Previous IPE

SOLVED PAPERS

MARCH -2019 (TS)

PREVIOUS PAPERS

IPE: MARCH-2019(TS)

MATHS-1B

Time : 3 Hours | Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **10 × 2 = 20**

1. Find $\underset{x\to 0}{\text{Lt}}\frac{a^x-1}{b^x-1}$
2. Find the value of p, if the straight lines 3x + 7y – 1 = 0 and 7x – py + 3 = 0 are mutually perpendicular.
3. If f(x) = log(tan e^x), then find f′(x).
4. Find the ratio in which the X Z - plane divides line joining A(–2, 3, 4) and B(1, 2, 3)
5. Reduce the equation x + 2y – 3z – 6 = 0 of the plane to the normal form.
6. Evaluate $\underset{x\to 0}{\text{Lt}}\frac{\log_e(1+5x)}{x}$ **7.** If f(x) = 1 + x + x^2 +............+ x^{100}, then find f'(1).
8. Find the angle which the straight line $y=\sqrt{3}x-4$ makes with the Y- axis.
9. Verify Rolle's theorem for the function y = f(x) = x^2 + 4 on [–3, 3]
10. If y = cosx then find Δy and dy when x = 60° and Δx = 1° = 0.0174 rad

SECTION-B

II. Answer any FIVE of the following SAQs: **5 × 4 = 20**

11. Check the continuity of $f(x)=\begin{cases}\frac{1}{2}(x^2-4) & \text{if } 0<x<2\\ 0 & \text{if } x=2\\ 2-8x^{-3} & \text{if } x>2\end{cases}$ at 2.
12. A(1, 2), B(2, –3), C(–2, 3) are 3 points. A point P moves such that $PA^2 + PB^2 = 2PC^2$. Show that the equation to the locus of P is 7x – 7y + 4 = 0.
13. A straight line through Q(√3,2) makes an angle π/6 with the positive direction of the X-axis. If the straight line intersects the line √3x – 4y + 8 = 0 at P, find the distance PQ.
14. When the axes are rotated through an angle α, find the transformed equation of xcosα + ysinα = p
15. S.T the tangent at any point θ on the curve x = c secθ, y = c tanθ is ysinθ = x – ccosθ.
16. Find the derivative of $\cos^2x$ from the first principle.
17. A container in the shape of an inverted cone has height 12 cm and radius 6cm at the top. If it is filled with water at the rate of 12 cm^3/sec, what is the rate of change in the height of water level when the tank is filled 8 cm?

SECTION-C

III. Answer any FIVE of the following LAQs: **5 × 7 = 35**

18. Find the orthocentre of the triangle whose vertices are (5, –2), (–1, 2), (1, 4)
19. Prove that the area of the triangle formed by the pair of lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $\frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$
20. Find the angle between the lines whose d.c's are related by $l + m + n = 0$ & $l^2 + m^2 - n^2 = 0$
21. If $x^{\log y} = \log x$ then show that $\frac{dy}{dx}=\frac{y}{x}\left(\frac{1-\log x\log y}{(\log x^2)}\right)$
22. If the tangent at a point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the coordinate axes in A,B then show that the length AB is a constant.
23. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line x + 2y = k are mutually perpendicular.
24. Find the maximum area of the rectangle that can be formed with fixed perimeter 20.

IPE TS MARCH-2019

SOLUTIONS

SECTION-A

1. Find $\underset{x\to 0}{Lt}\dfrac{a^x-1}{b^x-1}$

A: $\underset{x\to 0}{Lt}\dfrac{a^x-1}{b^x-1}=\underset{x\to 0}{Lt}\dfrac{\left(\dfrac{a^x-1}{x}\right)}{\left(\dfrac{b^x-1}{x}\right)}=\dfrac{\underset{x\to 0}{Lt}\left(\dfrac{a^x-1}{x}\right)}{\underset{x\to 0}{Lt}\left(\dfrac{b^x-1}{x}\right)}=\dfrac{\log_e a}{\log_e b}=\log_b a$

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2. Find the value of p, if the straight lines 3x + 7y –1 = 0 and 7x – py + 3 = 0 are mutually perpendicular.

A: Slope of the line 3x + 7y –1 = 0 is $m_1=-\left(\dfrac{a}{b}\right)=-\left(\dfrac{3}{7}\right)=-\dfrac{3}{7}$

Slope of the line 7x – py + 3 = 0 is $m_2=-\left(\dfrac{a}{b}\right)=-\left(\dfrac{7}{-p}\right)=\dfrac{7}{p}$

If the two lines are perpendicular then $m_1m_2=-1$

$\Rightarrow\left(\dfrac{-3}{\cancel{7}}\right)\left(\dfrac{\cancel{7}}{p}\right)=-1\Rightarrow-\dfrac{3}{p}=-1\Rightarrow p=3$

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3. If f(x) = log(tan e^x), then find f′(x).

A: Given f(x) = log (tan e^x) $\Rightarrow f'(x)=\dfrac{d}{dx}\log\left(\tan e^x\right)$

$=\dfrac{1}{\tan e^x}.\dfrac{d}{dx}\tan e^x=\dfrac{1}{\tan e^x}.\sec^2(e^x).\dfrac{d}{dx}(e^x)$

= cot (e^x) $\sec^2$ (e^x) . e^x

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4. Find the ratio in which the XZ - plane dividesline joiningA(–2, 3, 4) & B(1, 2, 3). Also find the point of Intersection.

A:
- We take $A(x_1, y_1, z_1)$ = (–2, 3, 4) and $B(x_2, y_2, z_2)$ = (1, 2, 3)
- ★ The ratio in which XZ plane divides $\overline{AB}$ is $-y_1 : y_2 = -3 : 2$
- ∴ the point P which divides A(–2, 3, 4) and B(1, 2, 3) in the ratio –3 : 2 is
- ★ $P=\left[\dfrac{mx_2+nx_1}{m+n},\dfrac{my_2+ny_1}{m+n},\dfrac{mz_2+nz_1}{m+n}\right]$
- ★ $=\left(\dfrac{-3\times1+2\times-2}{-3+2},\dfrac{-3\times2+2\times3}{-3+2},\dfrac{-3\times3+2\times4}{-3+2}\right)$
- $=\left(\dfrac{-7}{-1},\dfrac{0}{-1},\dfrac{-1}{-1}\right)=(7,0,1)$

5. Reduce the equation x+2y–3z–6=0 of the plane to the normal form.

A: Equation of the plane is $x + 2y - 3z - 6 = 0 \Rightarrow x + 2y - 3z = 6$

Dividing the above equation by $\sqrt{1^2+2^2+(-3)^2} = \sqrt{1+4+9} = \sqrt{14}$, we have

$\left(\frac{1}{\sqrt{14}}\right)x + \left(\frac{2}{\sqrt{14}}\right)y + \left(\frac{-3}{\sqrt{14}}\right)z = \frac{6}{\sqrt{14}}$, which is in the normal form.

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6. Evaluate $\underset{x\to 0}{\text{Lt}} \frac{\log(1+5x)}{x}$

A: $\underset{x\to 0}{\text{Lt}} \frac{\log(1+5x)}{x} = \underset{5x\to 0}{\text{Lt}} \frac{\log(1+5x)}{5x} \times 5 = 1\times 5 = 5$ $\left(\because \underset{x\to 0}{\text{Lt}} \frac{\log_e(1+x)}{x} = 1\right)$

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7. If $f(x) = 1 + x + x^2 + ... + x^{100}$, then find $f'(1)$.

A: $f(x) = 1 + x + x^2 + ... + x^{100} \Rightarrow f'(x) = 1 + 2x + 3x^2 .... + 100\,x^{99}$.

$\Rightarrow f'(1) = 1+2+3....+100 = \frac{100\times 101}{2} = 5050$ $\left[\because \Sigma n = \frac{n(n+1)}{2}\right]$

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8. Find the angle which the straight line $y = \sqrt{3}x - 4$ makes with the Y- axis.

A: Given line is $y=\sqrt{3}x-4$. This is in the form $y = mx + c \Rightarrow$ slope $m = \sqrt{3}$

$\Rightarrow \tan\theta = \sqrt{3} = \tan 60^\circ$ $\Rightarrow$ Angle made by the line with the X-axis is $\theta = 60^\circ$

$\therefore$ Angle made by the line with Y-axis is $90^\circ - 60^\circ = 30^\circ$

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9. Verify Rolle's theorem for the function $y = f(x) = x^2 + 4$ on [–3, 3]

A :
- Given $f(x) = x^2 + 4 \Rightarrow f'(x) = 2x$
- f(x) is (i) continuous on [–3, 3]
- (ii) differentiable in (–3, 3)
- ★ (iii) $f(-3) = (-3)^2 + 4 = 9 + 4 = 13$; $f(3) = 3^2 + 4 = 9 + 4 = 13$
- $\Rightarrow f(-3) = f(3)$
- ★ So, from Rolle's theorem, $f'(c)=0$

  $\Rightarrow 2c=0 \Rightarrow c=0$
- ★ $\therefore c=0\in(-3,3)$.
- Hence, Rolle's theorem is verified.

10. If $y = \cos x$ then find Δy and dy when $x = 60^\circ$ and $\Delta x = 1^\circ = 0.0174$ rad

A: (i) $\Delta y = f(x + \Delta x) - f(x)$

$= \cos(60^\circ + 1^\circ) - \cos 60^\circ = \cos 61^\circ - \cos 60^\circ$

From the tables the value of $\cos 1^\circ = 0.4848$

$\therefore \Delta y = 0.4848 - \frac{1}{2} = 0.4848 - 0.5 = -0.0152$

(ii) $dy = f'(x)\Delta x = (-\sin x)\Delta x$

$= (-\sin 60^\circ)(1^\circ) = \left(-\frac{\sqrt{3}}{2}\right)(0.0174)$

$= (-0.8660)(0.0174) = -0.01506$

SECTION-B

11. Check the continuity of $f(x) = \begin{cases} \frac{1}{2}(x^2-4) & \text{if } 0<x<2 \\ 0 & \text{if } x=2 \\ 2-8x^{-3} & \text{if } x>2 \end{cases}$ **at 2.**

A: (a) When $x < 2$, L.H.L= $\underset{x\to 2-}{\text{Lt}} f(x) = \underset{x\to 2-}{\text{Lt}} \frac{1}{2}(x^2-4) = \frac{1}{2}(4-4) = 0$ ....(1)

(b) When $x > 2$, R.H.L= $\underset{x\to 2+}{\text{Lt}} f(x) = \underset{x\to 2+}{\text{Lt}} (2-8x^{-3}) = \underset{x\to 2+}{\text{Lt}} \left(2-\frac{8}{x^3}\right) = 2-\frac{8}{8} = 2-1 = 1$.......(2)

From (1) & (2), L.H.L $\neq$ R.H.L

Hence, proved that f(x) is not continuous at 2.

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12. A(1, 2), B(2, –3), C(–2, 3) are 3 points. A point P moves such that $PA^2 + PB^2 = 2PC^2$. Show that the equation to the locus of P is $7x - 7y + 4 = 0$.

A: Given points are A = (1, 2), B = (2, –3), C = (–2, 3) and P = (x, y) is a point on the locus.

Given condition: $PA^2 + PB^2 = 2PC^2$

$\Rightarrow [(x-1)^2 + (y-2)^2] + [(x-2)^2 + (y+3)^2] = 2[(x+2)^2 + (y-3)^2]$

$\Rightarrow (x^2 + 1 - 2x) + (y^2 + 4 - 4y) + (x^2 + 4 - 4x) + (y^2 + 9 + 6y) = 2[(x^2 + 4 + 4x) + (y^2 + 9 - 6y)]$

$\Rightarrow \cancel{2x^2} + \cancel{2y^2} - 6x + 2y + 18 = \cancel{2x^2} + \cancel{2y^2} + 8x - 12y + 26$

$\Rightarrow -6x - 8x + 2y + 12y + 18 - 26 = 0$

$\Rightarrow -14x + 14y - 8 = 0 \quad \Rightarrow \cancel{-2}(7x - 7y + 4) = 0 \quad \Rightarrow 7x - 7y + 4 = 0$

Hence, locus of P is $7x - 7y + 4 = 0$

13. A straight line through Q(√3, 2) makes an angle π/6 with the positive direction of the X-axis. If the straight line intersects the line √3x – 4y + 8 = 0 at P, find the distance PQ

A: Given angle $\theta = \pi/6 = 30^o$

Given point $Q(x_1, y_1) = (\sqrt{3}, 2)$ and distance PQ = r, then

$$P = (x_1 + r\cos\theta, y_1 + r\sin\theta) = (\sqrt{3} + r\cos30^o, 2 + r\sin30^o)$$

$$= \left(\sqrt{3} + r\left(\frac{\sqrt{3}}{2}\right), 2 + r\left(\frac{1}{2}\right)\right)$$

$$= P\left(\sqrt{3} + \frac{\sqrt{3}r}{2}, 2 + \frac{r}{2}\right)$$

But, P lies on the line $\sqrt{3}x - 4y + 8 = 0$

$$\Rightarrow \sqrt{3}\left(\sqrt{3} + \frac{\sqrt{3}r}{2}\right) - 4\left(2 + \frac{r}{2}\right) + 8 = 0$$

$$\Rightarrow 3 + \frac{3r}{2} \cancel{-8} - 2r \cancel{+8} = 0$$

$$\Rightarrow 2r - \frac{3r}{2} = 3 \Rightarrow \frac{4r - 3r}{2} = 3$$

$$\Rightarrow \frac{r}{2} = 3 \Rightarrow r = 6$$

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14. When the axes are rotated through an angle α, find the transformed equation of xcosα + ysinα = p

A:
- Given original equation is $x\cos\alpha + y\sin\alpha = p$........(1)
- Angle of rotation $\theta = \alpha$, then
- ★ $x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos\alpha - Y\sin\alpha$

  $y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos\alpha + X\sin\alpha$
- From (1), transformed equation is
- $(X\cos\alpha - Y\sin\alpha)\cos\alpha + (Y\cos\alpha + X\sin\alpha)\sin\alpha = p$
- $\Rightarrow X\cos^2\alpha \cancel{- Y\sin\alpha\cos\alpha} \cancel{+ Y\cos\alpha\sin\alpha} + X\sin^2\alpha = p$
- $\Rightarrow X(\cos^2\alpha + \sin^2\alpha) = p \quad \Rightarrow X(1) = p \quad \Rightarrow X = p$

15. S.T the tangent at any point θ on the curve x=c secθ, y=c tanθ is ysinθ=x–ccosθ.

A: Slope of the tangent at any point θ(csecθ, ctanθ) on the curve is

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\frac{d}{d\theta}(c\tan\theta)}{\frac{d}{d\theta}(c\sec\theta)}=\frac{c\sec^2\theta}{c\sec\theta\tan\theta}=\frac{\sec\theta}{\tan\theta}=\frac{\cot\theta}{\cos\theta}=\frac{\cos\theta}{\sin\theta(\cos\theta)}=\frac{1}{\sin\theta}$$

∴ Equation of the tangent at θ(csecθ, ctanθ) with slope

$\frac{1}{\sin\theta}$ is $y-c\tan\theta=\frac{1}{\sin\theta}(x-c\sec\theta)$

$$\Rightarrow y-c\frac{\sin\theta}{\cos\theta}=\frac{1}{\sin\theta}\left(x-\frac{c}{\cos\theta}\right)\quad\Rightarrow y\sin\theta-\frac{c\sin^2\theta}{\cos\theta}=x-\frac{c}{\cos\theta}$$

$$\Rightarrow y\sin\theta=x-\frac{c}{\cos\theta}+\frac{c\sin^2\theta}{\cos\theta}=x-c\left[\frac{1-\sin^2\theta}{\cos\theta}\right]\Rightarrow y\sin\theta=x-c\left[\frac{\cos^2\theta}{\cos\theta}\right]$$

$\Rightarrow y\sin\theta=x-c\cos\theta.$ Hence, proved.

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16. Find the derivative of $\cos^2x$ from the first principle.

A: We take $f(x)=\cos^2 x$, then $f(x+h)=\cos^2(x+h)$

From the first principle, $f'(x)=\underset{h\to0}{Lt}\frac{f(x+h)-f(x)}{h}=\underset{h\to0}{Lt}\frac{\cos^2(x+h)-\cos^2 x}{h}$

$$=\underset{h\to0}{Lt}\frac{-\sin[(x+h)+x]\sin[(x+h)-x]}{h}\quad[\because \cos^2A-\cos^2B=-\sin(A+B)\sin(A-B)]$$

$$=\underset{h\to0}{Lt}\frac{-\sin(2x+h)\sin(h)}{h}=-\underset{h\to0}{Lt}\sin(2x+h).\underset{h\to0}{Lt}\frac{\sin(h)}{h}=-\sin(2x+0)(1)=-\sin 2x$$

17. A container in the shape of an inverted cone has height 12 cm and radius 6cm at the top. If it is filled with water at the rate of 12 cm^3/sec, what is the rate of change in the height of water level when the tank is filled 8 cm?

A : Let OC be height of water level at t sec.
Let OC = h, CD = r and volume = V.

Given that AB = 6 cm, OA = 12 cm, $\frac{dV}{dt} = 12\ cm^3/sec$

We have to find the rate of rise of the water level $\left(\frac{dh}{dt}\right)_{h=8}$, when h=8 cm

The triangles OAB and OCD are similar triangles.

$$\therefore \frac{CD}{AB} = \frac{OC}{OA} \Rightarrow \frac{r}{6} = \frac{h}{12} \Rightarrow r = \frac{h}{2} \quad ....(1)$$

Volume of the cone V is given by $V = \frac{\pi r^2 h}{3}$ ....(2)

From (1), we have $V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 \times h = \frac{\pi h^3}{12}$ ....(3)

Differentiating (3) w.r.to t, we get $\frac{dV}{dt} = \frac{\pi h^2}{4}.\frac{dh}{dt}$

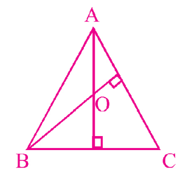
$$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \left(\frac{1}{\pi}\right)\frac{4}{8^2}(12) = \frac{3}{4\pi} cm/sec$$

Hence, the rate of change of water level is 3/4π cm/sec

SECTION-C

18. Find the orthocentre of the triangle whose vertices are (5, –2), (–1, 2), (1, 4)

A:
- Take O(x,y) as Orthocentre
- Vertices A = (5, –2), B = (–1, 2), C = (1, 4)



Step-1:Finding altitude through A(5, –2):

Slope of $\overline{BC}$ is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{1+1} = \frac{2}{2} = 1$

Its perpendicular slope is $\frac{-1}{m} = \frac{-1}{1} = -1$

Eq. of line through A(5,–2) with slope –1 is $y - y_1 = \frac{-1}{m}(x - x_1)$

$\Rightarrow y + 2 = -1(x - 5) \Rightarrow y + 2 = -x + 5$

$\Rightarrow x + y - 3 = 0$ .... (1)

Step-2: Finding altitude through B(–1,2):

Slope of $\overline{AC}$ is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-2)}{1-5} = \frac{6}{-4} = -\frac{3}{2}$

Its perpendicular slope is $\frac{-1}{m} = \frac{-1}{-\frac{3}{2}} = \frac{2}{3}$

Eq. of line through B(–1, 2) with slope $\frac{2}{3}$ is $y - y_1 = \frac{-1}{m}(x - x_1)$

$\Rightarrow y - 2 = \frac{2}{3}(x+1) \Rightarrow 3y - 6 = 2(x + 1)$

$\Rightarrow 3y - 6 = 2x + 2 \Rightarrow 2x - 3y + 2 + 6 = 0$

$\Rightarrow 2x - 3y + 8 = 0$ .... (2)

Step-3: Solving (1), (2), we get O;

$(1) \Rightarrow x + y - 3 = 0$

$(2) \Rightarrow 2x - 3y + 8 = 0$

$\therefore \frac{x}{1(8)-(-3)(-3)} = \frac{y}{(-3)(2)-(8)(1)} = \frac{1}{1(-3)-(2)(1)}$

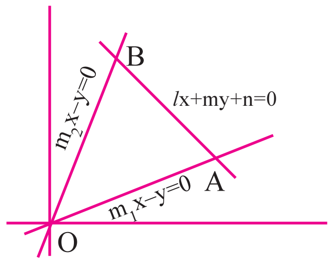
$\Rightarrow \frac{x}{8-9} = \frac{y}{-6-8} = \frac{1}{-3-2}$

$\Rightarrow \frac{x}{-1} = \frac{y}{-14} = \frac{1}{-5} \Rightarrow x = \frac{-1}{-5} = \frac{1}{5};\ y = \frac{-14}{-5} = \frac{14}{5}$

$\Rightarrow x = \frac{1}{5}, y = \frac{14}{5}$

$\therefore$ Orthocentre $O(x, y) = \left(\frac{1}{5}, \frac{14}{5}\right)$

19. Prove that the area of the triangle formed by the pair of lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is $\dfrac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$



A: ★ Let $ax^2+2hxy+by^2 \equiv (m_1x-y)(m_2x-y)$

- On equating like term coeff., we get

★ $m_1+m_2=-\dfrac{2h}{b},\quad m_1m_2=\dfrac{a}{b}$ ....(1)

- By solving $lx+my+n=0,\ m_1x-y=0$ we get A

$lx+my+n=0$

$m_1x-y+0=0$

$$\begin{array}{cccccc} & x & & y & & 1 \\ m & & n & & l & & m \\ -1 & & 0 & & m_1 & & -1 \end{array}$$

★ $\Rightarrow \dfrac{x}{m(0)-(-1)(n)}=\dfrac{y}{n(m_1)-l(0)}=\dfrac{1}{l(-1)-mm_1}$

- $\Rightarrow \dfrac{x}{n}=\dfrac{y}{nm_1}=\dfrac{1}{-l-mm_1} \Rightarrow A=\left(\dfrac{-n}{l+mm_1},\dfrac{-nm_1}{l+mm_1}\right)$

★ Similarly, we get $B=\left(\dfrac{-n}{l+mm_2},\dfrac{-nm_2}{l+mm_2}\right)$

★ The area of the triangle with vertices, $O(0,0),A(x_1,y_1),B(x_2,y_2)$ is $\Delta=\dfrac{1}{2}|x_1y_2-x_2y_1|$

★ $\therefore$ Area of $\Delta OAB=\dfrac{1}{2}\left|\left(\dfrac{-n}{l+mm_1}\right)\left(\dfrac{-nm_2}{l+mm_2}\right)-\left(\dfrac{-n}{l+mm_2}\right)\left(\dfrac{-nm_1}{l+mm_1}\right)\right|$

- $=\dfrac{1}{2}\left|\dfrac{n^2m_2-n^2m_1}{(l+mm_1)(l+mm_2)}\right|=\dfrac{1}{2}\left|\dfrac{n^2(m_2-m_1)}{l^2+lmm_2+lmm_1+m^2m_1m_2}\right|$

★ $=\dfrac{1}{2}\dfrac{n^2\sqrt{(m_1+m_2)^2-4m_1m_2}}{|l^2+lm(m_1+m_2)+m^2(m_1m_2)|}$ $\left[\because (a-b)=\sqrt{(a+b)^2-4ab}\right]$

★ $=\dfrac{1}{2}\dfrac{n^2\sqrt{\left(-\dfrac{2h}{b}\right)^2-4\left(\dfrac{a}{b}\right)}}{\left|l^2+lm\left(\dfrac{-2h}{b}\right)+m^2\left(\dfrac{a}{b}\right)\right|}=\dfrac{1}{2}\dfrac{n^2\sqrt{\dfrac{4h^2}{b^2}-4\dfrac{a}{b}}}{\left|l^2-\left(\dfrac{2hlm}{b}\right)+\left(\dfrac{am^2}{b}\right)\right|}=\dfrac{1}{2}\dfrac{n^2\sqrt{\dfrac{4h^2-4ab}{b^2}}}{\left|\dfrac{bl^2-2hlm+am^2}{b}\right|}$ [from (1)]

★ $=\dfrac{1}{2}\dfrac{n^2\sqrt{4h^2-4ab}}{|am^2-2hlm+bl^2|}=\dfrac{1}{2}\dfrac{2\left(n^2\sqrt{h^2-ab}\right)}{|am^2-2hlm+bl^2|}=\dfrac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$ sq. units

20. Find the angle between the lines whose d.c's are related by $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$

A:
- Given $l + m + n = 0 \Rightarrow l = -(m + n)$ .....(1),
- $l^2 + m^2 - n^2 = 0$ ......(2)
- Solving (1) & (2) we get

  $[-(m + n)]^2 + m^2 - n^2 = 0$

  $\Rightarrow (m^2 + \cancel{n^2} + 2mn) + m^2 - \cancel{n^2} = 0$

  $\Rightarrow 2m^2 + 2mn = 0 \Rightarrow \cancel{2}(m^2 + mn) = 0$

  $\Rightarrow m^2 + mn = 0 \Rightarrow m(m + n) = 0$

★ $\Rightarrow m = 0$ (or) $m + n = 0 \Rightarrow m = 0$ (or) $m = -n$

- **Case (i):** Put $m = 0$ in (1), then

  $l = -(0 + n) = -n$

  $\therefore l = -n$

Now, $l : m : n = -\cancel{n} : 0 : \cancel{n} = -1 : 0 : 1$

★So, d.r's of $L_1 = (a_1, b_1, c_1) = (-1, 0, 1)$ ......(3)

Case (ii): Put $m = -n$ in (1), then
$l = -(-n + n) = 0$
$\therefore l = 0$

Now, $l : m : n = 0 : -\cancel{n} : \cancel{n} = 0 : -1 : 1$
So, d.r's of $L_2 = (a_2, b_2, c_2) = (0, -1, 1)$ ...(4)

- If θ is the angle between the lines then from (3), (4), we get

★ $\cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}} = \dfrac{|(-1)(0) + (0)(-1) + 1(1)|}{\sqrt{((-1)^2 + 0^2 + 1^2)(0^2 + (-1)^2 + 1^2)}}$

★ $= \dfrac{1}{\sqrt{(2)(2)}} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2} = \cos 60^\circ \Rightarrow \theta = 60^\circ$

Hence angle between the lines is 60°.

21. If $x^{\log y} = \log x$ then show that $\frac{dy}{dx} = \frac{y}{x}\left(\frac{1-\log x \log y}{(\log x^2)}\right)$

A : Given that $x^{\log y} = \log x \Rightarrow \log(x^{\log y}) = (\log y)(\log x) \Rightarrow \log y(\log x) = \log(\log x)$

Differentiating w.r.t x, we have

$$\log y\left(\frac{1}{x}\right) + \log x\left(\frac{1}{y}\frac{dy}{dx}\right) = \frac{1}{\log x}\left(\frac{1}{x}\right) \Rightarrow \frac{\log y}{x} + \left(\frac{\log x}{y}\frac{dy}{dx}\right) = \frac{1}{x\log x}$$

$$\Rightarrow \left(\frac{\log x}{y}\right)\frac{dy}{dx} = \frac{1}{x\log x} - \frac{\log y}{x} = \frac{1}{x}\left(\frac{1}{\log x} - \log y\right)$$

$$\Rightarrow \frac{x}{y}\frac{dy}{dx} = \frac{1}{\log x}\left(\frac{1-\log x \log y}{\log x}\right) = \frac{1-\log x \log y}{(\log x)^2} \Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{1-\log x \log y}{(\log x)^2}\right)$$

22. If the tangent at a point on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ intersects the coordinate axes in A, B then show that the length AB is a constant.

A:
- ★ The point on the curve taken as $P(a\cos^3\theta, a\sin^3\theta)$
- • $x = a\cos^3\theta$ and $y = a\sin^3\theta$
- ★ $\therefore \dfrac{dy}{dx}=\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\dfrac{\frac{d}{d\theta}(a\sin^3\theta)}{\frac{d}{d\theta}(a\cos^3\theta)}=\dfrac{a.3\sin^2\theta(\cos\theta)}{a.3\cos^2\theta(-\sin\theta)}=-\dfrac{\sin\theta}{\cos\theta}$
- • So, slope of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ is $m=-\dfrac{\sin\theta}{\cos\theta}$
- ★ $\therefore$ Equation of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ having slope $-\dfrac{\sin\theta}{\cos\theta}$ is $y-y_1=m(x-x_1)$
- ★ $\Rightarrow y-a\sin^3\theta=-\dfrac{\sin\theta}{\cos\theta}(x-a\cos^3\theta)$
- • $\Rightarrow \dfrac{y-a\sin^3\theta}{\sin\theta}=-\dfrac{(x-a\cos^3\theta)}{\cos\theta}$
- • $\Rightarrow \dfrac{y}{\sin\theta}-\dfrac{a\sin^3\theta}{\sin\theta}=-\dfrac{x}{\cos\theta}+\dfrac{a\cos^3\theta}{\cos\theta}$
- • $\Rightarrow \dfrac{x}{\cos\theta}+\dfrac{y}{\sin\theta}=a\cos^2\theta+a\sin^2\theta=a(\cos^2\theta+\sin^2\theta)=a(1)$
- • $\Rightarrow \dfrac{x}{a\cos\theta}+\dfrac{y}{a\sin\theta}=1$
- ★ $\therefore$ A=$(a\cos\theta, 0)$, B=$(0, a\sin\theta)$
- • $\therefore AB=\sqrt{(a\cos\theta-0)^2+(0-a\sin\theta)^2}$

$$=\sqrt{a^2\cos^2\theta+a^2\sin^2\theta}=\sqrt{a^2(\cos^2\theta+\sin^2\theta)}=\sqrt{a^2(1)}=a$$

$\therefore$ Hence, proved that AB is a constant.

23. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = k$ are mutually perpendicular.

A:
- The given line is $x + 2y = k \Rightarrow \frac{x+2y}{k} = 1$ ...(1)
- Given curve is $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$............(2)
- Homogenising (1) & (2), we get

★ $2x^2 - 2xy + 3y^2 + 2x(1) - y(1) - (1)^2 = 0$

★ $\Rightarrow 2x^2 - 2xy + 3y^2 + 2x\left(\frac{x+2y}{k}\right) - y\left(\frac{x+2y}{k}\right) - \frac{(x+2y)^2}{k^2} = 0$

★ $\Rightarrow \frac{k^2(2x^2 - 2xy + 3y^2) + k(2x^2 + 4xy) - k(xy + 2y^2) - (x^2 + 4y^2 + 4xy)}{k^2} = 0$

★ $\Rightarrow k^2(2x^2 - 2xy + 3y^2) + k(2x^2 + 4xy) - k(xy + 2y^2) - (x^2 + 4y^2 + 4xy) = 0$

★ $\Rightarrow x^2(2k^2 + 2k - 1) + y^2(3k^2 - 2k - 4) + xy(-2k^2 + 3k - 4) = 0$

- If this pair or lines are perpendicular then

★ Coeff. x^2 + Coeff. y^2 =0

- $\Rightarrow (2k^2 \cancel{+2k} - 1) + (3k^2 \cancel{-2k} - 4) = 0 \Rightarrow 5k^2 - 5 = 0$
- $\Rightarrow \cancel{5}(k^2 - 1) = 0 \Rightarrow k^2 - 1 = 0 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

Hence, value of $k = \pm 1$

24. Find the maximum area of the rectangle that can be formed with fixed perimeter 20.

A: • For the rectangle, we take length = x, breadth = y

1) Given perimeter is 20 $\Rightarrow 2(x + y) = 20 \Rightarrow x + y = 10$

$\Rightarrow y = 10 - x$ .....(1)

2) Area of the rectangle is $A = xy$

From (1), $A(x) = xy = x(10 - x)$

$\therefore A(x) = 10x - x^2$ .....(2)

3) On diff. (2) w.r.to x, we get

$A'(x) = 10 - 2x$ ....(3)

4) At max. or min.,we have $A'(x) = 0$

$\Rightarrow 10 - 2x = 0 \Rightarrow x = 5$

Also, from (1), $y = 10 - x = 10 - 5 = 5$

5) Now, on diff. (3) w.r.to x, we get $A''(x) = -2$ ......(4)

6) At $x = 5$, from (4), we get $A''(5) < 0$

7) $\therefore$ Area is maximum at $x = 5$ and $y = 5$

Hence, maximum area of the rectangle is $A = xy = 5(5) = 25$ sq.units