Previous IPE

SOLVED PAPERS

MARCH -2023 (TS)

PREVIOUS PAPERS

IPE: MARCH-2023(TS)

Time : 3 Hours **MATHS-1B** Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: $10 \times 2 = 20$

1. Find the slope of the straight line passing through the points (3, 4), (7, –6).
2. Transform the following straight line equation into normal form $3x + 4y = 5$.
3. Find the centroid of the tetrahedron whose vertices are (2,3, –4), (–3, 3, –2), (–1, 4, 2), (3, 5, 1)
4. Write the equation of the plane $4x – 4y + 2z + 5 = 0$ in the intercept form.
5. Evaluate $\underset{x\to 2}{Lt}\left[\frac{1}{x-2}-\frac{4}{x^2-4}\right]$
6. Evaluate $\underset{x\to 0}{Lt}\frac{e^{7x}-1}{x}$
7. If $y = \log(\sin(\log x))$, find dy/dx.
8. Find the derivative of $Sin^{-1}(3x-4x^3)$ w.r.to x.
9. Find the slope of the tangent to the curve $y = 5x^2$ at (–1, 5).
10. Find Δy and dy for the function $y = x^2 + x$, when $x = 10, \Delta x = 0.1$

SECTION-B

II. Answer any FIVE of the following SAQs: $5 \times 4 = 20$

11. The ends of the hypotenuse of a right angled triangle are (0, 6) and (6, 0). Find the equation of locus of its third vertex.
12. Find the transformed equation of $x^2+2\sqrt{3}\,xy-y^2 = 2a^2$, when the axes are rotated through an angle $\pi/6$.
13. If Q(h,k) is the foot of the perpendicular of $P(x_1, y_1)$ on the line ax+by+c=0 then prove that $(h-x_1) : a = (k-y_1) : b = -(ax_1+by_1+c) : (a^2+b^2)$.
14. Evaluate $\underset{x\to 0}{Lt}\frac{1-\cos 2mx}{\sin^2 nx}\ (m, n \in Z)$
15. Find the derivative of tan2x from the first principle.
16. If the increase in the side of a square is 2% then find the approximate percentage of increase in the area of the square.
17. Find the length of subtangent, subnormal at a point 't' on the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.

SECTION-C

III. Answer any FIVE of the following LAQs: $5 \times 7 = 35$

18. Find the orthocentre of the triangle whose vertices are (–2,–1),(6, –1), (2, 5)
19. If θ is the angle between the pair of lines $ax^2+2hxy+by^2=0$ then $\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$
20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line $x+2y=k$ are mutually perpendicular.
21. Find the angle between two diagonals of a cube.
22. If $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$ then prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$
23. Show that the curves $y^2 = 4(x+1)$, $y^2 = 36(9-x)$ intersect orthogonally.
24. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?

IPE TS MARCH-2023

SOLUTIONS

SECTION-A

1. Find the slope of the straight line passing through the points (3, 4), (7, –6).

A: Given $A = (x_1, y_1) = (3, 4)$; $B = (x_2, y_2) = (7, -6)$

Slope of the line joining A (3,4) and (7,–6) is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-4}{7-3} = \frac{-10}{4} = \frac{-5}{2}$

---

2. Transform the equation 3x + 4y = 5 into Normal form

A: Given equation is $3x + 4y = 5$ $\quad [\because p \geq 0]$

Dividing by $\sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$, we have $\frac{3}{5}x + \frac{4}{5}y = \frac{5}{5} = 1$

Here $p = 1, \cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}$. Hence $\tan\alpha = \frac{4}{3} \Rightarrow \alpha = \mathrm{Tan}^{-1}\left(\frac{4}{3}\right) \in Q_1$

---

3. Find the centroid of the tetrahedron whose vertices are (2, 3 –4) (–3, 3, –2), (–1, 4, 2), (3, 5, 1)

A: Let, A = (2, 3, –4), B = (–3, 3, –2), C = (–1, 4, 2), D = (3, 5, 1)

$$\text{Centroid } G = \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

$$= \left(\frac{2 - 3 - 1 + 3}{4}, \frac{3 + 3 + 4 + 5}{4}, \frac{-4 - 2 + 2 + 1}{4}\right) = \left(\frac{1}{4}, \frac{15}{4}, \frac{-3}{4}\right)$$

---

4. Write the equation of the plane 4x – 4y + 2z + 5 = 0 in the intercept form.

A: The given equation of the plane is $4x - 4y + 2z + 5 = 0 \Rightarrow 4x - 4y + 2z = -5$

$$\Rightarrow \frac{4x}{-5} + \frac{-4y}{-5} + \frac{2z}{-5} = 1 \Rightarrow \frac{x}{\left(\frac{-5}{4}\right)} + \frac{y}{\left(\frac{5}{4}\right)} + \frac{z}{\left(\frac{-5}{2}\right)} = 1$$ which is in the intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

5. Evaluate $\underset{x\to 2}{Lt}\left[\frac{1}{x-2}-\frac{4}{x^2-4}\right]$

A: $\underset{x\to 2}{Lt}\left[\frac{1}{x-2}-\frac{4}{x^2-4}\right]=\underset{x\to 2}{Lt}\frac{x+2-4}{x^2-4}=\underset{x\to 2}{Lt}\frac{x-2}{(x-2)(x+2)}=\underset{x\to 2}{Lt}\frac{1}{x+2}=\frac{1}{2+2}=\frac{1}{4}$

---

6. Evaluate $\underset{x\to 0}{Lt}\frac{e^{7x}-1}{x}$

A: $\underset{x\to 0}{Lt}\frac{e^{7x}-1}{x}=\underset{7x\to 0}{Lt}\frac{e^{7x}-1}{7x}\times 7=(1)7=7$ $\left(\because \underset{x\to 0}{Lt}\frac{e^x-1}{x}=1\right)$

---

7. If y = log (sin(logx)), find dy/dx.

A: We take y = log(sin(logx)), then $\frac{dy}{dx}=\frac{d}{dx}\log[\sin(\log x)]$

$=\frac{1}{\sin(\log x)}\frac{d}{dx}\sin(\log x)\quad =\frac{1}{\sin(\log x)}\cos(\log x).\frac{d}{dx}\log x$

$=\frac{\cos(\log x)}{\sin(\log x)}.\frac{1}{x}=\frac{\cot(\log x)}{x}$

---

8. Find the derivative of $Sin^{-1}(3x-4x^3)$

A: We take $x=\sin\theta$, then $\theta=Sin^{-1}x$

$\therefore Sin^{-1}(3x-4x^3)=Sin^{-1}(3\sin\theta-4\sin^3\theta)\ =Sin^{-1}(\sin 3\theta)=3\theta=3(Sin^{-1}x)$

$\therefore \frac{d}{dx}(3Sin^{-1}x)\ =3\frac{d}{dx}Sin^{-1}x=3\left(\frac{1}{\sqrt{1-x^2}}\right)=\frac{3}{\sqrt{1-x^2}}$

---

9. Find the slope of the tangent to the curve $y=5x^2$ at (–1, 5)

A: Given equation is $y=5x^2\Rightarrow\frac{dy}{dx}=10x$

Slope of the tangent at (–1, 5) is m = 10(–1) = –10

10. Find Δy and dy for the function $y = x^2 + x$, when $x = 10$, $\Delta x = 0.1$

A:
- We take $y=f(x) = x^2 + x$ and $x = 10$, $\Delta x = 0.1$
- ★ (i) $\Delta y = f(x + \Delta x) - f(x)$
- ★ $=[(x + \Delta x)^2 + (x + \Delta x)] - x^2 - x$
- ★ $=[\cancel{x^2} + (\Delta x)^2 + 2x\Delta x] + \cancel{x} + \Delta x - \cancel{x^2} - \cancel{x}$
- $= (\Delta x)^2 + 2x\Delta x + \Delta x$
- ★ $= \Delta x(\Delta x + 2x + 1)$
- $= 0.1\ (0.1 + 2(10) + 1)$
- ★ $= (0.1)(0.1 + 21) = (0.1)(21.1) = 2.11$
- ★ (ii) $dy = f'(x)\Delta x = (2x + 1)\ \Delta x$
- ★ $=[2(10) + 1)(0.1)] = 21(0.1) = 2.1$

SECTION-B

11. The ends of the hypotenuse of a right angled triangle are (0,6) and (6,0). Find the equation of locus of its third vertex.

Sol: We take A=(0,6), B=(6,0) and P= (x,y) is a point on the locus.

Given condition: $\angle APB=90^\circ$

$\Rightarrow PA^2+PB^2=AB^2$

$\Rightarrow [(x-0)^2+(y-6)^2]+[(x-6)^2+(y-0)^2]=(0-6)^2+(6-0)^2$

$\Rightarrow x^2+(y^2+\cancel{36}-12y)+(x^2+\cancel{36}-12x)+y^2 = \cancel{36}+\cancel{36}$

$\Rightarrow 2x^2+2y^2-12x-12y=0 \Rightarrow \cancel{2}(x^2+y^2-6x-6y)=0$

$\Rightarrow x^2+y^2-6x-6y=0$

Hence, locus of P is $x^2+y^2-6x-6y=0$

---

12. Find the transformed equation of $x^2+2\sqrt{3}xy-y^2=2a^2$, when the axes are rotated through an angle $\pi/6$.

Sol:
- Given equation (original) is $x^2+2\sqrt{3}xy-y^2=2a^2$ ....(1)
- Angle of rotation $\theta=\pi/6=30^\circ$, then

★ $x=X\cos\theta-Y\sin\theta \Rightarrow x=X\cos30^\circ-Y\sin30^\circ = X\left(\frac{\sqrt{3}}{2}\right)-Y\left(\frac{1}{2}\right) \Rightarrow x=\frac{\sqrt{3}X-Y}{2}$

★ $y=Y\cos\theta+X\sin\theta \Rightarrow y=Y\cos30^\circ+X\sin30^\circ = Y\left(\frac{\sqrt{3}}{2}\right)+X\left(\frac{1}{2}\right) \Rightarrow y=\frac{\sqrt{3}Y+X}{2}$

- From (1), transformed equation is
- $\left(\frac{\sqrt{3}X-Y}{2}\right)^2+2\sqrt{3}\left(\frac{\sqrt{3}X-Y}{2}\right)\left(\frac{\sqrt{3}Y+X}{2}\right)-\left(\frac{\sqrt{3}Y+X}{2}\right)^2=2a^2$
- $\Rightarrow \frac{(\sqrt{3}X-Y)^2+2\sqrt{3}(\sqrt{3}X-Y)(\sqrt{3}Y+X)-(\sqrt{3}Y+X)^2}{4}=2a^2$

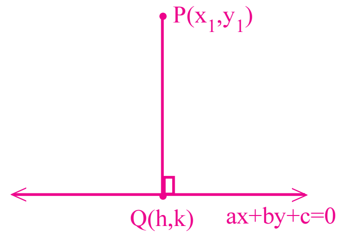
★ $\Rightarrow (3X^2+Y^2-2\sqrt{3}XY)+2\sqrt{3}(3XY+\sqrt{3}X^2-\sqrt{3}Y^2-XY) \quad -(3Y^2+X^2+2\sqrt{3}XY)=4(2a^2)$

★ $\Rightarrow 3X^2+Y^2\cancel{-2\sqrt{3}XY}\cancel{+6\sqrt{3}XY}+6X^2-6Y^2 \quad \cancel{-2\sqrt{3}XY}-3Y^2-X^2\cancel{-2\sqrt{3}XY}=8a^2$

- $\Rightarrow 8X^2-8Y^2=8a^2$
- $\Rightarrow \cancel{8}(X^2-Y^2)=\cancel{8}(a^2)$
- $\Rightarrow X^2-Y^2=a^2$.

13. If Q(h,k) is the foot of the perpendicular of $P(x_1,y_1)$ on the line ax+by+c=0 then prove that $(h-x_1) : a=(k-y_1):b = -(ax_1+by_1+c) : (a^2+b^2)$.

Sol : Given $P=(x_1,y_1)$,Q=(h,k)



Slope of $\overline{PQ}$ is $m_1 = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{k-y_1}{h-x_1}$

Slope of the line ax+by+c=0 is $m_2 = -\dfrac{a}{b}$

Now $m_1m_2=-1$ [$\because$ the 2 lines are perpendicular]

$$\Rightarrow \left(\frac{k-y_1}{h-x_1}\right)\left(\not-\frac{a}{b}\right) = \not-1 \Rightarrow \left(\frac{k-y_1}{h-x_1}\right)\left(\frac{a}{b}\right) = 1 \Rightarrow \frac{k-y_1}{h-x_1} = \frac{b}{a} \Rightarrow \frac{k-y_1}{b} = \frac{h-x_1}{a}$$

We take $\dfrac{h-x_1}{a} = \dfrac{k-y_1}{b} = r$ ....(1)

$\therefore \dfrac{h-x_1}{a} = r \Rightarrow h-x_1 = ar \Rightarrow h = x_1 + ar$;

$\dfrac{k-y_1}{b} = r \Rightarrow k-y_1 = br \Rightarrow k = y_1 + br$

☺ SOLUTION STEPS☺

1) Find slopes of PQ & line
2) Apply $m_1m_2=-1$
3) Find h,k
4) Put (h,k) in line equation & simplify

But Q(h,k) lies on ax+by+c=0 $\Rightarrow$ ah+bk+c=0

$\Rightarrow a(x_1+ar)+b(y_1+br)+c=0 \Rightarrow ax_1+a^2r+by_1+b^2r+c=0 \Rightarrow a^2r+b^2r+ax_1+by_1+c=0$

$\Rightarrow r(a^2+b^2) = -(ax_1+by_1+c) \Rightarrow r = \dfrac{-(ax_1+by_1+c)}{a^2+b^2}$............(2)

From (1) & (2), $\dfrac{h-x_1}{a} = \dfrac{k-y_1}{b} = \dfrac{-(ax_1+by_1+c)}{a^2+b^2}$

---

14. Evaluate $\displaystyle \underset{x\to 0}{\text{Lt}} \frac{1-\cos 2mx}{\sin^2 nx}$

Sol : $\displaystyle \underset{x\to 0}{\text{Lt}} \frac{2\sin^2 mx}{\sin^2 nx} = 2 \underset{x\to 0}{\text{Lt}} \left(\frac{\sin mx}{mx}\right)^2 \left(\frac{nx}{\sin nx}\right)^2 . \frac{m^2x^2}{n^2x^2} = 2(1)^2(1)^2\frac{m^2}{n^2} = \frac{2m^2}{n^2}$

15. Find the derivative of tan2x from the first principle.

Sol:
- We take f(x)=tan(2x) , then
- ★ f(x+h)=tan2(x+h)=tan(2x+2h)
- From the first principle,

$$\star\ f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

$$\bullet = \underset{h\to 0}{Lt} \frac{\tan(2x+2h)-\tan(2x)}{h}$$

$$\star = \underset{h\to 0}{Lt} \frac{1}{h}\left[\frac{\sin(2x+2h)}{\cos(2x+2h)} - \frac{\sin(2x)}{\cos(2x)}\right]$$

$$\bullet = \underset{h\to 0}{Lt} \frac{1}{h}\left[\frac{\sin(2x+2h)\cos(2x)-\cos(2x+2h)\sin(2x)}{\cos(2x+2h)\cos(2x)}\right]$$

$$\star = \underset{h\to 0}{Lt} \frac{1}{h}\frac{\sin[(\cancel{2x}+2h)-\cancel{2x}]}{\cos(2x+2h)\cos(2x)} \qquad [\because \sin A\cos B - \cos A\sin B = \sin(A-B)]$$

$$\star = \underset{h\to 0}{Lt} \frac{\sin 2h}{h}\ \underset{h\to 0}{Lt} \frac{1}{\cos(2x+2h)\cos(2x)}$$

$$\star = 2.\frac{1}{\cos^2(2x)} = 2\sec^2(2x)$$

---

16. If the increase in the side of a square is 2% then find the approximate percentage of increase in the area of the square.

Sol: We take x as side of the square.

$$\text{Given}\ \frac{dx}{x}\times 100 = 2$$

Area of the square $A = x^2 \Rightarrow dA=2xdx$

$$\Rightarrow \frac{dA}{A} = \frac{2\cancel{x}dx}{\cancel{x^2}} \Rightarrow \frac{dA}{A} = 2\left(\frac{dx}{x}\right)$$

$$\therefore \frac{dA}{A}\times 100 = 2\left(\frac{dx}{x}\right)\times 100 = 2(2)=4$$

17. Find the length of subtangent, subnormal at a point t on the curve x=a(cost+tsint), y=a(sint–tcost)

Sol:
- Given x=a(cost+tsint).
- On differentiating w.r.t to t, we get
- $\frac{dx}{dt} = a\frac{d}{dt}[(\cos t + t\sin t)]$

★ $=a[-\sin t + [t\cos t + \sin t(1)]]$ (Applying UV formula on tsint)

- $\therefore \frac{dx}{dt} = a(t\cos t)$
- Also given y=a(sint–tcost), on differentiating w.r.t to t, we get
- $\frac{dy}{dt} = a\frac{d}{dt}[(\sin t - t\cos t)]$

★ $= a[\cos t - [t(-\sin t) + \cos t(1)]]$ (Applying UV formula on tcost)

- $\therefore \frac{dy}{dt} = a(t\sin t)$

★ $\therefore m = \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{a(t\sin t)}{a(t\cos t)} = \tan t$

- So, m=tant and given y=a(sint–tcost).

★ **(i) Length of sub tangent** $= \left|\frac{y}{m}\right| = \left|\frac{a(\sin t - t\cos t)}{\tan t}\right| = |a(\sin t - t\cos t)\cot t|$

★ **(ii) Length of subnormal** $= |y.m| = |a(\sin t - t\cos t)\tan t|$

☺ SOLUTION STEPS☺

1) Find $\frac{dx}{dt}$ & $\frac{dy}{dt}$

2) Find $m=\frac{dy}{dx}$

3) Write m & y

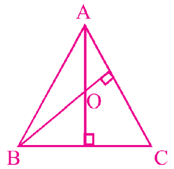
4) $S.T=\left|\frac{y}{m}\right|$

5) S.N=|ym|

SECTION-C

18. Find the orthocentre of the triangle whose vertices are(–2, –1),(6, –1),(2, 5)

A:
- Take O(x, y) as Orthocentre
- Vertices A = (–2, –1), B = (6, –1), C = (2, 5)



Step-1: Finding altitude through A(–2,–1):

Slope of $\overline{BC}$ is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5+1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$

Its perpendicular is $\frac{-1}{m} = \frac{-1}{-\frac{3}{2}} = \frac{2}{3}$

Eq. of line through A(–2,–1) with slope $\frac{2}{3}$ is

$y - y_1 = \frac{-1}{m}(x - x_1)$

$\Rightarrow y+1 = \frac{2}{3}(x+2) \Rightarrow 3y+3 = 2x+4$

$\Rightarrow 2x - 3y + 1 = 0$.........(1)

Step-2: Finding altitude through B(6,–1):

Slope of $\overline{AC}$ is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5+1}{2+2} = \frac{6}{4} = \frac{3}{2}$

Its perpendicular slope is $\frac{-1}{m} = \frac{-1}{\frac{3}{2}} = -\frac{2}{3}$

Eq. of line through B(6,–1) with slope $-\frac{2}{3}$ is $y - y_1 = \frac{-1}{m}(x - x_1)$

$\Rightarrow y+1 = -\frac{2}{3}(x-6) \Rightarrow 3y+3 = -2x+12$

$\Rightarrow 2x + 3y - 9 = 0$.........(2)

Step-3: Solving (1), (2), we get 'O';

(1) $\Rightarrow 2x - 3y + 1 = 0$

(2) $\Rightarrow 2x + 3y - 9 = 0$

(1)+(2) $\Rightarrow 4x - 8 = 0 \Rightarrow 4x = 8 \Rightarrow x = 2$

(1) $\Rightarrow 2(2) - 3(y) + 1 = 0 \Rightarrow 3y = 5 \Rightarrow y = 5/3$

$\Rightarrow x = 2$, $y = 5/3$

$\therefore$ Orthocentre O (x, y) =(2 ,5/3).

19. If θ is the angle between the pair of lines $ax^2+2hxy+by^2=0$ then $\cos\theta = \frac{a+b}{\sqrt{(a-b)^2+4h^2}}$

Proof: Let the separate equations of $ax^2+2hxy+by^2=0$ be $l_1x+m_1y=0$ .....(1) and $l_2x+m_2y=0$ ......(2)

$$\therefore ax^2+2hxy+by^2 \equiv (l_1x+m_1y)(l_2x+m_2y)$$

Comparing bot h sides, we get $l_1l_2 = a$, $l_1m_2+l_2m_1 = 2h$, $m_1m_2 = b$.

If θ is an angle between the lines (1) and (2) then

$$\cos\theta = \frac{l_1l_2+m_1m_2}{\sqrt{(l_1^2+m_1^2).(l_2^2+m_2^2)}} = \frac{l_1l_2+m_1m_2}{\sqrt{l_1^2l_2^2+m_1^2m_2^2+l_1^2m_2^2+l_2^2m_1^2}}$$

$$= \frac{l_1l_2+m_1m_2}{\sqrt{(l_1l_2-m_1m_2)^2+\cancel{2l_1l_2m_1m_2}+(l_1m_2+l_2m_1)^2-\cancel{2l_1l_2m_1m_2}}}$$

$$= \frac{a+b}{\sqrt{(a-b)^2+(2h)^2}} = \frac{a+b}{\sqrt{(a-b)^2+4h^2}} \qquad \left[\begin{array}{l}\because a^2+b^2=(a-b)^2+2ab\\ \quad a^2+b^2=(a+b)^2-2ab\end{array}\right]$$

20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = k$ are mutually perpendicular.

A:

- The given line is $x + 2y = k \Rightarrow \frac{x+2y}{k} = 1$ ...(1)
- Given curve is $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$............(2)
- Homogenising (1) & (2), we get

★ $2x^2 - 2xy + 3y^2 + 2x(1) - y(1) - (1)^2 = 0$

★ $\Rightarrow 2x^2 - 2xy + 3y^2 + 2x\left(\frac{x+2y}{k}\right) - y\left(\frac{x+2y}{k}\right) - \frac{(x+2y)^2}{k^2} = 0$

★ $\Rightarrow \frac{k^2(2x^2 - 2xy + 3y^2) + k(2x^2 + 4xy) - k(xy + 2y^2) - (x^2 + 4y^2 + 4xy)}{k^2} = 0$

★ $\Rightarrow k^2(2x^2 - 2xy + 3y^2) + k(2x^2 + 4xy) - k(xy + 2y^2) - (x^2 + 4y^2 + 4xy) = 0$

★ $\Rightarrow x^2(2k^2 + 2k - 1) + y^2(3k^2 - 2k - 4) + xy(-2k^2 + 3k - 4) = 0$

- If this pair or lines are perpendicular then

★ Coeff. x^2 + Coeff. $y^2 = 0$

- $\Rightarrow (2k^2 + 2k - 1) + (3k^2 - 2k - 4) = 0 \Rightarrow 5k^2 - 5 = 0$
- $\Rightarrow 5(k^2 - 1) = 0 \Rightarrow k^2 - 1 = 0 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

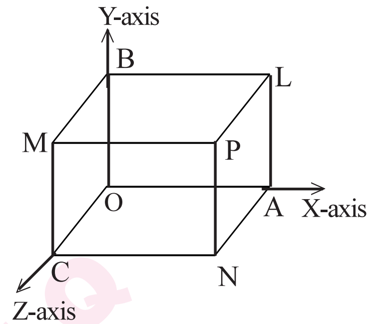
Hence, value of $k = \pm 1$

21. Find the angle between two diagonals of a cube

A: ★ Consider a cube of side 'a' with vertices O, A, B, C, L, M, N, P where O = (0, 0, 0)

★ Take A,B,C are on the X-axis, Y-axis, Z-axis, then

A = (a, 0, 0), B = (0, a, 0), C = (0, 0, a)



★ Take L,M,N on the XY-plane, YZ-plane, ZX-plane, then

L = (a, a, 0), M = (0, a, a), N = (a, 0, a)

★ Take P in the XYZ space, then P = (a, a, a)

★ Take the 2 diagonals $\overline{OP}$, $\overline{CL}$.

- d.r's of $\overline{OP}$ = (a – 0, a – 0, a – 0) = (a, a, a) = (a_1, b_1, c_1)
- d.r's of $\overline{CL}$ = (a – 0, a – 0, 0 – a) = (a, a, –a) = (a_2, b_2, c_2)
- So, angle between the two diagonals is given by

★ $$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}$$

$$= \frac{|a(a) + a(a) + a(-a)|}{\sqrt{(a^2 + a^2 + a^2)(a^2 + a^2 + a^2)}}$$

$$= \frac{a^2}{\sqrt{(3a^2)(3a^2)}} = \frac{a^2}{3a^2} = \frac{1}{3}$$

★ $$\therefore \cos\theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$$

So, that angle between diagonals of a cube is $\cos^{-1}\frac{1}{3}$

22. If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ then prove that $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

A:

- Given $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$
- ★ We take x=sinα, y=sinβ, then
- $\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$
- $\Rightarrow \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$
- ★ $\Rightarrow \frac{\cos\alpha+\cos\beta}{\sin\alpha-\sin\beta}=a$
- ★ $\Rightarrow \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}=a$

  $\left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right.$

  $\left.\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$
- $\Rightarrow \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}=a \Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right)=a$
- ★ $\Rightarrow \frac{\alpha-\beta}{2}=\mathrm{Cot}^{-1}(a)$
- $\Rightarrow \alpha-\beta=2\mathrm{Cot}^{-1}(a)$
- ★ But sinα = x $\Rightarrow \alpha=\mathrm{Sin}^{-1}x$ and $y=\sin\beta \Rightarrow \beta=\mathrm{Sin}^{-1}y$
- ★ $\therefore \mathrm{Sin}^{-1}x-\mathrm{Sin}^{-1}y=2\mathrm{Cot}^{-1}(a)$
- On diff. w.r.to x, we get
- ★ $\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$
- $\Rightarrow \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}} \Rightarrow \boxed{\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}}$.

Hence proved.

23. Show that the curves $y^2 = 4(x+1)$, $y^2 = 36(9-x)$ intersect orthogonally.

A: 1) Finding Point of intersection:

Given $y^2 = 4(x+1)$....(1) and $y^2 = 36(9-x)$ ....(2)

From (1) & (2) we have

$4(x+1) = 36(9-x) \Rightarrow x+1 = 9(9-x) \Rightarrow x+1 = 81-9x \Rightarrow 10x = 80 \Rightarrow x = 8$

Put x = 8 in (1), then $y^2 = 4(8+1) \Rightarrow y^2 = (4)(9) = 36 \Rightarrow y = \pm 6$

$\therefore$ The two points of intersection are P(8, 6), Q(8, –6)

2) Finding Derivatives:

Now $y^2 = 4(x+1) \Rightarrow \cancel{2}y\frac{dy}{dx} = \cancel{4} \Rightarrow y\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$ ....(3)

Also given $y^2 = 36(9-x) \Rightarrow \cancel{2}y\frac{dy}{dx} = -\cancel{36} \Rightarrow y\frac{dy}{dx} = -18 \Rightarrow \frac{dy}{dx} = \frac{-18}{y}$ ....(4)

3) Finding Slopes at P(8,6):

From (3), slope of tangent at P(8,6) is $m_1 = \frac{2}{y} = \frac{\cancel{2}}{\cancel{6}} = \frac{1}{3}$

From (4), slope of tangent at P(8,6) is $m_2 = \frac{-18}{y} = \frac{-\cancel{18}}{\cancel{6}} = -3$

Product of slopes is $(m_1)(m_2) = \left(\frac{1}{3}\right)(-3) = -1$

$\therefore$ Given curves intersect orthogonally at P(8,6)

4) Finding Slopes at Q(8,–6):

From (3), slope of the tangent at Q(8,–6) is $m_1 = \frac{2}{y} = \frac{\cancel{2}}{-\cancel{6}} = -\frac{1}{3}$

From (4), slope of the tangent at Q (8,–6) is $m_2 = \frac{-18}{y} = \frac{-\cancel{18}}{-\cancel{6}} = 3$

Product of slopes is $m_1 m_2 = \left(-\frac{1}{3}\right)(3) = -1$

$\therefore$ Given curves intersect orthogonally at Q(8, –6)

24. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?

A: For the cube, we take length of the edge =x , Volume=V and Surface area=S

Given $\frac{dV}{dt} = 9cm^3 / sec$ and x = 10cm

Volume of the cube $V=x^3$ On diff. w.r.t 't', we get $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$$\Rightarrow 9 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$$

Surface area $S = 6x^2$ On diff. w.r.t 't', we get

$$\frac{dS}{dt} = 12x \frac{dx}{dt} = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x} = \frac{36}{10} = 3.6\ cm^2 / sec$$