3.MATRICES

(2 x 2) + (1 x 4) + (2 x 7) = 22 Marks

IMP FORMULAS, KEY CONCEPTS

1) BASIC TERMINOLOGY

1.1) MATRIX: An ordered rectangular array of elements is called a matrix.

Ex 1: $A = [1\ 2\ 3]_{1\times 3}$; **Ex 2:** $A = \begin{bmatrix}1\\2\end{bmatrix}_{2\times 1}$; **Ex 3:** $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}_{2\times 2}$; **Ex 4:** $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}_{3\times 3}$

Note: A matrix having m rows and n columns is said to be of **order $m \times n$**

1.2) Equality of matrices: Matrices A,B are said to be equal, written as A=B if

(i) A, B are of same order (ii) the corresponding elements in A,B are equal

Ex: $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ but $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \neq \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} \neq \begin{bmatrix}1 & 2 & 5\\3 & 4 & 6\end{bmatrix}$

2) TYPES OF MATRICES-I

2.1) Null matrix or zero matrix: If each element of a matrix is zero then it is called a null matrix.

Ex: $O = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

2.2) Square matrix: A matrix in which the number of rows is equal to number of columns, is called a square matrix. A square matrix of order n×n is also called a square matrix of order n.

Ex: $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ is a square matrix of order 2= $\begin{bmatrix}2 & 5 & 0\\1 & 2 & 4\\0 & 8 & 7\end{bmatrix}$ is a square matrix of order 3

2.3) Principal diagonal: If A is a square matrix then the diagonal in A, from the first element of first row to the last element of the last row is called the principal diagonal of A.

2.4) Trace of a matrix: If A is a square matrix then the sum of the elements in the principal diagonal of A is called trace of A and it is denoted by Tr(A)

Ex: If $A = \begin{bmatrix}1 & 2 & 3\\4 & -5 & -7\\6 & 8 & 10\end{bmatrix}$ then $Tr(A) = 1 + (-5) + 10 = 6$

2.5) Diagonal matrix: If each non-diagonal element of a square matrix is zero then the matrix is called a diagonal matrix. i.e., a matrix $A = [a_{ij}]_{nxn}$ where $a_{ij} = 0$ for $i \neq j$ is called a diagonal matrix and it is denoted by diag $(a_{11}\ a_{22}\ ....\ a_{nn})$

Ex: $diag(1,2) = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, $diag(1,3,2) = \begin{bmatrix}1 & 0 & 0\\0 & 3 & 0\\0 & 0 & 2\end{bmatrix}$, $diag(5,0,1) = \begin{bmatrix}5 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$

IMP FORMULAS, KEY CONCEPTS

MULTIPLICATION OF MATRICES

1) Multiplication of Matrices: Two Matrices A and B are said to be conformable for multiplication in that order written as AB if the number of columns of A is equal to the number of rows of B.

Row - by - Column multiplication: To find the product AB, the elements in the row(s) of the first matrix A should be multiplied by the corresponding elements in the column(s) of the second matrix B.

If the order of A is mxn and order of B is nxp, then the order of the product matrix AB is mxp.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}_{3\times2} = \begin{bmatrix} 1(7)+2(9)+3(11) & 1(8)+2(10)+3(12) \\ 4(7)+5(9)+6(11) & 4(8)+5(10)+6(12) \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}_{2\times2}$

2) Special properties of Multiplication of Matrices unlike Multiplication of Reals:

1) Matrix multiplication is not commutative i.e., ingeneral $AB \neq BA$.
But, we can find matrices A, B such that AB = BA.

Note: If AB is defined then BA may not always be defined. Even if AB and BA both are defined (even when A,B are square matrices of same order) then AB and BA may not be equal.

Rem 1: If AB = BA then A,B are said to be commutative.

Rem 2: If AB = –BA then A,B are said to be anticommutative.

Rem 3: If A & I are square matrices of same order, I being a unit matrix then AI = IA = A. Hence this unit matrix is called Identity matrix under multiplication.

2) If AB = O then either A or B need not be equal to O

Ex: $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus zero divisors (matrix) exist in 'matrix Algebra'.

3) If AB= AC, $A \neq O$, then B need not be equal to C, i.e., cancellation laws do not hold true.

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$

Note: Cancellation laws hold true, if A is a non-singular matrix.

4) Let AB = B then A need not be the unit matrix I. **Ex:** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

5) If AB = O then BA need not be equal to O.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \neq O$

IMP FORMULAS, KEY CONCEPTS

TRANSPOSE, DETERMINANT, ADJOINT & INVERSE OF A MATRIX

1) TRANSPOSE OF A MATRIX: The matrix obtained by interchanging the rows and columns of a given matrix is called the Transpose of the given matrix. Transpose of A is denoted by A^T or A'

Ex 1: If $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}_{3\times2}$ then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}_{2\times3}$ **Ex 2:** If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} \Rightarrow A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times2}$

1) SYMMETRIC MATRIX: A square matrix A is said to be a symmetric matrix if $A^T = A$.

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Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 0 & -1 & 4 \end{bmatrix} C = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

2) SKEW-SYMMETRIC MATRIX:

A square matrix A is said to be a skew-symmetric matrix if $A^T = -A$.

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$

3) DETERMINANT OF A MATRIX

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then determinant of A is denoted by det A or |A| or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Ex: $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3(2) - (1)(4) = 6 - 4 = 2$

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ then its determinant |A| is given as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

Def: If $|A| = 0$ then A is said to be a **singular matrix** and if $|A| \neq 0$ then A is said to be a **non-singular matrix**.

4) MINOR OF AN ELEMENT: If a_{ij} is an element which is in the i^{th} row and j^{th} column of a square matrix A, then the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A is called minor of a_{ij} and it is denoted by M_{ij}.

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then M_{11} = Minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

M_{12} = Minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$ and so on.

5) COFACTOR OF AN ELEMENT: If a_{ij} is an element which is in the i^{th} row and j^{th} column of a square matrix A, then the product of $(-1)^{i+j}$ and the minor of a_{ij} is called cofactor of a_{ij} and it is denoted by A_{ij}.

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $A_{11} = (-1)^{1+1} M_{11} = +1\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

$A_{12} =$ Cofactor of $a_{12} = (-1)^{1+2} M_{11} = -1\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$ and so on.

6) ADJOINT OF A MATRIX: The transpose of the matrix obtained by replacing the elements of a square matrix A by the corresponding cofactors is called the adjoint matrix of A.

The adjoint of the matrix A is denoted by Adj A or adj A.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

7) INVERSE OF A MATRIX: A square matrix A is said to be an invertible matrix if there exists a square matrix B such that AB = BA = I and the matrix B is called inverse of A.

- If A is a nonsingular matrix then $A^{-1} = \dfrac{1}{\det A}(\text{Adj } A)$

☞ **Inverse of a 2 × 2 non-singular matrix**:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ Determinant of A = ad – bc ≠ 0

$\Rightarrow$ Minor matrix of $A = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \Rightarrow$ Cofactor matrix of $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$\Rightarrow$ Adjoint matrix of $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow$ Inverse matrix of $A = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if ad – bc ≠ 0

Ex: If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ then, $A^{-1} = \dfrac{1}{14 - 15}\begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} = -1\begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$

IMP FORMULAS, KEY CONCEPTS

PROPERTIES OF DETERMINANTS

1) Result 1: If A is a square matrix, then det A = det A^T. Thus $|A| = |A^T|$

Example: $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3(2) - 1(4) = 2$

$A^T = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \Rightarrow |A^T| = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 3(2) - 4(1) = 2 \quad \therefore |A| = |A^T|$

2) Result 2: The sign of the determinant of a square matrix changes if any two rows (or columns) in the matrix are interchanged.

Example: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1\times 4) - (3\times 2) = -2$ *and* $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = (3\times 2) - (1\times 4) = 2$

3) Result 3: If any two rows (or columns) of a square matrix are identical, the value of the determinant of the matrix is zero.

Example: $\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 7 \\ 2 & 4 & -6 \end{vmatrix} = 0$. Because elements in R_1 and R_3 are proportional

4) Result 4: If all the elements of a row (or column) of a square matrix are multiplied by a number k then the value of the determinant of the matrix obtained is k times the determinant of the given matrix. (For a 3 x 3 matrix A, $|kA| = k^3|A|$)

Example: If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = p$ then $\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = kp, \begin{vmatrix} ka & kb & kc \\ ld & le & lj \\ g & h & i \end{vmatrix} = (kl)p, \begin{vmatrix} ka & kb & kc \\ ld & le & lj \\ mg & mh & mi \end{vmatrix} = (klm)p$

Also $k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & kb & c \\ d & ke & f \\ g & kh & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{vmatrix}$ etc.,

☞ $k\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{bmatrix}$ & $\begin{vmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{vmatrix} = k^3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

5) Result 5: If each element in a row (or column) of a square matrix is the sum of the two numbers, then its determinant can be expressed as the sum of the determinants of two square matrices.

Example: $\begin{vmatrix} a_1 & b_1 + x & c_1 \\ a_2 & b_2 + y & c_2 \\ a_3 & b_3 + z & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & x & c_1 \\ a_2 & y & c_2 \\ a_3 & z & c_3 \end{vmatrix}$

6) **Result 6:** If the elements of a row (or column) of a square matrix are added with k times the corresponding elements of another row (or column), then the value of the determinant of the matrix obtained is same as the value of determinant of the given matrix.

Example: $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1+ky_1 & y_1 & z_1 \\ x_2+ky_2 & y_2 & z_2 \\ x_3+ky_3 & y_3 & z_3 \end{vmatrix}$

☞ The value of the determinant of a matrix do not change with the operations like R_1+R_2, R_1-R_2, R_1+kR_2, $R_1+R_2+R_3$ or C_1+C_2, C_1-C_2, C_1+kC_2, $C_1+C_2+C_3$

7) **Result 7:** If the elements of a row/column of a square matrix are proportional to some other row i.e.,k times the elements of another row/column, then the determinant of the matrix is zero.

Example: $\begin{vmatrix} a_1 & kb_1 & b_1 \\ a_2 & kb_2 & b_2 \\ a_3 & kb_3 & b_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = 0$

8) **Result 8:** If the elements of a square matrix are polynomials in x and two rows (or columns) are identical when x=a, then (x–a) is a factor of the determinant of the matrix.

9) **Result 9:** The determinant of triangular matrix is the product of the diagonal elements of the matrix $\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3$

10) **Result 10:** If A and B are two square matrices of same type, then det (AB)=det A.det B.

Notation: While evaluating determinants, we use the following notation.

(i) $R_1 \to kR_1$ means that the elements of R_1 are multiplied by k.

(ii) $R_1 \to R_1+kR_2$ means that the elements of R_1 are added with k times the corresponding elements of R_2.

Remark: The sum of the products of the elements of any row (or column) of a square matrix with the cofactors of the corresponding elements of another row (or column) of the matrix is zero.

☞ **The following observations are helpful to avoid the confusion between the matrix representation and determinant representation.**

(i) Elements of matrix are enclosed between square brackets while elements of determinants are enclosed between vertical lines.

(ii) Matrix is an array of numbers but determinant of a matrix is a real value.

(iii) In matrices we deal with square and rectangular matrices, but in determinants we deal with the determinants of square matrices only.

(iv) Distinction should be made clearly especially w.r.to scalar multiplication property.(Result 4)

(v) The process of matrix multiplication is unique but the process of determinant multiplication can be done in several ways.

IMP FORMULAS, KEY CONCEPTS

RANK OF A MATRIX

1) **Sub Matrix:** The matrix obtained by deleting some rows or columns or both of a given matrix is called a sub matrix of the given matrix.

2) **Minor:** The determinant of a square submatrix of a given matrix is called minor. If the order of the square submatrix is r then the minor is called r-rowed minor or minor of order r.

3) **Rank of a Matrix:** A positive integer r is said to be the rank of a non-zero matrix A, denoted by rank(A) or Γ(A) if

(i) there is atleast one non-zero minor of order r

(ii) the minor of order (r+1) should be zero

Thus the rank of a matrix is the order of any highest order non-zero minor of the matrix.

Rem-1 : The Rank of a zero matrix is defined as zero.

Rem-2 : Rank exists for a rectangular matrix also.

Hint 1: In order to find the rank of a matrix, we have to search for a non-zero minor in the descending order.

Hint 2: Let A be a non-zero matrix of order 3x3 then if

(i) A is non-singular $\Rightarrow$ Rank (A)=3

(ii) A is singular and atleast one of its 2×2 submatrix is non singular $\Rightarrow$ rank(A)=2

(iii) A is a singular and every 2×2 submatrix is also singular $\Rightarrow$ rank (A)=1

I. Find the rank of each of the following matrices

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 0 & -4 \\ 2 & -1 & 3 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Sol: (i) Consider $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1(1) - 0 = 1 \neq 0$ $\therefore$ Rank(A) = 2

(ii) Consider a 2×2 minor $\begin{vmatrix} 1 & -4 \\ 2 & 3 \end{vmatrix} = 3 + 8 = 11 \neq 0$ $\therefore$ Rank(A) = 2

(iii) Consider $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 1(0) - 0 = 0$. Also, a 1×1 minor is $|1| = 1 \neq 0$ $\therefore$ Rank(A) = 1

(iv) $A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \det A = 1(6-0) - 4(4-0) - 1(2-0) = 6 - 16 - 2 = -12 \neq 0$

Here, $\det A \neq 0$ and order of A is 3×3. $\therefore$ Rank (A) = 3

IMP FORMULAS, KEY CONCEPTS

CONSISTENT AND INCONSISTENT SYSTEM OF EQUATIONS

In this section, we study the existence and nature of solutions of a Homogenous system and non-Homogeneous system of linear equations in 3 variables.

A system of linear equations is said to be
(i) consistent, if it has a solution
(ii) inconsistent, if it has no solution.

Consider the following system of equations in 3 variables

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \qquad \text{........(1)} \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

The corresponding matrix equation is AX = D, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$ is the coefficient matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ is the variable matrix $$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$ is the constant matrix

Also, the matrix written as $[A\ D] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ is called the augmented matrix

If D is a zero matrix then the system given by (1), is said to be a Homogeneous system
If D is a non-zero matrix then the system given by (1), is said to be a non-Homogeneous system.

1) **Homogeneous system:** Consider the following homogeneous system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

The equivalent matrix equation is AX = O where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is clear that $x = 0, y = 0, z = 0$ is always a solution and this is known as the trivial solution of the homogeneous system. Thus, a homogeneous system is always consistent.

But for some Homogeneous systems, there may exist some non-trivial (infinitely many) solutions also. Thus we are interested to find, whether a given system of Homogeneous equations possess non-trivial solution or not. For this we use the following theorem.

Theorem: The system of equations in 3 variables AX = O has
(i) the trivial solution only if rank(A) = 3
(ii) a non-trivial solution (infinite number of solutions) if rank(A) $\neq$ 3

IMP FORMULAS, KEY CONCEPTS

3(G) CRAMER'S RULE, MATRIX INVERSION METHOD & GAUSS JORDAN METHOD

1) A given system of equations can be solved by some matrix methods viz., (1) Cramer's rule (2) Matrix inversion method (3) Gauss-Jordan method..

1.1) Cramer's Rule: Let the given system of equations be written in the matrix equation form as AX = B. Let A be non-singular and

$$|A| = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then the solution of the system is given by $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

1.2) Matrix inversion method: Let the given system of equations be written in the matrix equation form as AX = D and if A is non-singular then the solution of system is given by $X = A^{-1}D$.

If |A| = 0, then the system may be consistent, but the above 2 methods fail to arrive any solution, then Gauss Jordon method paves the way, in such situations.

1.3) Gauss Jordan method: Working Procedure:

STEP 1: Write Augmented matrix of the given system.

$$[AB] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

STEP 2: Apply appropriate elementary row transformations on the augmented matrix to reduce it into the form $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$

Then the solution is given by $x = \alpha, y = \beta, z = \gamma$.

STEP 3: The elementary transformations on the agumented matrix [AB] ultimately reduces it into any one of the following forms. Hence the solution can be determined accordingly.

(i) $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$ The solution is $x = \alpha, y = \beta,. z = \gamma$

(ii) $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & k & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Here, the system has 'infinitely many solutions'.

(iii) $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 0 & \gamma \end{bmatrix}$ Here, the system has 'no solution'.

1. By using Cramer's rule solve 2x – y + 3z =8, –x+2y+z =4, 3x+y–4z =0

Sol: Given equations can be written as AX = D, where

$$A=\begin{bmatrix}2 & -1 & 3\\ -1 & 2 & 1\\ 3 & 1 & -4\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, D=\begin{bmatrix}8\\ 4\\ 0\end{bmatrix}$$

Now, $$\Delta=\det A=\begin{vmatrix}2 & -1 & 3\\ -1 & 2 & 1\\ 3 & 1 & -4\end{vmatrix}=2(-8-1)+1(4-3)+3(-1-6)=-18+1-21=-38$$

$$\Delta_1=\begin{vmatrix}8 & -1 & 3\\ 4 & 2 & 1\\ 0 & 1 & -4\end{vmatrix}=2(-8-1)+1(-16-0)+3(4-0)=-72-16+12=-76$$

$$\Delta_2=\begin{vmatrix}2 & 8 & 3\\ -1 & 4 & 1\\ 3 & 0 & -4\end{vmatrix}=2(-16-0)-8(4-3)+3(-0-12)=-32-8+36=-76$$

$$\Delta_3=\begin{vmatrix}2 & -1 & 8\\ -1 & 2 & 4\\ 3 & 1 & -0\end{vmatrix}=2(0-4)+1(0-12)+8(-1-6)=-8-12-56=-76$$

$$\therefore \text{By Carmer's rule } x=\frac{\Delta_1}{\Delta}=\frac{-76}{-38}=2;\quad y=\frac{\Delta_2}{\Delta}=\frac{-76}{-38}=2;\quad z=\frac{\Delta_3}{\Delta}=\frac{-76}{-38}=2$$

$\therefore$ The solution is x = 2, y = 2, z = 2

2 By using Matrix inversion method, solve x–y+3z=5, 4x+2y–z=0, –x+3y+z=5 .

Sol: Matrix equation of the given system of equations is AX=D, where

$$A=\begin{bmatrix}1&-1&3\\4&2&-1\\-1&3&1\end{bmatrix};X=\begin{bmatrix}x\\y\\z\end{bmatrix},D=\begin{bmatrix}5\\0\\5\end{bmatrix}\quad \therefore \text{ The solution of AX=D is } X=A^{-1}D$$

First we find A^{-1}

$$|A|=\begin{vmatrix}1&-1&3\\4&2&-1\\-1&3&1\end{vmatrix}=1(2+3)+1(4-1)+3(12+2)=1(5)+1(3)+3(14)=5+3+42=50\neq0$$

The co-factor matrix of A is

$$\begin{bmatrix}+\begin{vmatrix}2&-1\\3&1\end{vmatrix}&-\begin{vmatrix}4&-1\\-1&1\end{vmatrix}&+\begin{vmatrix}4&2\\-1&3\end{vmatrix}\\-\begin{vmatrix}-1&3\\3&1\end{vmatrix}&+\begin{vmatrix}1&3\\-1&1\end{vmatrix}&-\begin{vmatrix}1&-1\\-1&3\end{vmatrix}\\+\begin{vmatrix}-1&3\\2&-1\end{vmatrix}&-\begin{vmatrix}1&3\\4&-1\end{vmatrix}&+\begin{vmatrix}1&-1\\4&2\end{vmatrix}\end{bmatrix}=\begin{bmatrix}(2+3)&-(4-1)&(12+2)\\-(-1-9)&(1+3)&-(3-1)\\(1-6)&-(-1-12)&(2+4)\end{bmatrix}=\begin{bmatrix}5&-3&14\\10&4&-2\\-5&13&6\end{bmatrix}$$

$$\Rightarrow \text{Adj } A=\begin{bmatrix}5&10&-5\\-3&4&13\\14&-2&6\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj}A)=\frac{1}{50}\begin{bmatrix}5&10&-5\\-3&4&13\\14&-2&6\end{bmatrix}$$

$$\text{Now, } X=A^{-1}D=\frac{1}{50}\begin{bmatrix}5&10&-5\\-3&4&13\\14&-2&6\end{bmatrix}\begin{bmatrix}5\\0\\5\end{bmatrix}=\frac{1}{50}\begin{bmatrix}25+0-25\\-15+0+65\\70+0+30\end{bmatrix}=\frac{1}{50}\begin{bmatrix}0\\50\\100\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\therefore X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}$$

3. Solve the equations x+y+z=9, 2x+5y+7z=52, 2x+y-z=0, by Gauss-Jordan method

Sol : The matrix equation corresponding to the given system of equations be AX = D,

The augmented matrix is $[AD] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

$$[AD]=\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -4 & -8 & -52 \end{bmatrix} \begin{matrix} (\because R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_2) \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 1 & 2 & 13 \end{bmatrix} (\because R_3 \to (-1/4)R_3)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 3 & 5 & 34 \end{bmatrix} (\because R_{32})$$

$$= \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -1 & -5 \end{bmatrix} (\because R_3 \to R_3 - 3R_2)$$

$$= \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -5 \end{bmatrix} \begin{matrix} (\because R_1 \to R_1 + R_3) \\ R_2 \to R_2 + 2R_3) \end{matrix}$$

$$[AD] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \begin{matrix} (\because R_1 \to R_1 - R_2 \\ \\ R_3 \to (-1)R_3 \end{matrix}$$

$\therefore$ from the last augumented matrix, we get x = 1, y = 3, z = 5

4. Solve the equations x + y + z = 1, 2x + 2y + 3z = 6, x + 4y + 9z = 3, by Gauss-Jordan method

Sol: Augumented Matrix of the given system of equations is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 6 \\ 1 & 4 & 9 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 8 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 0 & -30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Thus [AD] is reduced to Form I of Gauss Jordan Solution.

Hence there exists a Unique solution.

$\therefore$ From (I), the solution is x = 7, y = –10, z = 4.

5. Solve the equations x – 3y – 8z = –10, 3x + y – 4z = 0, 2x + 5y + 6z = 13, by Gauss-Jordan method

Sol: The augmented matrix is $[AD] = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} \\ R_2(1/10) \\ R_3(1/11) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 4R_2 \\ \\ \\ \end{matrix} \quad ....(I)$$

Thus [AD] is reduced to Form II with all zeroes in the third row.

Thus Rank (A) = 2; Rank[AD] = 2 < 3 (No. of Unknowns)

$\therefore$ The System is Consistent with Infinite number of solutions.

From (I) we have $x + y = 2$ .....(i) , $y + 2z = 3$ ....(ii)

Let $z = k, k \in R$ (ii) $\Rightarrow y = 3 - 2z = 3 - 2k$;

$$(i) \Rightarrow x = 2 - y = 2 - (3 - 2k) = 2 - 3 + 2k = 2k - 1$$

$\therefore$ The solutions $x = -1 + 2k, y = 3 - 2k, z = k, \; k \in R$

BULLET MASTER'S

MATH BEATS!

Matrices ⇒ Inter Students' Most Favourite Chapter

Matrices ⇒ Most Scoring Chapter in Maths-1A (2 + 2 + 4 + 7 + 7 = 22 Marks)

MATRIX FORM OF IPE - 2022 TOPPERS (STAR COLLEGE)

Name / Subject	Eng	Skt	1 A	1 B	Phy	Chem	Total
Rahamathunnisa	96	99	75	75	60	60	465
S.Srinija	96	99	75	75	60	60	465
T.Theerdhasri	96	99	75	73	60	60	465
V.Durga Sushma	95	99	75	75	60	60	464
T.Rajasri	94	99	75	75	60	60	463
M.S.Reshma Sai	96	99	75	75	60	58	463

6×7

BULLET MASTER'S

MATH BEATS!

PECULIAR PROPERTIES of Matrix Multiplication

In Real Numbers

1) $ab = ba$ $(2 \times 3 = 3 \times 2)$
2) $ab = 0 \Rightarrow a = 0$ (or) $b = 0$

   Ex: $2x = 0 \Rightarrow x = 0$
3) $ab = ac,\ a \neq 0 \Rightarrow b = c$

   Ex: $\not{2}x = \not{2}y \Rightarrow x = y$
4) $ab = b$ then $a = 1,\ b \neq 0$

   Ex: $2x=2 \Rightarrow x=1$

In Matrices

1) $AB \neq BA$ (Generally); Rarely $AB = BA$
2) $AB = O$ need not imply $A = O$ (or) $B = O$

   Ex: $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
3) $AB = AC,\ A \neq O \Rightarrow B$ need not $= C$

   Ex: $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$
4) $AB = B$ then A need not be the unit matrix I.

   Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

మొత్తానికి Matrix Multiplication కొంచెం తేడాగానే కనిపిస్తుంది...... కొంచెం కాదు బాగా....