f(x)}=\log f(x)\right]$

$\therefore I.F=e^{\int P(x)dx}=e^{\log(1+x^3)}=1+x^3$

$\therefore$ The solution is $y(I.F)=\int(I.F)Q(x)dx$

$\Rightarrow y(1+x^3)=\int \cancel{(1+x^3)}\left(\frac{1+x^2}{\cancel{1+x^3}}\right)dx=\int(1+x^2)dx=x+\frac{x^3}{3}+c$

$\therefore y(1+x^3)=x+\frac{x^3}{3}+c$

SECTION-C

18. Find the values of c if the points (2, 0), (0, 1), (4, 5), (0, c) are concyclic.

Sol: Let A = (2, 0), B = (0, 1), C = (4, 5), D = (0, c)

We take $S(x_1, y_1)$ as the centre of the circle $\Rightarrow SA = SB = SC$

Now, $SA = SB \Rightarrow SA^2 = SB^2 \Rightarrow (x_1-2)^2+(y_1-0)^2=(x_1-0)^2+(y_1-1)^2$

$\Rightarrow (\cancel{x_1^2} - 4x_1 + 4) + (\cancel{y_1^2}) = (\cancel{x_1^2}) + (\cancel{y_1^2} - 2y_1 + 1)$

$\Rightarrow 4x_1 - 2y_1 + 1 - 4 = 0 \Rightarrow 4x_1 - 2y_1 - 3 = 0$.........(1)

Also, $SB = SC \Rightarrow SB^2 = SC^2 \Rightarrow (x_1-0)^2+(y_1-1)^2=(x_1-4)^2+(y_1-5)^2$

$\Rightarrow (\cancel{x_1^2}) + (\cancel{y_1^2} - 2y_1 + 1) = (\cancel{x_1^2} - 8x_1 + 16) + (\cancel{y_1^2} - 10y_1 + 25)$

$\Rightarrow 8x_1 - 2y_1 + 10y_1 + 1 - 16 - 25 = 0 \Rightarrow 8x_1 + 8y_1 - 40 = 0 \Rightarrow 8(x_1 + y_1 - 5) = 0 \Rightarrow x_1 + y_1 - 5 = 0$......(2)

Solving (1) & (2) we get the centre $S(x_1, y_1)$

$2 \times (2) \Rightarrow 2x_1 + 2y_1 - 10 = 0$...........(3)

$(1) + (3) \Rightarrow 6x_1 - 13 = 0. \Rightarrow 6x_1 = 13 \Rightarrow x_1 = 13/6$

$(2) \Rightarrow y_1 = 5 - x_1 = 5 - \frac{13}{6} = \frac{30-13}{6} = \frac{17}{6} \Rightarrow y_1 = \frac{17}{6}$

$\therefore$ Centre of the circle is $S(x_1, y_1) = \left(\frac{13}{6}, \frac{17}{6}\right)$

Also, we have A = (2,0) Hence, radius $r = SA \Rightarrow r^2 = SA^2$

$\therefore r^2 = SA^2 = \left(2 - \frac{13}{6}\right)^2 + \left(0 - \frac{17}{6}\right)^2 = \left(\frac{12-13}{6}\right)^2 + \left(\frac{17}{6}\right)^2 = \left(\frac{1}{36}\right) + \left(\frac{289}{36}\right) = \frac{290}{36}$

$\therefore$ Circle with Centre, $\left(\frac{13}{6}, \frac{17}{6}\right)$ and $r^2 = \frac{290}{36}$ is $\left(x - \frac{13}{6}\right)^2 + \left(y - \frac{17}{6}\right)^2 = \frac{290}{36}$

Put D(0, c) in the above equation $\Rightarrow \left(0 - \frac{13}{6}\right)^2 + \left(c - \frac{17}{6}\right)^2 = \frac{290}{36} \Rightarrow \left(c - \frac{17}{6}\right)^2 = \frac{290}{36} - \frac{169}{36} = \frac{121}{36}$

$\Rightarrow \left(\frac{6c-17}{6}\right)^2 = \frac{121}{36} \Rightarrow \frac{(6c-17)^2}{\cancel{36}} = \frac{11^2}{\cancel{36}} \Rightarrow 6c - 17 = \pm 11$

$\Rightarrow 6c = \pm 11 + 17 \Rightarrow 6c = 28 \Rightarrow c = \frac{\cancel{28}}{\cancel{6}} = \frac{14}{3}$ (or) $\cancel{6}c = \cancel{6} \Rightarrow c = 1$

$\therefore$ c= 14/3 (or) 1

19. Find the equation to the pair of transverse common tangents to the circles $x^2+y^2-4x-10y+28=0$ and $x^2+y^2+4x-6y+4=0$

Sol: For the circle $x^2 + y^2 - 4x - 10y + 28 = 0$, centre $C_1 = (2, 5)$,

$$\text{radius } r_1 = \sqrt{(-2)^2 + (-5)^2 - 28} = \sqrt{1} = 1$$

For the circle $x^2 + y^2 + 4x - 6y + 4 = 0$, centre $C_2 = (-2, 3)$,

$$\text{radius } r_2 = \sqrt{2^2 + (-3)^2 - 4} = \sqrt{9} = 3$$

The internal centre of similitude, I divides C_1C_2 internally in the ratio $r_1 : r_2 = 1:3$

$$\therefore I = \left(\frac{1(-2)+3(2)}{1+3}, \frac{1(3)+3(5)}{1+3}\right) = \left(\frac{4}{4}, \frac{18}{4}\right) = \left(1, \frac{9}{2}\right)$$

The equation to the pair of transverse common tangents is $S_1^2 = S_{11}(S)$

$$\Rightarrow \left[x + \frac{9}{2}y - 2(x+1) - 5\left(y + \frac{9}{2}\right) + 28\right]^2 = \left(1 + \frac{81}{4} - 4 - 45 + 28\right)(x^2 + y^2 - 4x - 10y + 28)$$

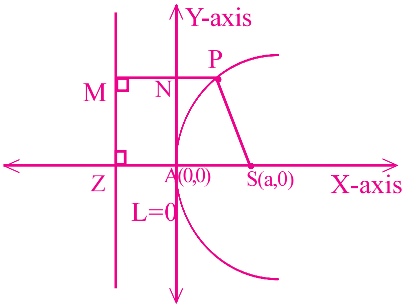
$$\Rightarrow \left(-x - \frac{y}{2} + \frac{7}{2}\right)^2 = \frac{1}{4}(x^2 + y^2 - 4x - 10y + 28) \Rightarrow \frac{(2x - y + 7)^2}{4} = \frac{1}{4}(x^2 + y^2 - 4x - 10y + 28)$$

$$\Rightarrow (2x + y - 7)^2 = x^2 + y^2 - 4x - 10y + 28$$

$$\Rightarrow 4x^2 + y^2 + 49 + 4xy - 28x - 14y = x^2 + y^2 - 4x - 10y + 28 \Rightarrow 3x^2 + 4xy - 24x - 4y + 21 = 0$$

20. Derive the standard form of the parabola.

Sol: Let S be the focus and L=0 be the directrix of the parabola.



Let Z be the projection of S on to the directrix

Let A be the mid point of SZ

$\Rightarrow SA = AZ \Rightarrow \dfrac{SA}{AZ} = 1$

$\Rightarrow$ A is a point on the parabola.

Take AS, the principle axis of the parabola as X-axis and

the line perpendicular to AS through A as the Y-axis $\Rightarrow A=(0,0)$

Let $AS = a \Rightarrow S = (a,0),\ Z = (-a,0)$

$\Rightarrow$ the equation of the directrix is $x = -a \Rightarrow x + a = 0$

Let $P(x_1, y_1)$ be any point on the parabola.

N be the projection of P on to the Y-axis.

M be the projection of P on to the directrix.

Here $PM=PN+NM = x_1+a$ $\quad (\because PN = x - \text{coordinate of } P \text{ and } NM = AZ = AS = a)$

Now, by the focus directrix property of the parabola,

we have $\dfrac{SP}{PM} = 1 \Rightarrow SP = PM \Rightarrow SP^2 = PM^2$

$\Rightarrow (x_1 - a)^2 + (y_1 - 0)^2 = (x_1 + a)^2$

$\Rightarrow y_1^2 = (x_1 + a)^2 - (x_1 - a)^2$

$\Rightarrow y_1^2 = 4ax_1$ $\quad [\because (a+b)^2 - (a-b)^2 = 4ab]$

$\therefore$ The equation of locus of $P(x_1, y_1)$ is $y^2 = 4ax$

21. Evaluate the reduction formula for $I_n = \int \sin^n x dx$ and hence find $\int \sin^4 x dx$

Sol: Given, $I_n = \int \sin^n x dx = \int \sin^{n-1} x(\sin x)dx$.

We take First function $u = \sin^{n-1} x$ and Second function $v = \sin x \Rightarrow \int v = -\cos x$

From By parts rule, we have

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cos x + (n\text{-}1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x \cos x \ +(n-1)[\int \sin^{n-2} xdx - \int \sin^n xdx]$$

$$= -\sin^{n-1} x \cos x + (n-1)[I_{n-2} - I_n]$$

$$I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - nI_n + I_n$$

$$\Rightarrow nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} + \cancel{I_n} - \cancel{I_n}$$

$$\Rightarrow I_n = \frac{-\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right)I_{n-2} \quad ....(1)$$

Put n = 4, 2,0 successively in (1), we get

$$I_4 = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}I_2$$

$$= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left[-\frac{\sin x \cos x}{2} + \frac{1}{2}I_0\right]$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3}{8}I_0$$

$$= -\frac{\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3}{8}x + c \qquad [\because I_0 = x]$$

22. Evaluate $\int \frac{x+1}{x^2+3x+12}dx$

Sol: Let $x+1 = A\frac{d}{dx}(x^2+3x+12) + B$ ....(i)

$\therefore x+1 = A(2x+3)+B \Rightarrow x+1 = 2Ax + (3A+B)$

Equating the coefficients of 'x', we get $2A=1 \Rightarrow A=1/2$

Equating the constant terms, we get $3A+B=1 \Rightarrow B=1-3A=1-\frac{3}{2} \Rightarrow B=-\frac{1}{2}$

Putting $A=\frac{1}{2}, B=\frac{-1}{2}$ in (i), we get Nr. $x+1=\frac{1}{2}\frac{d}{dx}(x^2+3x+12)-\frac{1}{2}$

$$\therefore I = \int \frac{x+1}{x^2+3x+12}dx = \frac{1}{2}\int \frac{\frac{d}{dx}(x^2+3x+12)}{x^2+3x+12}dx - \frac{1}{2}\int \frac{dx}{x^2+3x+12}$$

$$= \frac{1}{2}\log\left|x^2+3x+12\right| - \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{39}}{2}\right)^2}$$

$$= \frac{1}{2}\log\left|x^2+3x+12\right| - \frac{1}{\cancel{2}}\frac{\cancel{2}}{\sqrt{39}}\text{Tan}^{-1}\left(\frac{x+\frac{3}{2}}{\frac{\sqrt{39}}{2}}\right)+c$$

$$= \frac{1}{2}\log\left|x^2+3x+12\right| - \frac{1}{\sqrt{39}}\text{Tan}^{-1}\left(\frac{2x+3}{\sqrt{39}}\right)+c$$

$\because x^2+3x+12 = x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+12$

$= \left(x^2+3x+\left(\frac{3}{2}\right)^2\right)-\frac{9}{4}+12 = \left(x+\frac{3}{2}\right)^2-\frac{9+48}{4}$

$= \left(x+\frac{3}{2}\right)^2+\frac{39}{4}$

$\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\text{Tan}^{-1}\frac{x}{a}$

23. **Evaluate** $\int_{0}^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

Sol: Here, we take the substitution sinx–cosx=t. Then (cosx+sinx)dx=dt.

Now x = 0 $\Rightarrow$ t = sin0–cos0 = 0–1= –1 and $x = \frac{\pi}{4} \Rightarrow t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

Also, $(\sin x - \cos x)^2 = t^2 \Rightarrow \sin^2 x + \cos^2 x - 2\sin x\cos x = t^2 \Rightarrow 1 - \sin 2x = t^2 \Rightarrow \sin 2x = 1 - t^2$

$\therefore 9 + 16\sin 2x = 9 + 16(1 - t^2) = 9 + 16 - 16t^2 = 25 - 16t^2$

$$\therefore I = \int_{0}^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx = \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \int_{-1}^{0} \frac{dt}{5^2 - (4t)^2}$$

$$= \frac{1}{4} \cdot \frac{1}{2(5)} \cdot \log\left[\frac{5 + 4t}{5 - 4t}\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log\left[\frac{5+0}{5-0}\right] - \log\left[\frac{5-4}{5+4}\right]\right] = \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right]$$

$$= \frac{1}{40}\left[0 - \log 9^{-1}\right] = \frac{1}{40}[\log 9] = \frac{\log 3^2}{40} = \frac{2\log 3}{40} = \frac{\log 3}{20}$$

24. Solve $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

Sol: Given that $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0 \Rightarrow (1+e^{x/y})dx = -e^{x/y}\left(1-\frac{x}{y}\right)dy$

$$\Rightarrow \frac{dy}{dx} = -\frac{(1+e^{x/y})}{e^{x/y}\left(1-\frac{x}{y}\right)} \Rightarrow \frac{dx}{dy} = -\left[\frac{e^{x/y}(1-(x/y))}{1+e^{x/y}}\right] \text{.....(1)}$$

Put $\frac{x}{y} = v \Rightarrow x = vy$ differentiating w.r.to y, we have $\frac{dx}{dy} = v + y.\frac{dv}{dy}$

$$\therefore (1) \Rightarrow v + y\left(\frac{dv}{dy}\right) = -\left[\frac{e^{v}(1-v)}{1+e^{v}}\right]$$

$$\Rightarrow y\left(\frac{dv}{dy}\right) = -\left[\frac{e^{v}(1-v)}{1+e^{v}} + v\right] = -\left[\frac{e^{v} - v.e^{v} + v + ve^{v}}{1+e^{v}}\right] = -\left[\frac{v+e^{v}}{1+e^{v}}\right]$$

$$\Rightarrow \left(\frac{1+e^{v}}{v+e^{v}}\right)dv = -\left(\frac{1}{y}\right)dy \Rightarrow \int \frac{1+e^{v}}{v+e^{v}}dv = -\int \frac{1}{y}dy$$

$$\Rightarrow \log|v+e^{v}| = -\log|y| + \log|c| \Rightarrow \log|v+e^{v}| + \log|y| = \log|c|$$

$$\Rightarrow \log|(v+e^{v})y| = \log|c| \Rightarrow \left[\frac{x}{y} + e^{x/y}\right]y = c \Rightarrow x + ye^{x/y} = c$$