Previous IPE

SOLVED PAPERS

MARCH-2019 (AP)

PREVIOUS PAPERS

IPE: MARCH-2019(AP)

MATHS-1A

Time : 3 Hours Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: $10 \times 2 = 20$

1. If A={–2,–1,0,1,2} and f:A→B is a surjection defined by $f(x)=x^2+x+1$ then find B.
2. If f(x)=2x–1, $g(x)=\frac{x+1}{2}$ for all x∈R, find (gof)(x)
3. If $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix}=\begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$ then find the values of x, y, z and a **4.** Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$
5. Let $\bar{a}=2\bar{i}+4\bar{j}-5\bar{k}, \bar{b}=\bar{i}+\bar{j}+\bar{k}, \bar{c}=\bar{j}+2\bar{k}$. Find the unit vector in the opposite direction of $\bar{a}+\bar{b}+\bar{c}$
6. Find the vector equation of the plane passing through the points $\bar{i}-2\bar{j}+5\bar{k}, -5\bar{j}-\bar{k}, -3\bar{i}+5\bar{j}$
7. If the vectors $\lambda\bar{i}-3\bar{j}+5\bar{k}, 2\lambda\bar{i}-\lambda\bar{j}-\bar{k}$ are perpendicular to each other find λ.
8. If $\sin\theta=\frac{4}{5}$ and θ is not in the first quadrant, find the value of $\cos\theta$.
9. If θ is not an integral multiple of $\frac{\pi}{2}$, prove that tan θ + 2 tan 2θ + 4 tan 4θ + 8cot 8θ = cotθ
10. Show that $\text{Tanh}^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\log_e 3$

SECTION-B

II. Answer any FIVE of the following SAQs: $5 \times 4 = 20$

11. If $A=\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then show that adj A = $3A^T$. Also find A^{-1}
12. Show that the line joining the pair of points $6\bar{a}-4\bar{b}+4\bar{c}, -4\bar{c}$ and the line joining the pair of points $-\bar{a}-2\bar{b}-3\bar{c}, \bar{a}+2\bar{b}-5\bar{c}$ intersect at the point $-4\bar{c}$, when $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors.
13. If $\bar{a}=2\bar{i}+\bar{j}-\bar{k}, \bar{b}=-\bar{i}+2\bar{j}-4\bar{k}, \bar{c}=\bar{i}+\bar{j}+\bar{k}$ then find $(\bar{a}\times\bar{b}).(\bar{b}\times\bar{c})$
14. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$
15. Given p≠±q, show that the solutions of cos pθ+cosqθ=0 form two series each of which is in A.P. Find also the common difference of each A.P.
16. Prove that $\text{Tan}^{-1}\frac{1}{2}+\text{Tan}^{-1}\frac{1}{5}+\text{Tan}^{-1}\frac{1}{8}=\frac{\pi}{4}$ **17.** If a=(b+c)cosθ, then prove that $\sin\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$

SECTION-C

III. Answer any FIVE of the following LAQs: $5 \times 7 = 35$

18. If f:A→B, g:B→C be bijections, then prove that $(gof)^{-1}=f^{-1}og^{-1}$
19. Using Mathematical induction, prove that for all $n \in N$, $a+ar+ar^2+....+$n terms $=\frac{a(r^n-1)}{r-1}, r\neq 1$
20. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2=\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}=(a^3+b^3+c^3-3abc)^2$
21. $x-y+3z=5, 4x+2y-z=0, -x+3y+z=5$, solve the system of equation by using Cramer'srule.
22. Show that the volume of a tetrahedron with $\bar{a}, \bar{b}$ and $\bar{c}$ as coterminousedges is $\frac{1}{6}\left|[\bar{a}\ \bar{b}\ \bar{c}]\right|$.
23. If A+B+C=180°, then show that sin2A+sin2B+sin2C = – 4sinAsinBsinC.
24. In a ΔABC if a=13, b=14, c=15 then show that $R=\frac{65}{8}, r=4, r_1=\frac{21}{2}, r_2=12, r_3=14$

IPE AP MARCH-2019

SOLUTIONS

SECTION-A

1. If A = {–2, –1, 0, 1, 2} and f : A → B is a surjection defined by $f(x) = x^2 + x + 1$ then find B.

A: Given A = {–2, –1, 0, 1, 2} and $f(x) = x^2 + x + 1$

$\therefore f(-2) = (-2)^2 - 2 + 1 = 4 - 2 + 1 = 3;$

$f(-1) = (-1)^2 - 1 + 1 = 1;$

$f(0) = 0^2 + 0 + 1 = 1;$

$f(1) = 1^2 + 1 + 1 = 3;$

$f(2) = 2^2 + 2 + 1 = 7$

$\therefore B = f(A) = \{3, 1, 1, 3, 7\} = \{3, 1, 7\}$ [$\because$ For a surjection, Range f(A)= Codomain B]

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2. If f(x) = 2x – 1, $g(x) = \frac{x+1}{2}$ for all x∈R, find (i) (gof)(x) (ii) (fog)(x)

A: (i) $(gof)(x) = g(f(x)) = g(2x-1) = \frac{(2x-1)+1}{2} = \frac{2x}{2} = x$

(ii) $(fog)(x) = f(g(x)) = f\left(\frac{x+1}{2}\right) = \cancel{2}\left(\frac{x+1}{\cancel{2}}\right) - 1 = x + 1 - 1 = x$

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3. If $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$ then find the values of x, y, z and a.

A: Given $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$

On equating corresponding elements, we get $x - 3 = 5 \Rightarrow x = 5 + 3 = 8;\ 2y - 8 = 2$

$\Rightarrow 2y = 2 + 8 \Rightarrow \cancel{2}y = \cancel{10} \Rightarrow y = 5$

$z + 2 = -2 \Rightarrow z = -2 - 2 = -4;$

$a - 4 = 6 \Rightarrow a = 6 + 4 \Rightarrow a = 10$

$\therefore x = 8, y = 5, z = -4, a = 10$

4. Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

A: We take $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 1(6-4) - 2(4-0) + 3(2-0) = 2 - 8 + 6 = 0$

$\therefore |A| = 0.$

Take a 2×2 minor, $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

$\therefore$ Rank(A) = 2.

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5. Let $\bar{a} = 2\bar{i} + 4\bar{j} - 5\bar{k}, \bar{b} = \bar{i} + \bar{j} + \bar{k}, \bar{c} = \bar{j} + 2\bar{k}$. Find the unit vector in the opposite direction of $\bar{a} + \bar{b} + \bar{c}$

A: Given $\bar{a} = 2\bar{i} + 4\bar{j} - 5\bar{k}$, $\bar{b} = \bar{i} + \bar{j} + \bar{k}$, $\bar{c} = 0\,\bar{i} + \bar{j} + 2\,\bar{k}$, then $\bar{a} + \bar{b} + \bar{c}$

$= (2\bar{i} + 4\bar{j} - 5\bar{k}) + (\bar{i} + \bar{j} + \bar{k}) + (0\bar{i} + \bar{j} + 2\bar{k}) \quad = 3\bar{i} + 6\bar{j} - 2\bar{k}$

$\Rightarrow |\bar{a} + \bar{b} + \bar{c}| = \sqrt{3^2 + 6^2 + (-2)^2} \quad = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$

$\therefore$ Opposite Unit vector $= \dfrac{-(\bar{a} + \bar{b} + \bar{c})}{|\bar{a} + \bar{b} + \bar{c}|} \quad = \dfrac{-(3\bar{i} + 6\bar{j} - 2\bar{k})}{7}$

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6. Find the vector equation of the plane passing through the points $\bar{i} - 2\bar{j} + 5\bar{k}$, $-5\bar{j} - \bar{k}$, $-3\bar{i} + 5\bar{j}$

A: Given $A(\bar{a}) = \bar{i} - 2\bar{j} + 5\bar{k}$, $B(\bar{b}) = -5\bar{j} - \bar{k}$, $C(\bar{c}) = -3\bar{i} + 5\bar{j}$

Vector equation of the plane is $\bar{r} = (1 - s - t)\bar{a} + s\bar{b} + t\bar{c}$, $s, t \in R$

$\therefore \bar{r} = (1 - s - t)(\bar{i} - 2\bar{j} + 5\bar{k}) + s(-5\bar{j} - \bar{k}) + t(-3\bar{i} + 5\bar{j})$, $s, t \in R$

7. If the vectors $\lambda\bar{i}-3\bar{j}+5\bar{k},\ 2\lambda\bar{i}-\lambda\bar{j}-\bar{k}$ are perpendicular to each other find λ.

A: Let $\bar{a}=\lambda\bar{i}-3\bar{j}+5\bar{k}, \bar{b}=2\lambda\bar{i}-\lambda\bar{j}-\bar{k}$

Given $\bar{a}\perp\bar{b}\Rightarrow\bar{a}.\bar{b}=0$

$\therefore(\lambda\bar{i}-3\bar{j}+5\bar{k}).(2\lambda\bar{i}-\lambda\bar{j}-\bar{k})=0$

$\Rightarrow\lambda(2\lambda)-3(-\lambda)+5(-1)=0$

$\Rightarrow 2\lambda^2+3\lambda-5=0\ \Rightarrow(\lambda-1)(2\lambda+5)=0$

$\Rightarrow(\lambda-1)=0\Rightarrow\lambda=1$ (or) $(2\lambda+5)=0$

$\Rightarrow 2\lambda=-5\Rightarrow\lambda=\frac{-5}{2}$

$\therefore\lambda=1 \text{ or } -5/2$

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8. If sinθ=4/5and θ is not in the first quadrant, find the value of cosθ

A: Given sin θ = 4/5 which is positive and θ is not in Q_1 $\therefore\theta\in Q_2$

So 'θ' lies in Q_2 cosθ is -ve

$$\therefore\cos\theta=-\sqrt{1-\sin^2\theta}=-\sqrt{1-\left(\frac{4}{5}\right)^2}=-\sqrt{1-\frac{16}{25}}=-\sqrt{\frac{25-16}{25}}=-\sqrt{\frac{9}{25}}=-\frac{3}{5}$$

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9. If θ is not an integral multiple of $\frac{\pi}{2}$, prove that tan θ+2 tan 2θ+4 tan 4θ+8cot 8θ =cotθ

A: We know that tan A = cot A – 2 cot 2A

∴ L.H.S = tan θ + 2 tan 2θ + 4 tan 4θ + 8 cot 8θ

= (cotθ – 2 cot 2θ) + 2 (cot 2θ – 2 cot 4θ) + 4 (cot 4θ – 2 cot 8θ) + 8 cot 8θ.

= cot θ – 2 cot 2θ + 2 cot 2θ – 4 cot 4θ + 4 cot 4θ – 8 cot 8θ + 8 cot 8θ

= cot θ = R.H.S

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10. Show that $\text{Tanh}^{-1}\frac{1}{2}=\frac{1}{2}\log_e 3$

A: We know $\text{Tanh}^{-1}x=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

$$\therefore\text{Tanh}^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)=\frac{1}{2}\log_e\left(\frac{\frac{3}{\not 2}}{\frac{1}{\not 2}}\right)=\frac{1}{2}\log_e(3)$$

SECTION-B

11. If $A=\begin{bmatrix}-1&-2&-2\\2&1&-2\\2&-2&1\end{bmatrix}$ then show that adj A = $3A^T$. Also find A^{-1}

A: Given that $A=\begin{bmatrix}-1&-2&-2\\2&1&-2\\2&-2&1\end{bmatrix}$ $\Rightarrow \det A = -1\begin{vmatrix}1&-2\\-2&1\end{vmatrix}+2\begin{vmatrix}2&-2\\2&1\end{vmatrix}-2\begin{vmatrix}2&1\\2&-2\end{vmatrix}$

$$= -(1-4)+2(2+4)-2(-4-2) = 3+2(6)+2(6) = 3+12+12 = 27$$

Cofactor matrix of A

$$=\begin{bmatrix}\begin{vmatrix}1&-2\\-2&1\end{vmatrix} & -\begin{vmatrix}2&-2\\2&1\end{vmatrix} & \begin{vmatrix}2&1\\2&-2\end{vmatrix}\\ -\begin{vmatrix}-2&-2\\-2&1\end{vmatrix} & \begin{vmatrix}-1&-2\\2&1\end{vmatrix} & -\begin{vmatrix}-1&-2\\2&-2\end{vmatrix}\\ \begin{vmatrix}-2&-2\\1&-2\end{vmatrix} & -\begin{vmatrix}-1&-2\\2&-2\end{vmatrix} & \begin{vmatrix}-1&-2\\2&1\end{vmatrix}\end{bmatrix}=\begin{bmatrix}-3&-6&-6\\6&3&-6\\6&-6&3\end{bmatrix}\Rightarrow \text{Adj } A=\begin{bmatrix}-3&6&6\\-6&3&-6\\-6&-6&3\end{bmatrix} \text{------(1)}$$

Also, $A=\begin{bmatrix}-1&-2&-2\\2&1&-2\\2&-2&1\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}-1&2&2\\-2&1&-2\\-2&-2&1\end{bmatrix}\Rightarrow 3A^T=3\begin{bmatrix}-1&2&2\\-2&1&-2\\-2&-2&1\end{bmatrix}=\begin{bmatrix}-3&6&6\\-6&3&-6\\-6&-6&3\end{bmatrix}$ ---(2)

$\therefore$ from (1) & (2); Adj $A = 3A^T$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj}A)=\frac{1}{27}\begin{bmatrix}-3&6&6\\-6&3&-6\\-6&-6&3\end{bmatrix}=\frac{3}{27}\begin{bmatrix}-1&2&2\\-2&1&-2\\-2&-2&1\end{bmatrix}=\frac{1}{9}\begin{bmatrix}-1&2&2\\-2&1&-2\\-2&-2&1\end{bmatrix}$$

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12. Show that the line joining the pair of points $6\bar{a}-4\bar{b}+4\bar{c}, -4\bar{c}$ and the line joining the pair of points $-\bar{a}-2\bar{b}-3\bar{c}, \bar{a}+2\bar{b}-5\bar{c}$ intersect at the point $-4\bar{c}$, when $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors.

A: Let $P=-4\bar{c}$ and $Q=6\bar{a}-4\bar{b}+4\bar{c}$

Vector equation of the line joining the points P,Q is

$\bar{r}=(1-t)(-4\bar{c})+t(6\bar{a}-4\bar{b}+4\bar{c}), t\in R \Rightarrow \bar{r}=(6t)\bar{a}-(4t)\bar{b}+(8t-4)\bar{c}$ ....(1)

Let $T=-\bar{a}-2\bar{b}-3\bar{c}$ and $S=\bar{a}+2\bar{b}-5\bar{c}$

Vector equation of the line joining the points T,S is

$\bar{r}=(1-s)(-\bar{a}-2\bar{b}-3\bar{c})+s(\bar{a}+2\bar{b}-5\bar{c}), s\in R \Rightarrow \bar{r}=(2s-1)\bar{a}+(4s-2)\bar{b}+(-2s-3)\bar{c}$ ....(2)

If the two lines intersect each other at $P(\bar{r})$ then from (1) & (2), we have

$(6t)\bar{a}-(4t)\bar{b}+(8t-4)\bar{c}=(2s-1)\bar{a}+(4s-2)\bar{b}+(-2s-3)\bar{c}$

Equating the corresponding coefficients of $\bar{a}, \bar{b}, \bar{c}$, we have

$6t = 2s-1 \Rightarrow 6t-2s=-1$ ....(3) $\quad -4t = 4s-2 \Rightarrow 4t+4s=2$ ....(4)

$8t-4 = -2s-3 \Rightarrow 8t+2s=1$ ......(5)

Adding (3) & (5) we get $14t+0=0 \Rightarrow 14t=0 \Rightarrow t=0$

$\therefore$ Substituting the value of t=0 in (1), the point of intersection of the lines is obtained as $-4\bar{c}$

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13. If $\bar{a} = 2\bar{i}+\bar{j}-\bar{k}, \bar{b} = -\bar{i}+2\bar{j}-4\bar{k}, \bar{c} = \bar{i}+\bar{j}+\bar{k}$ then find $(\bar{a}\times\bar{b}).(\bar{b}\times\bar{c})$

A: Given that $\bar{a} = 2\bar{i}+\bar{j}-\bar{k}, \bar{b} = -\bar{i}+2\bar{j}-4\bar{k}$ and $\bar{c} = \bar{i}+\bar{j}+\bar{k}$

Now $\bar{a}x\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -1 \\ -1 & 2 & -4 \end{vmatrix} = \bar{i}(-4+2)-\bar{j}(-8-1)+\bar{k}(4+1) = -2\bar{i}+9\bar{j}+5\bar{k}$

Also, $\bar{b}x\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \bar{i}(2+4)-\bar{j}(-1+4)+\bar{k}(-1-2) = 6\bar{i}-3\bar{j}-3\bar{k}$

Now $(\bar{a}x\bar{b}).(\bar{b}x\bar{c}) = (-2\bar{i}+9\bar{j}+5\bar{k}).(6\bar{i}-3\bar{j}-3\bar{k}) = (-2)(6)+(9)(-3)+5(-3) = -12-27-15 = -54$

$\therefore \quad (\bar{a}x\bar{b}).(\bar{b}x\bar{c}) = -54$

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14. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right) = \frac{1}{16}$

A: L.H.S $= \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$

$$= (1+\cos 18^o)(1+\cos 54^o)(1+\cos 126^o)(1+\cos 162^o)$$

$$= (1+\cos 18^o)(1+\cos 54^o)(1+\cos(180^o-54^o))(1+\cos(180^o-18^o))$$

$$= (1+\cos 18^o)(1+\cos 54^o)(1-\cos 54^o)(1-\cos 18^o) = (1-\cos^2 18^o)(1-\cos^2 54^o)$$

$$= \sin^2 18^o \sin^2 54^o = \left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2 = \left(\frac{5-1}{16}\right)^2 = \left(\frac{4}{16}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{R.H.S}$$

15. Given $p \neq \pm q$, show that the solutions of $\cos p\theta + \cos q\theta = 0$ form two series each of which is in A.P. Find also the common difference of each A.P.

A : Given $\cos p\theta + \cos q\theta = 0$

$$\Rightarrow 2\cos\left[\left(\frac{p+q}{2}\right)\theta\right]\cos\left[\left(\frac{p-q}{2}\right)\theta\right] = 0 \qquad \left[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$\Rightarrow \cos\left(\frac{p+q}{2}\right)\theta = 0 \text{ (or) } \cos\left(\frac{p-q}{2}\right)\theta = 0$$

(i) $\cos\left(\frac{p+q}{2}\right)\theta = 0 = \cos\frac{\pi}{2}$

$$\Leftrightarrow \left(\frac{p+q}{2}\right)\theta = 2n\pi \pm \frac{\pi}{2} = (4n \pm 1)\frac{\pi}{2} \Leftrightarrow \theta = \frac{(4n \pm 1)\pi}{(p+q)}, n \in Z$$

The solutions are : $\ldots\ldots, -\frac{\pi}{p+q}, \frac{\pi}{p+q}, \frac{3\pi}{p+q}, \frac{5\pi}{p+q}, \ldots$

These solutions form an A.P with common difference $\frac{2\pi}{(p+q)}$

(ii) $\cos\left(\frac{p-q}{2}\right)\theta = 0 = \cos\frac{\pi}{2}$

$$\Leftrightarrow \left(\frac{p-q}{2}\right)\theta = 2n\pi \pm \frac{\pi}{2} = (4n \pm 1)\frac{\pi}{2} \Leftrightarrow \theta = \frac{(4n \pm 1)\pi}{(p-q)}, n \in Z$$

The solutions are: $\ldots\ldots, -\frac{\pi}{p-q}, \frac{\pi}{p-q}, \frac{3\pi}{p-q}, \frac{5\pi}{p-q}, \ldots$

These solutions form another A.P with common difference $\frac{2\pi}{(p-q)}$

16. Prove that $\text{Tan}^{-1}\frac{1}{2}+\text{Tan}^{-1}\frac{1}{5}+\text{Tan}^{-1}\frac{1}{8}=\frac{\pi}{4}$

A: We know, $\text{Tan}^{-1}x+\text{Tan}^{-1}y=\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\therefore \text{Tan}^{-1}\frac{1}{2}+\text{Tan}^{-1}\frac{1}{5}=\text{Tan}^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}\right)=\text{Tan}^{-1}\left(\frac{\frac{5+2}{10}}{\frac{10-1}{10}}\right)=\text{Tan}^{-1}\frac{7}{9}$$

$$\therefore \text{L.H.S}=\text{Tan}^{-1}\frac{7}{9}+\text{Tan}^{-1}\frac{1}{8}$$

$$=\text{Tan}^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}\right)=\text{Tan}^{-1}\left(\frac{\frac{56+9}{72}}{\frac{72-7}{72}}\right)=\text{Tan}^{-1}\left(\frac{65}{65}\right)=\text{Tan}^{-1}1=\frac{\pi}{4}=\text{R.H.S}$$

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17. If a = (b + c) cosθ, then prove that $\sin\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$

A: Given a = (b + c)cosθ , then $\cos\theta=\frac{a}{b+c}\Rightarrow \cos^2\theta=\frac{a^2}{(b+c)^2}$

$\therefore \sin^2\theta = 1-\cos^2\theta$

$$=1-\frac{a^2}{(b+c)^2}=\frac{(b+c)^2-a^2}{(b+c)^2}=\frac{(b^2+c^2+2bc)-a^2}{(b+c)^2}$$

$$=\frac{2bc+(b^2+c^2-a^2)}{(b+c)^2}=\frac{2bc+2bc\cos A}{(b+c)^2} \qquad \left(\because \frac{b^2+c^2-a^2}{2bc}=\cos A\right)$$

$$=\frac{2bc(1+\cos A)}{(b+c)^2}=\frac{2bc.\left(2\cos^2\frac{A}{2}\right)}{(b+c)^2}=\frac{4bc\cos^2\frac{A}{2}}{(b+c)^2}$$

$$\therefore \sin\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$$

SECTION-C

18. If $f : A \to B$, $g : B \to C$ are two bijective functions then prove that $(gof)^{-1} = f^{-1}og^{-1}$

A: **Part -1:** Given that $f : A \to B$, $g : B \to C$ are two bijective functions, then

(i) $gof : A \to C$ is bijection $\Rightarrow (gof)^{-1} : C \to A$ is also a bijection

(ii) $f^{-1} : B \to A$, $g^{-1} : C \to B$ are both bijections $\Rightarrow (f^{-1}og^{-1}) : C \to A$ is also a bijection.

So, $(gof)^{-1}$ and $f^{-1}og^{-1}$, both have same domain 'C'

Part–2: Given $f : A \to B$ is bijection, then $f(a) = b \Rightarrow a = f^{-1}(b)$........(1), [Here $a \in A$, $b \in B$]

$g : B \to C$ is bijection, then $g(b) = c \Rightarrow b = g^{-1}(c)$........(2), [Here $b \in B$, $c \in C$]

$gof : A \to C$ is bijection, then $gof(a) = c \Rightarrow a = (gof)^{-1}(c)$.........(3)

Now, $(f^{-1}og^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a$ ....(4), [From (1) & (2)]

$\therefore (gof)^{-1}(c) = (f^{-1}og^{-1})(c), \forall c \in C$, [from (3) & (4)]

Hence, we proved that $(gof)^{-1} = f^{-1}og^{-1}$

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19. Prove that $a + ar + ar^2 + ... + n$ terms $= \dfrac{a(r^n - 1)}{r - 1}, r \neq 1$

A : The n^{th} term of the Geometric series $a, ar, ar^2...$ is ar^{n-1}. $\therefore T_n = ar^{n-1}$

Let $S(n) : a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$

Step 1: L.H.S of S(1)=a and R.H.S of $S(1) = \dfrac{a(r^1 - 1)}{r - 1} = a$

L.H.S of S(1)= R.H.S of S(1) $\Rightarrow$ S(1) is true

Step 2: Assume that S(k) is true for kÎN $S(k) : a + ar + ar^2 + ... + ar^{k-1} = \dfrac{a(r^k - 1)}{r - 1}$ ....(1)

Step 3: We show that S(k+1) is true

On adding $ar^{(k+1)-1} = ar^k$ to both sides of (1), we get

$$\text{L.H.S of } S(k+1) = (a + ar + ar^2 + ... + ar^{k-1}) + ar^k = \frac{a(r^k - 1)}{r - 1} + ar^k, \text{ [From (1)]}$$

$$= \frac{a(r^K - 1) + (r - 1)ar^K}{r - 1} = \frac{ar^K - a + rar^K - ar^K}{r - 1} = \frac{r.ar^K - a}{r - 1}$$

$$= \frac{a.r^{k+1} - a}{r - 1} = \frac{a(r^{k+1} - 1)}{r - 1} = \text{R.H.S of } S(k+1)$$

$\therefore$ L.H.S of S(k+1)=R.H.S of S(k+1) $\Rightarrow$ S(k+1) is true whenever S(k) is true

Hence, by the principle of Mathematical induction, the given statement is true $\forall n \in N$

20. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$

A: We take $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= a(bc-a^2) - b(b^2-ac) + c(ab-c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= -(a^3+b^3+c^3-3abc)$$

$$\Rightarrow \Delta^2 = (a^3+b^3+c^3-3abc)^2 \text{..............}(1)$$

Again $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

On applying C_{23} on the first determinent, we get

$$= -\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+cb+bc & \cancel{-ab}+c^2\cancel{+ab} & \cancel{-ac}+\cancel{ac}+b^2 \\ \cancel{-ab}\cancel{+ab}+c^2 & -b^2+ac+ca & \cancel{-bc}+a^2+\cancel{cb} \\ \cancel{-ca}+b^2\cancel{+ac} & \cancel{-cb}\cancel{+bc}+a^2 & -c^2+ba+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \text{................}(2)$$

From (1) and (2), the given result is proved.

21. By using Cramers rule solve $x-y+3z=5$, $4x+2y-z=0$, $x+3y+z=5$.

A: Given equations in the matrix equation form: AX = D, where

$$A=\begin{bmatrix}1 & -1 & 3\\ 4 & 2 & -1\\ 1 & 3 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, D=\begin{bmatrix}5\\ 0\\ 5\end{bmatrix}$$

$$\Delta=\det A=\begin{vmatrix}1 & -1 & 3\\ 4 & 2 & -1\\ 1 & 3 & 1\end{vmatrix}$$

$$=1(2+3)+1(4+1)+3(12-2)$$

$$=5+5+30=40\neq 0$$

$\Rightarrow$ A is non singular

$$\Delta_1=\begin{vmatrix}5 & -1 & 3\\ 0 & 2 & -1\\ 5 & 3 & 1\end{vmatrix}$$

$$=5(2\times 1-(-1)\times 3)+1(0\times 1-(-1)\times 5)+3(0\times 3-5\times 2)$$

$$=25+5-30=0;$$

$$\Delta_2=\begin{vmatrix}1 & 5 & 3\\ 4 & 0 & -1\\ 1 & 5 & 1\end{vmatrix}$$

$$=1(0\times 1-(-1)\times 5)-5(4\times 1-(-1)\times 1)+3(4\times 5-0\times 1)$$

$$=5-25+60=40;$$

$$\Delta_3=\begin{vmatrix}1 & -1 & 5\\ 4 & 2 & 0\\ 1 & 3 & 5\end{vmatrix}$$

$$=1(2\times 5-0\times 3)+1(4\times 5-0\times 1)+5(4\times 3-2\times 1)$$

$$=10+20+50=80$$

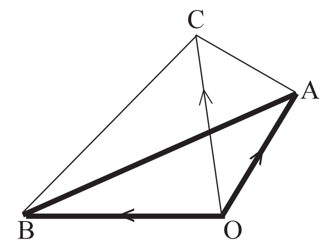
$\therefore$ By Cramer's rule,

$$x=\frac{\Delta_1}{\Delta}=\frac{0}{40}=0;\ \ y=\frac{\Delta_2}{\Delta}=\frac{40}{40}=1 \text{ and } z=\frac{\Delta_3}{\Delta}=\frac{80}{40}=2$$

$\therefore$ The solution is $x=0$, $y=1$, $z=2$

22. Show that the volume of a tetrahedron with $\bar{a}, \bar{b}$ and $\bar{c}$ as coterminous edges is $\frac{1}{6}\left|[\bar{a}\ \bar{b}\ \bar{c}]\right|$.

Proof: Let $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}$ be the co-terminus edges of a tetrahedron OABC such that $[\bar{a}\ \bar{b}\ \bar{c}]$ is a right handed vector traid.



The volume V of the tetrahedron OABC is

$$V = \frac{1}{3}(\text{Area of } \Delta OAB) \text{ x } (\text{length of the perpendicular from C to the plane OAB})$$

$$\text{Area of } \Delta OAB = \frac{1}{2}(\bar{a} \times \bar{b})$$

Length of the perpendicular from C to the plane OAB=Projection of C in the direction of $\bar{a} \times \bar{b}$

$$= \frac{(\bar{a} \times \bar{b}).\bar{c}}{|\bar{a} \times \bar{b}|} = \frac{[\bar{a}\bar{b}\bar{c}]}{|\bar{a} \times \bar{b}|}$$

$$\therefore V = \frac{1}{3}\left(\frac{1}{2}|\bar{a} \times \bar{b}| \frac{[\bar{a}\bar{b}\bar{c}]}{|\bar{a} \times \bar{b}|}\right) = \frac{1}{6}|[\bar{a}\bar{b}\bar{c}]|$$

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23. If A + B + C =0 then prove that sin2A + sin2B + sin2C = –4 sin A sin B sin C.

Sol: L.H.S = sin2A+sin2B+sin2C

$$= 2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \sin 2C \quad \left(\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right)$$

= 2sin(A+B)cos(A–B)+sin2C

= 2 sin (0–C) cos(A–B)+sin 2C [∵ A+B+C=0]

= –2sinC cos(A–B)+2sinCcosC [∵ sin2θ=2sinθcosθ]

= –2sinC[cos(A–B)–cosC]

= –2sinC[cos(A–B)–cos(A+B)]

= –2sinC (2sinA sinB) [∵ cos(A–B)–cos(A+B)=2sinAsinB]

= –4sinA sinB sinC = R.H.S

24. In a ΔABC if $a = 13, b = 14, c = 15$ then show that $R = \frac{65}{8}, r = 4, r_1 = \frac{21}{2}, r_2 = 12, r_3 = 14$

A: Given $a = 13, b = 14, c = 15$, then

$$2s = a + b + c = 13 + 14 + 15 = 42 \Rightarrow 2s = 42 \Rightarrow s = 21$$

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(21-13)(21-14)(21-15)}$

$$= \sqrt{21 \times (8)(7)(6)} = \sqrt{(3\times 7)(4\times 2)(7)(3\times 2)} = \sqrt{3^2 \times 4^2 \times 7^2} = 3\times 4\times 7 = 84$$

(i) $R = \frac{abc}{4\Delta} = \frac{13\times 14\times 15}{4\times 84} = \frac{65}{8}$

(ii) $r = \frac{\Delta}{s} = \frac{84}{21} = 4$;

(iii) $r_1 = \frac{\Delta}{s-a} = \frac{84}{21-13} = \frac{84}{8} = \frac{21}{2}$

(iv) $r_2 = \frac{\Delta}{s-b} = \frac{84}{21-14} = \frac{84}{7} = 12$

(v) $r_3 = \frac{\Delta}{s-c} = \frac{84}{21-15} = \frac{84}{6} = 14$