Previous IPE

SOLVED PAPERS

MARCH-2020 (TS)

PREVIOUS PAPERS

IPE: MARCH-2020(TS)

Time : 3 Hours **MATHS-1B** Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: $10 \times 2 = 20$

1. Find the value of p, if the straight lines $x + p=0$, $y+2=0$ and $3x +2y + 5= 0$ are concurrent.
2. Find the length of the perpendicular drawn from the point (3,4) to the straight line $3x-4y+10=0$
3. Show that the points (1,2,3), (7, 0, 1), (–2, 3, 4) are collinear
4. Find a traid of d.c's of the normal to the plane $x+2y+2z-4=0$ 5. Evaluate $\underset{x\to 0}{Lt}\frac{e^{\sin x}-1}{x}$
6. Find $\underset{x\to\infty}{Lt}\frac{8|x|+3x}{3|x|-2x}$ 7. Find the derivative of $y = x\,Tan^{-1}x$. 8. If $y= ae^{nx}+be^{-nx}$, then prove that $y''=n^2y$.
9. If $y=5x^2 +6x + 6$, then find Δy and dy when $x=2$, $\Delta x=0.001$
10. Define the strictly increasing function and strictly decreasing function on an interval I.

SECTION-B

II. Answer any FIVE of the following SAQs: $5 \times 4 = 20$

11. A(2,3) and B(–3,4) be two given points. Find the equation of the locus of P so that the area of the triangle PAB is 8.5 sq.units.
12. When the axes are rotated through an angle 45°, the transformed equation of a curve is $17x^2-16xy+17y^2=225$. Find the original equation of the curve.
13. Find the image of (1,2) in the straight line $3x+4y-1=0$.
14. Show that $f(x)=\begin{cases}\frac{\cos ax-\cos bx}{x^2} & \text{if } x\neq 0\\ \frac{1}{2}(b^2-a^2) & \text{if } x=0\end{cases}$, is continuous at 0. ,
15. Find the derivative of $f(x)=\sin 2x$ using the first principle.
16. Find the length of subtangent, subnormal at a point t on the curve $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$
17. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?

SECTION-C

III. Answer any FIVE of the following LAQs: $5 \times 7 = 35$

18. If p and q are the lengths of the perpendiculars from the origin to the straight lines $x\sec\alpha + y\,\text{cosec}\,\alpha = a$ and $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$, prove that $4p^2+q^2 =a^2$.
19. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines then Prove that (i) $h^2=ab$ (ii) $af^2=bg^2$ (iii) the distance between the parallel lines is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2-bc}{b(a+b)}}$
20. Show that the lines joining the origin to the points of intersection of the curve $x^2-xy+y^2+3x+3y-2=0$ and the straight line $x-y-\sqrt{2}=0$ are mutually perpendicular.
21. Find the angle between two non-parallel lines whose direction cosines satisfy the equations $3l+m+5n=0$ and $6mn-2nl+5lm=0$.
22. If $y = Tan^{-1}\left(\frac{2x}{1-x^2}\right)+Tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)-Tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$ show that $\frac{dy}{dx}=\frac{1}{1+x^2}$
23. Find the angle between the curves $2y^2-9x = 0$, $3x^2+4y = 0$ (in Q_4).
24. From a rectangular sheet of dimensions 30cm x 80cm, four equal squares of sides x cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. What is the value of x, so that the volume of the box is the greatest?

IPE TS MARCH-2020

SOLUTIONS

SECTION-A

1. Find the value of p, if the straight lines x + p = 0, y + 2 = 0, 3x + 2y + 5 = 0 are concurrent.

A:
- Given line $x + p = 0 \Rightarrow x = -p$;
  $y+2 = 0 \Rightarrow y = -2$
- $\therefore$ Point of intersection is (–p, –2)
- But (–p, –2) lies on 3x + 2y + 5 = 0 [$\because$ Given lines are concurrent]

★ $\Rightarrow 3(-p) + 2(-2) + 5 = 0 \Rightarrow -3p - 4 + 5 = 0$

★ $\Rightarrow -3p + 1 = 0 \Rightarrow -3p = -1 \Rightarrow 3p = 1 \Rightarrow p = \frac{1}{3}$

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2. Find the length of the perpendicular drawn from the point (3,4) to the straight line 3x – 4y + 10 = 0

A: Perpendicular distance from (3, 4) to 3x – 4y + 10 = 0 is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

$$= \frac{|3(3) - 4(4) + 10|}{\sqrt{3^2 + (-4)^2}} = \frac{|9 - 16 + 10|}{\sqrt{25}} = \frac{19 - 16}{5} = \frac{3}{5}$$

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3. Show that the points (1, 2, 3), (7, 0, 1), (–2, 3, 4) are collinear.

A:
- We take A = (1, 2, 3), B = (7, 0, 1), C = (–2, 3, 4), then
- $AB = \sqrt{(7-1)^2 + (0-2)^2 + (1-3)^2} = \sqrt{36+4+4} = \sqrt{44} = 2\sqrt{11}$
- $BC = \sqrt{(-2-7)^2 + (3-0)^2 + (4-1)^2} = \sqrt{81+9+9} = \sqrt{99} = 3\sqrt{11}$
- $AC = \sqrt{(-2-1)^2 + (3-2)^2 + (4-3)^2} = \sqrt{9+1+1} = \sqrt{11}$

★ Now, $AB + AC = 2\sqrt{11} + \sqrt{11} = 3\sqrt{11} = BC$

$\therefore$ A, B, C are collinear

4. Find a traid of d.c's of the normal to the plane x + 2y + 2z – 4 = 0

A: The d.c's of the plane ax + by + cz + d = 0 are $\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

$\therefore$ d.c's of x + 2y + 2z – 4 = 0 are $\pm\left(\frac{1}{\sqrt{1+4+4}}, \frac{2}{\sqrt{1+4+4}}, \frac{2}{\sqrt{1+4+4}}\right) = \pm\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

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5. Evaluate $\underset{x\to 0}{\text{Lt}} \frac{e^{\sin x}-1}{x}$

A: $\underset{x\to 0}{\text{Lt}} \frac{e^{\sin x}-1}{x} = \underset{x\to 0}{\text{Lt}} \frac{e^{\sin x}-1}{\sin x}.\left(\frac{\sin x}{x}\right) = \underset{\sin x\to 0}{\text{Lt}} \frac{e^{\sin x}-1}{\sin x} \underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = 1.1 = 1$

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6. Find $\underset{x\to\infty}{\text{Lt}} \frac{8|x|+3x}{3|x|-2x}$

A: If $x \to \infty$, then $x > 0$. Hence $|x| = x$

$\therefore \underset{x\to\infty}{\text{Lt}} \frac{8|x|+3x}{3|x|-2x} = \underset{x\to\infty}{\text{Lt}} \frac{8x+3x}{3x-2x} = \underset{x\to\infty}{\text{Lt}} \frac{11x}{x} = \underset{x\to\infty}{\text{Lt}}\ 11 = 11$

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7. Find the derivative of x Tan^{-1} x.

A: Let us apply the uv formula

$$\frac{d}{dx}(x\text{Tan}^{-1}x) = x\frac{d}{dx}\text{Tan}^{-1}x + \text{Tan}^{-1}x\frac{d}{dx}(x) = x\left(\frac{1}{1+x^2}\right) + \text{Tan}^{-1}x(1) = \frac{x}{1+x^2} + \text{Tan}^{-1}x$$

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8. If $y = ae^{nx} + be^{-nx}$, then prove that $y'' = n^2y$.

A: Given $y = ae^{nx} + be^{-nx}$; on diff. w.r.t x, we get

$y' = ae^{nx}(n) + be^{-nx}(-n)$. on diff. again w.r.t x, we get $y'' = ae^{nx}(n)(n) + be^{-nx}(-n)(-n)$

$\Rightarrow y'' = n^2ae^{nx} + n^2be^{-nx} = n^2(ae^{nx} + be^{-nx}) = n^2y \qquad \therefore y'' = n^2y.$

9. If $y=5x^2+6x+6$, then find Δy and dy when $x=2, \Delta x=0.001$

Sol: Let $y=f(x)=5x^2+6x+6$, given that $x=2, \Delta x=0.001$

(i) $\Delta y= f(x+\Delta x)-f(x) =(5(x+\Delta x)^2+6(x+\Delta x)+6)-(5x^2+6x+6)$

$=5(x^2+(\Delta x)^2+2x\Delta x)+6x+6\Delta x+6)-(5x^2+6x+6)$

$=5x^2+10x\Delta x+5(\Delta x)^2+6x+6\Delta x+6-5x^2-6x-6= 10x\Delta x+5(\Delta x)^2+6\Delta x=\Delta x(10x+5\Delta x+6)$

$=(0.001)(10(2)+5(0.001)+6)=(0.001)(26+0.005)$ $=(0.001)(26.005)=0.026005$

(ii) $dy=f'(x)\Delta x=(10x+6)(0.001)=(10(2)+6)(0.001)=(26)(0.001)=0.0026$

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10. Define the strictly increasing function and strictly decreasing function on an interval I.

A: Let f be a function defined in a neighbourhood of apoint a. Then f is said to be

i) increasing at a or locally increasing at a if $\exists$ a $\delta > 0$ such that $x \in (a-\delta, a)$ $\Rightarrow f(x) < f(a)$ and $x \in (a, a+\delta)$ $\Rightarrow f(x) > f(a)$

ii) decreasing at a or locally decreasing at a if $\exists$ a $\delta > 0$ such that $x \in (a-\delta, a) \Rightarrow f(x) > f(a)$ and $x \in (a, a+\delta)$ $\Rightarrow f(x) < f(a)$

Ex 1: i) $f(x) = x^3$ is locally increasing at $x = 0$

Ex 2: ii) $f(x) = x^2$ is locally decreasing at $x = -1$, locally increasing at $x = 1$.

$f(x) = x^2$ is neither increasing nor decreasing at $x = 0$

at a if $\exists$ a $\delta > 0$ such that $x \in (a-\delta, a)$ $\Rightarrow f(x) > f(a)$ and $x \in (a, a+\delta)$ $\Rightarrow f(x) < f(a)$

Ex 1: i) $f(x) = x^3$ is locally increasing at $x = 0$

Ex 2: ii) $f(x) = x^2$ is locally decreasing at $x = -1$, locally increasing at $x = 1$.

SECTION-B

11. A(2, 3) and B(–3, 4) be two given points. Find the equation of the locus of P so that the area of the triangle PAB is 8.5 sq.units.

A: Let the locus point P = (x, y)

Given points A = (2, 3), B = (–3, 4)

Given condition:

Area of Δ PAB = 8.5sq.units

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix} = 8.5$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2+3 & 2-x \\ 3-4 & 3-y \end{vmatrix} = 8.5$$

$$\Rightarrow \begin{vmatrix} 5 & 2-x \\ -1 & 3-y \end{vmatrix} = 2(8.5)$$

$$\Rightarrow |5(3-y) + 1(2-x)| = 17$$

$$\Rightarrow |15 - 5y + 2 - x| = 17$$

$$\Rightarrow |17 - 5y - x| = 17 \quad \Rightarrow |(x + 5y - 17)| = 17$$

$$\Rightarrow x + 5y - 17 = \pm 17$$

$$\Rightarrow x + 5y - 17 = 17 \quad \text{(or)} \quad x + 5y - 17 = -17$$

$$\Rightarrow x + 5y - 34 = 0 \quad \text{(or)} \quad x + 5y = 0$$

$$\Rightarrow (x + 5y - 34)(x + 5y) = 0$$

Hence, locus of P is $(x + 5y - 34)(x + 5y) = 0$

12. When the axes are rotated through an angle 45º, the transformed equation of a curve is $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.

A: ★ Given transformed(new) equation is taken as $17X^2 - 16XY + 17Y^2 = 225$ ....(1)

- Angle of rotation $\theta = 45^\circ$, then

★ $X = x\cos\theta + y\sin\theta = x\cos45^\circ + y\sin45^\circ$

$$= x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) \Rightarrow X = \frac{x+y}{\sqrt{2}}$$

$Y = y\cos\theta - x\sin\theta = y\cos45^\circ - x\sin45^\circ$

$$= y\left(\frac{1}{\sqrt{2}}\right) - x\left(\frac{1}{\sqrt{2}}\right) \Rightarrow Y = \frac{y-x}{\sqrt{2}}$$

- From (1), original equation is

- $$17\left(\frac{x+y}{\sqrt{2}}\right)^2 - 16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{y-x}{\sqrt{2}}\right) + 17\left(\frac{y-x}{\sqrt{2}}\right)^2 = 225$$

★ $$\Rightarrow 17\left(\frac{x^2+y^2+2xy}{2}\right) - 16\left(\frac{y^2-x^2}{2}\right) + 17\left(\frac{y^2+x^2-2xy}{2}\right) = 225 \quad [\because (x+y)(y-x) = (y+x)(y-x) = y^2 - x^2]$$

★ $$\Rightarrow \frac{17x^2 + 17y^2 \cancel{+34xy} - 16y^2 + 16x^2 + 17x^2 + 17y^2 \cancel{-34xy}}{2} = 225$$

★ $$\Rightarrow 50x^2 + 18y^2 = 2(225) \Rightarrow \cancel{2}(25x^2 + 9y^2) = \cancel{2}(225) \Rightarrow 25x^2 + 9y^2 = 225$$

Hence, required original equation is $25x^2+9y^2=225$.

13. Find the image of (1, 2) in the straight line 3x + 4y – 1 = 0

A: Let, (h, k) be the image of (1, 2) w.r.to 3x + 4y – 1 = 0

Here $(x_1, y_1) = (1, 2)$, a = 3, b = 4, c = –1.

$$\therefore \frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

$$\Rightarrow \frac{h-1}{3} = \frac{k-2}{4} = \frac{-2[3(1)+4(2)-1]}{3^2+4^2} = \frac{-2(10)}{25} = -2\left(\frac{2}{5}\right) = -\frac{4}{5}$$

Now, $\frac{h-1}{3} = -\frac{4}{5} \Rightarrow h-1 = -\frac{12}{5} \Rightarrow h = 1-\frac{12}{5} = \frac{5-12}{5} = -\frac{7}{5}$

Also $\frac{k-2}{4} = -\frac{4}{5} \Rightarrow k-2 = -\frac{16}{5} \Rightarrow k = 2-\frac{16}{5} = \frac{10-16}{5} = \frac{-6}{5}$

$\therefore$ The image is $(h,k) = \left(-\frac{7}{5}, -\frac{6}{5}\right)$

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14. Show that $f(x) = \begin{cases} \dfrac{\cos ax - \cos bx}{x^2} & \text{if } x \neq 0 \\ \dfrac{1}{2}(b^2-a^2) & \text{if } x = 0 \end{cases}$**, is continuous at 0.**

A: (a) Given $f(0) = \frac{b^2-a^2}{2}$ ....(1)

(b) $\underset{x\to 0}{\text{Lt}}\, f(x) = \underset{x\to 0}{\text{Lt}} \frac{\cos ax - \cos bx}{x^2} = \underset{x\to 0}{\text{Lt}} \frac{-2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{ax-bx}{2}\right)}{x^2}$

$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$

$$= -2\,\underset{x\to 0}{\text{Lt}}\left(\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\frac{\sin\left(\frac{a-b}{2}\right)x}{x}\right) = -2\left(\underset{x\to 0}{\text{Lt}}\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\underset{x\to 0}{\text{Lt}}\frac{\sin\left(\frac{a-b}{2}\right)x}{x}\right)$$

$$= -2\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right) = -\left(\frac{a^2-b^2}{2}\right) = \frac{b^2-a^2}{2} \text{..........(2)} \qquad \left(\because \underset{x\to 0}{\text{Lt}}\frac{\sin kx}{x} = k\right)$$

$\therefore$ From (1) & (2), $\underset{x\to 0}{\text{Lt}}\, f(x) = f(0)$

Hence, proved that f(x) is continuous at x = 0

15. Find the derivative of sin2x from the first principle.

A: • We take f(x) = sin2x, then

★ f(x + h) = sin2(x + h) = sin (2x + 2h)

• From the first principle,

★ $f'(x) = \underset{h\to 0}{Lt} \dfrac{f(x+h)-f(x)}{h}$

★ $= \underset{h\to 0}{Lt} \dfrac{\sin(2x+2h)-\sin 2x}{h}$

★ $= \underset{h\to 0}{Lt} \dfrac{1}{h}\left(2\cos\left(\dfrac{(2x+2h)+2x}{2}\right)\sin\left(\dfrac{(2x+2h)-2x}{2}\right)\right)$ $\left[\because \sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)\right]$

★ $= 2 \underset{h\to 0}{Lt} \dfrac{1}{h}\cos\left(\dfrac{4x+2h}{2}\right)\sin\left(\dfrac{2h}{2}\right)$

★ $= 2 \underset{h\to 0}{Lt} \dfrac{1}{h}\cos\left(\dfrac{2(2x+h)}{2}\right)\sin(h)$

• $= 2 \underset{h\to 0}{Lt} \dfrac{1}{h}\cos(2x+h)\sin(h)$

★ $= 2 \underset{h\to 0}{Lt} \cos(2x+h). \underset{h\to 0}{Lt} \dfrac{\sinh}{h}$

★ $= 2\cos(2x+0)(1) = 2\cos 2x$

16. Find the length of subtangent, subnormal at a point t on the curve x = a(cost + t sint), y = a(sint – t cost)

A:
- Given x = a(cost + t sint).
- On differentiating w.r.to t, we get
- $\frac{dx}{dt} = a\frac{d}{dt}[(\cos t + t\sin t)]$

★ $= a\left[\cancel{-\sin t} + [t\cos t + \cancel{\sin t(1)}]\right]$ (Applying UV formula on tsint)

- $\therefore \frac{dx}{dt} = a(t\cos t)$
- Also given y = a(sint – t cost), on differentiating w.r.to t, we get
- $\frac{dy}{dt} = a\frac{d}{dt}[(\sin t - t\cos t)]$

★ $= a\left[\cancel{\cos t} - [t(-\sin t) + \cancel{\cos t(1)}]\right]$ (Applying UV formula on tcost)

- $\therefore \frac{dy}{dt} = a(t\sin t)$

★ $\therefore m = \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\cancel{a}(\cancel{t}\sin t)}{\cancel{a}(\cancel{t}\cos t)} = \tan t$

- So, m = tant and given y = a(sint – t cost).

★ **(i) Length of sub tangent** $= \left|\frac{y}{m}\right| = \left|\frac{a(\sin t - t\cos t)}{\tan t}\right| = |a(\sin t - t\cos t)\cot t|$

★ **(ii) Length of subnormal** $= |y.m| = |a(\sin t - t\cos t)\tan t|$

17. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?

A:
- For the cube, we take
- length of the edge = x , Volume = V and Surface area = S

★ Given $\frac{dV}{dt} = 9$ and x = 10 cm

★ Volume of the cube $V = x^3$

★ On diff. w.r.t 't', we get $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

★ $\Rightarrow 9 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{\cancel{9}}{\cancel{3}x^2} = \frac{3}{x^2}$

★ Surface area $S = 6x^2$

★ On diff. w.r.t 't', we get $\frac{dS}{dt} = 12x \frac{dx}{dt}$

★ $= 12\cancel{x}\left(\frac{3}{\cancel{x^2}}\right) = \frac{36}{x} = \frac{36}{10} = 3.6\,cm^2/sec$

SECTION-C

18. If p and q are the lengths of the perpendiculars from the origin to the straight lines x secα +y cosec α = a and x cosα – y sinα = a cos2α, prove that $4p^2 + q^2 = a^2$.

A: Givent line x secα + y cosec α = a

$$\Rightarrow \frac{x}{\cos\alpha}+\frac{y}{\sin\alpha}=a \Rightarrow \frac{x\sin\alpha+y\cos\alpha}{\cos\alpha\sin\alpha}=a \Rightarrow x\sin\alpha+y\cos\alpha=a\sin\alpha\cos\alpha$$

The perpendicular distance O(0, 0) to $x\sin\alpha+y\cos\alpha-a\sin\alpha\cos\alpha=0$ is

$$p=\frac{|-a\sin\alpha\cos\alpha|}{\sqrt{\sin^2\alpha+\cos^2\alpha}}=\frac{a\sin\alpha\cos\alpha}{1}=a\sin\alpha\cos\alpha$$

Now $2p=a(2\sin\alpha\cos\alpha)=a\sin 2\alpha \Rightarrow 4p^2=a^2\sin^2 2\alpha$.........(1)

The perpendicular distance from O(0,0) to $x\cos\alpha-y\sin\alpha-a\cos 2\alpha=0$ is

$$q=\frac{|-a\cos 2\alpha|}{\sqrt{\cos^2\alpha+\sin^2\alpha}}=\frac{a\cos 2\alpha}{1}=a\cos 2\alpha$$

$\therefore q^2=a^2\cos^2 2\alpha$........(2)

From (1) & (2), we have

$$4p^2+q^2=a^2\sin^2 2\alpha+a^2\cos^2 2\alpha=a^2(\sin^2 2\alpha+\cos^2 2\alpha)=a^2(1)=a^2$$

19. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines then Prove that (i) $h^2 = ab$ (ii) $af^2 = bg^2$ (iii) the distance between the parallel lines is $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ (or) $2\sqrt{\frac{f^2 - bc}{b(a+b)}}$

A: ★ Let, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \equiv (lx + my + n_1)(lx + my + n_2)$

- On equating like term coeff. , we get

★ $a = l^2,\ b = m^2,\ h = lm,\ 2g = l(n_1 + n_2),\ 2f = m(n_1 + n_2),\ c = n_1 n_2$

★ (i) $h^2 = (lm)^2 = l^2 m^2 = ab \Rightarrow h^2 = ab$

★ (ii) $af^2 = l^2\left(\frac{m(n_1+n_2)}{2}\right)^2 = \frac{l^2 m^2 (n_1+n_2)^2}{4} = \frac{m^2 l^2 (n_1+n_2)^2}{4} = m^2\left(\frac{l(n_1+n_2)}{2}\right)^2 = bg^2$

★ (iii) Distance between $lx + my + n_1 = 0,\ lx + my + n_2 = 0$ is $\frac{|n_1 - n_2|}{\sqrt{l^2 + m^2}}$

★ $= \frac{\sqrt{(n_1+n_2)^2 - 4n_1 n_2}}{\sqrt{a+b}} = \sqrt{\frac{\left(\frac{2g}{l}\right)^2 - 4c}{a+b}} = \sqrt{\frac{\frac{4g^2}{l^2} - 4c}{a+b}} = \sqrt{\frac{\frac{4g^2}{a} - 4c}{a+b}}$

$= \sqrt{\frac{4g^2 - 4ac}{a(a+b)}} = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$

★ Similarly, by taking $n_1 + n_2 = \frac{2f}{m}$ we get, the distance between the lines $2\sqrt{\frac{f^2 - bc}{b(a+b)}}$

20. Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and the straight line $x - y - \sqrt{2} = 0$ are mutually perpendicular.

A:
- Given line is $x - y = \sqrt{2}$

$$\Rightarrow \frac{x-y}{\sqrt{2}} = 1 \quad ...(1)$$

- Given curve is

$$x^2 - xy + y^2 + 3x + 3y - 2 = 0 \text{...........}(2)$$

- Homogenising (1)&(2), we get

$$x^2 - xy + y^2 + 3x(1) + 3y(1) - 2(1)^2 = 0$$

$$\Rightarrow x^2 - xy + y^2 + 3x\left(\frac{x-y}{\sqrt{2}}\right) + 3y\left(\frac{x-y}{\sqrt{2}}\right) - \not{2}\frac{(x-y)^2}{\not{2}} = 0$$

$$\Rightarrow \frac{\sqrt{2}x^2 - \sqrt{2}xy + \sqrt{2}y^2 + 3x(x-y) + 3y(x-y) - \sqrt{2}(x-y)^2}{\sqrt{2}} = 0$$

$$\Rightarrow \sqrt{2}x^2 - \sqrt{2}xy + \sqrt{2}y^2 + 3x^2 \cancel{-3xy} \cancel{+3yx}$$

$$-3y^2 - \sqrt{2}(x^2 + y^2 - 2xy) = 0$$

$$\Rightarrow \sqrt{2}x^2 - \sqrt{2}xy + \sqrt{2}y^2 + 3x^2 - 3y^2$$

$$-\sqrt{2}x^2 - \sqrt{2}y^2 + 2\sqrt{2}xy = 0$$

$$\Rightarrow 3x^2 - 3y^2 + \sqrt{2}xy = 0$$

- Here, coeff. of x^2 + coeff. of y^2 is 3–3=0

$\therefore$ The pair of lines are perpendicular

21. Find the angle between the lines whose dc's are related by $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.

A: Given $3l + m + 5n = 0 \Rightarrow m = -3l - 5n$ .....(1) ,

$6mn - 2nl + 5lm = 0$ .....(2)

Solving (1) & (2) we get

$6n(-3l - 5n) - 2nl + 5l(-3l - 5n) = 0$

$\Rightarrow -18ln - 30n^2 - 2ln - 15l^2 - 25ln = 0$

$\Rightarrow -15l^2 - 45ln - 30n^2 = 0$

$\Rightarrow -15(l^2 + 3ln + 2n^2) = 0 \Rightarrow l^2 + 3ln + 2n^2 = 0$

$\Rightarrow l^2 + ln + 2ln + 2n^2 = 0 \Rightarrow l(l + n) + 2n(l + n) = 0$

$\Rightarrow (l + n)(l + 2n) = 0 \Rightarrow l = -n$ or $l = -2n$

Case (i): Put $l = -n$ in (1), then

$m = -3(-n) - 5n = 3n - 5n = -2n$

$\therefore m = -2n$

Now, $l : m : n = -\not{n} : -2\not{n} : \not{n}$

$= -1 : -2 : 1 = 1:2:-1$

So, d.r's of $L_1 = (a_1, b_1, c_1) = (1, 2, -1)$.....(3)

Case (ii): Put $l = -2n$ in (1), then

$m = -3(-2n) - 5n = 6n - 5n = n \quad \therefore m = n$

Now, $l : m : n = -2\not{n} : \not{n} : \not{n}$

$= -2 : 1 : 1 = 2 : -1 : -1$

So, d.r's of $L_2 = (a_2, b_2, c_2) = (2, -1, -1)$ ....(4)

If θ is the angle between the lines then from (3), (4), we get

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}$$

$$= \frac{|1(2) + 2(-1) + (-1)(-1)|}{\sqrt{(1^2 + 2^2 + (-1)^2)(2^2 + (-1)^2 + (-1)^2)}} = \frac{|\not{2} - \not{2} + 1|}{\sqrt{(6)(6)}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

$$\therefore \cos\theta = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\frac{1}{6}$$

Hence, angle between the lines is $\cos^{-1}\frac{1}{6}$

22. If $y = \operatorname{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) + \operatorname{Tan}^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) - \operatorname{Tan}^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$ show that $\frac{dy}{dx} = \frac{1}{1+x^2}$

A: Put $x = \tan\theta$

$$\therefore\ y = \operatorname{Tan}^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \operatorname{Tan}^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) - \operatorname{Tan}^{-1}\left(\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}\right)$$

$$= \operatorname{Tan}^{-1}(\tan 2\theta) + \operatorname{Tan}^{-1}(\tan 3\theta) - \operatorname{Tan}^{-1}(\tan 4\theta) = 2\theta + 3\theta - 4\theta = \theta = \operatorname{Tan}^{-1}x \quad (\because x = \tan\theta)$$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}\operatorname{Tan}^{-1}x = \frac{1}{1+x^2}$$

23. Find the angle between the curves $2y^2-9x=0, 3x^2+4y=0$ (in Q_4).

A: 1) Finding Point of intersection:

Given $2y^2-9x=0 \Rightarrow 9x=2y^2 \Rightarrow x=\frac{2}{9}y^2$ .....(1)

$3x^2+4y=0$ ....(2)

Solving (1) & (2) we get, P

$$3.\left(\frac{2}{9}.y^2\right)^2+4y=0 \Rightarrow 3\left(\frac{4}{81}\right)y^4+4y=0 \quad \Rightarrow \frac{4}{27}y^4+4y=0 \Rightarrow 4y\left(\frac{y^3}{27}+1\right)=0$$

$$\Rightarrow y\left(\frac{y^3+27}{27}\right)=0 \quad \Rightarrow y(y^3+27)=0$$

$\Rightarrow y=0$ (or) $y^3=-27=(-3)^3 \quad \Rightarrow y=-3$

Put $y=-3$ in (1), then $x=\frac{2}{9}(-3)^2=\frac{2}{9}.9=2$

So, point of intersection is $P(x,y)=(2,-3)\in Q_4$

2) Finding derivatives:

Given $2y^2-9x=0$, on diff. w.r.t x we get

$$4y\frac{dy}{dx}-9=0 \Rightarrow 4y\frac{dy}{dx}=9 \Rightarrow \frac{dy}{dx}=\frac{9}{4y} \text{ ....(3)}$$

Also given $3x^2+4y=0$, on diff. w.r.t x we get $6x+4\frac{dy}{dx}=0 \Rightarrow 4\frac{dy}{dx}=-6x$

$$\Rightarrow \frac{dy}{dx}=\frac{-6x}{4}=\frac{-3x}{2} \Rightarrow \frac{dy}{dx}=\frac{-3x}{2} \text{ .....(4)}$$

3) Finding Slopes at P:

From (3) slope of the tangent at (2, –3) is $m_1=\frac{9}{4y}=\frac{9}{4(-3)}=\frac{9}{-12}=\frac{-3}{4}$ ....(5)

From (4) slope of the tangent at (2, –3) is $m_2=\frac{-3x}{2}=\frac{-3(2)}{2}=-3$ ....(6)

4) Finding angle at P:

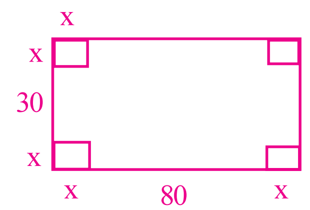
If θ is the angle between the curves then from (5) & (6), we have

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{-3}{4}+3}{1+\left(\frac{-3}{4}\right)(-3)}\right|=\left|\frac{\frac{-3+12}{4}}{1+\frac{9}{4}}\right|=\left|\frac{\frac{9}{4}}{\frac{13}{4}}\right|=\frac{9}{13}$$

$$\therefore \tan\theta=\frac{9}{13} \Rightarrow \theta=\mathrm{Tan}^{-1}\left(\frac{9}{13}\right)$$

Hence, angle between the given curves is $\theta=\mathrm{Tan}^{-1}(9/13)$

24. From a rectangular sheet of dimensions 30cm × 80cm, four equal squares of sides x cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. What is the value of x, so that the volume of the box is the greatest?



☺ I'm Open Box Q'

A:
- ★ For the open box, we take
  height $h = x$
  length $l = 80 - 2x$
  breadth $b = 30 - 2x$
- ★ Volume $V = lbh = (80-2x)(30-2x)(x)$
- $= 2(40-x)\,2(15-x)(x)$
- $= 4(40-x)(15-x)(x) = 4(600 - 40x - 15x + x^2)x$
- $= 4(600 - 55x + x^2)x = 4(x^3 - 55x^2 + 600x)$
- ★ $V(x) = 4(x^3 - 55x^2 + 600x)$ .....(2)
- On diff. (2) w.r.to x, we get,
- $V'(x) = 4(3x^2 - 110x + 600)$ .....(3)
- At max. or min., we have $V'(x) = 0 \Rightarrow \not{4}(3x^2 - 110x + 600) = 0$
- ★ $\Rightarrow 3x^2 - 90x - 20x + 600 = 0$
- $\Rightarrow 3x(x-30) - 20(x-30) = 0 \Rightarrow (3x-20)(x-30) = 0$
- $\Rightarrow 3x = 20$ (or) $x = 30 \Rightarrow x = 20/3$ (or) $x = 30$
- Now, on diff. (3), w.r.to x, we get
- ★ $V''(x) = 4(6x - 110)$ ....(4)
- ★ At $x = \frac{20}{3}$, from (4), we get
- ★ $V''\left(\frac{20}{3}\right) = 4\left(\not{6}\left(\frac{20}{\not{3}}\right) - 110\right) = 4(40-110) = 4(-70) = -280$
- Thus, $V''\left(\frac{20}{3}\right) < 0$
- $\therefore$ V(x) has maximum value at $x = \frac{20}{3}$ cm