Previous IPE

SOLVED PAPERS

MARCH-2023 (AP)

PREVIOUS PAPERS

IPE: MARCH-2023(AP)

MATHS-2B

Time : 3 Hours | Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **10 × 2 = 20**

1. Find a if $2x^2+ay^2-3x+2y-1=0$ represents a circle,also find its radius.
2. Find the value of k if the points (1,3), (2,k) are conjugate w.r.to the circle $x^2 + y^2 = 35$.
3. Find the equation of the radical axis of $x^2 + y^2 - 5x + 6y + 12 = 0$, $x^2 + y^2 + 6x - 4y - 14 = 0$.
4. Find the value of k, if the line $2y = 5x + k$ is a tangent to the parabola $y^2 = 6x$.
5. If the eccentricity of a hyperbola is 5/4, then find the eccentricity of its conjugate hyperbola.
6. Evaluate $\int\sqrt{1-\cos 2x}\,dx$
7. Evaluate $\int e^x(\tan x+\log \sec x)dx$
8. Evaluate $\int_0^2 |1-x|dx$
9. Evaluate $\int_0^{\pi/2} \sin^6 x\cos^4 x\,dx$
10. Find the order of the differential equation of the family of all circles with their centres at the origin.

SECTION-B

II. Answer any FIVE of the following SAQs: **5 × 4 = 20**

11. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2 + y^2 - 2x + 4y - 20 = 0$ and $x^2 + y^2 - 2x - 8y + 1 = 0$ are in the ratio 2:1 then show that the equation of locus of P is $x^2 + y^2 - 2x - 12y + 8 = 0$.
12. Find the equation and length of the common chord of the two circles. $x^2 + y^2 + 3x + 5y + 4 = 0$ and $x^2 + y^2 + 5x + 3y + 4 = 0$
13. Find the eccentricity, coordinates of foci of the ellipse $3x^2 + y^2 - 6x - 2y - 5 = 0$
14. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentricity e,passes through one end of the minor axis, then show that $e^4 + e^2 = 1$
15. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$, which are (i) parallel and (ii) perpendicular to the line $y = x - 7$
16. Evaluate $\int_0^{\pi/2}\frac{dx}{4+5\cos x}$
17. Solve $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

SECTION-C

III. Answer any FIVE of the following LAQs: **5 × 7 = 35**

18. Find the equation of the circle passing through (4, 1) ,(6, 5) and having the centre on theline $4x + y - 16 = 0$.
19. Find the equation of the circle which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at (5, 5) with radius 5.
20. Show that the equation of the parabola in the standard form is $y^2 = 4ax$.
21. Evaluate $\int\frac{3\sin x+\cos x+7}{\sin x+\cos x+1}dx$
22. If $I_n = \int\cos^n x\,dx$, then show that $I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}I_{n-2}$ and hence deduce the value of $\int\cos^4 x\,dx$
23. Evaluate $\int_0^1\frac{\log(1+x)}{1+x^2}dx$
24. Solve $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$

IPE AP MARCH-2023
SOLUTIONS

SECTION-A

1. Find value of 'a' if $2x^2+ay^2-3x+2y-1=0$ represents a circle and also find its radius.

Sol: In the equation of a circle, we know

Coefficient of x^2=Coefficient of $y^2 \Rightarrow 2=a$

$\therefore$ Circle is $2x^2+2y^2-3x+2y-1=0 \quad \Rightarrow x^2+y^2-\frac{3}{2}x+y-\frac{1}{2}=0 \Rightarrow g=\frac{-3}{4}, f=\frac{1}{2}, c=\frac{-1}{2}$

Radius r= $\sqrt{\left(\frac{-3}{4}\right)^2+\left(\frac{1}{2}\right)^2+\frac{1}{2}} = \sqrt{\frac{9}{16}+\frac{1}{4}+\frac{1}{2}} = \sqrt{\frac{9+4+8}{16}} = \frac{\sqrt{21}}{4}$

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2. Find the value of k if the points (1,3), (2,k) are conjugate w.r.to the circle $x^2+y^2=35$.

Sol: The points (1,3), (2,k) are conjugate to w.r.t the circle S= $x^2+y^2-35=0 \Rightarrow S_{12}=0$

$\Rightarrow x_1x_2+y_1y_2-35=0 \Rightarrow (1)(2)+(3)(k)-35=0 \Rightarrow 3k=33 \Rightarrow k=11$

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3. Find the equation of the radical axis of the given circles $x^2+y^2-5x+6y+12=0$, $x^2+y^2+6x-4y-14=0$.

A: The given equations are $S\equiv x^2+y^2-5x+6y+12=0$ and $S'\equiv x^2+y^2+6x-4y-14=0$.

Now the equation of radical axis of the given circles is $S-S'=0$

$\Rightarrow -5x-6x+6y+4y+12+14=0 \Rightarrow -11x+10y+26=0$

$\Rightarrow 11x-10y-26=0$

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4. Find the value of k if the line 2y=5x+k is a tangent to the parabola $y^2=6x$.

Sol: Given parabola is $y^2=6x \quad \Rightarrow 4a=6 \Rightarrow a=\frac{6}{4}=\frac{3}{2}$

Given line is $2y=5x+k \Rightarrow y=\frac{5}{2}x+\frac{k}{2}$ Comparing with y=mx+c we get $m=\frac{5}{2}, c=\frac{k}{2}$

Tangential condition is c=a/m

$\Rightarrow \frac{k}{2}=\frac{\frac{3}{\cancel{2}}}{\frac{5}{\cancel{2}}} \Rightarrow \frac{k}{2}=\frac{3}{5} \Rightarrow k=2\left(\frac{3}{5}\right)=\frac{6}{5}$

5. If eccentricity of a hyperbola is 5/4, then find eccentricity of its conjugate hyperbola.

Sol: Let e= 5/4 and the eccentricity of the conjugate hyperbola be e_1

Then, $\frac{1}{e^2}+\frac{1}{e_1^2}=1 \Rightarrow \frac{1}{(5/4)^2}+\frac{1}{e_1^2}=1 \Rightarrow \frac{1}{e_1^2}=1-\frac{16}{25}=\frac{9}{25} \Rightarrow e_1^2=\frac{25}{9} \Rightarrow e_1=\frac{5}{3}$

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6. Evaluate $\int\sqrt{1-\cos 2x}\,dx$

Sol: $I=\int\sqrt{1-\cos 2x}\,dx=\int\sqrt{2\sin^2 x}\,dx=\sqrt{2}\int\sin x\,dx=-\sqrt{2}\cos x+c$

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7. Evaluate $\int e^x(\tan x+\log\sec x)dx$

Sol: Here, f(x)=logsecx $\Rightarrow$ f'(x)= $\frac{1}{\sec x}\sec x\tan x=\tan x$

$\therefore \int e^x(\log\sec x+\tan x)dx=e^x\log\sec x+c$

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8. Evaluate $\int_0^2|1-x|dx$

Sol : From the definition of Modulus function,

we have $|1-x|=1-x$ for $1-x\ge 0\Rightarrow x-1\le 0\Rightarrow x\le 1$

$|1-x|=-(1-x)=x-1$ for $1-x<0 \qquad \Rightarrow x-1>0\Rightarrow x>1$

$\therefore \int_0^2|1-x|dx=\int_0^1(1-x)dx+\int_1^2(x-1)dx$

$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^2=\left(1-\frac{1}{2}\right)+\left(\frac{4}{2}-2\right)-\left(\frac{1}{2}-1\right) \quad =\frac{1}{2}+0-\left(-\frac{1}{2}\right)=\frac{1}{2}+\frac{1}{2}=1$

9. Evaluate $\int_0^{\pi/2} \sin^6 x \cos^4 x dx$

Sol : $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx \quad = \frac{[(5)(3)(1)][(3)(1)]}{(10)(8)(6)(4)(2)} \frac{\pi}{2} = \frac{3\pi}{512}$

☞ Note the factor $\frac{\pi}{2}(\because m, n \text{ are even})$

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10. Find the order of the differential equation of the family of all circles with their centres at the origin.

Sol: The equation of the family of circles with centres at origin is $x^2+y^2=r^2$, (r)

Differentiating w.r.to x, we get $2x + 2y\frac{dy}{dx} = 0 \Rightarrow x + y\frac{dy}{dx} = 0$

$\therefore$ order is 1

SECTION-B

11. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2+y^2-2x+4y-20=0$ and $x^2+y^2-2x-8y+1=0$ are in the ratio 2:1 then show that the equation of locus of P is $x^2+y^2-2x-12y+8=0$

Sol: Let $P(x_1,y_1)$ be a point on the locus.

Let PT_1 = length of the tangent P to $S= x^2+y^2-2x+4y-20=0$

and PT_2 = length of the tangent P to $S' = x^2+y^2-2x-8y+1=0$

Given that $PT_1 : PT_2 = 2:1 \Rightarrow \frac{PT_1}{PT_2} = \frac{2}{1} \Rightarrow 1PT_1 = 2PT_2 \Rightarrow (PT_1)^2 = 4(PT_2)^2$

$\Rightarrow [x_1^2 + y_1^2 - 2x_1 + 4y_1 - 20] = 4[x_1^2 + y_1^2 - 2x_1 - 8y_1 + 1]$ $\quad [\because PT_1 = \sqrt{S_{11}}, PT_2 = \sqrt{S'_{11}}\ ]$

$\Rightarrow x_1^2 + y_1^2 - 2x_1 + 4y_1 - 20 = 4x_1^2 + 4y_1^2 - 8x_1 - 32y_1 + 4$

$\Rightarrow 3x_1^2 + 3y_1^2 - 6x_1 - 36y_1 + 24 = 0 \Rightarrow x_1^2 + y_1^2 - 2x_1 - 12y_1 + 8 = 0$

$\therefore$ Equation of locus of $P(x_1,y_1)$ is $x^2+y^2-2x-12y+8=0$

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12. Find the equation and length of the common chord of the two circles.

$x^2 + y^2 + 3x + 5y + 4 = 0$ and $x^2 + y^2 + 5x + 3y + 4 = 0$

Sol: Given circles are $S \equiv x^2 + y^2 + 3x + 5y + 4 = 0$ and $S' \equiv x^2 + y^2 + 5x + 3y + 4 = 0$

Equation of the common chord is $S-S' = 0 \Rightarrow -2x + 2y = 0 \Rightarrow x-y = 0$

Consider the circle $S \equiv x^2 + y^2 + 3x + 5y + 4 = 0$. Its centre $C\left(\frac{-3}{2}, \frac{-5}{2}\right)$

Radius $r = \sqrt{\frac{9}{4} + \frac{25}{4} - 4} = \sqrt{\frac{18}{4}} = \sqrt{\frac{9}{2}}$

Length of the perpendicular from $C\left(\frac{-3}{2}, \frac{-5}{2}\right)$ to the line $x-y = 0$ is $p = \frac{\left|\frac{-3}{2} + \frac{5}{2}\right|}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$

$\therefore$ Length of the common chord $= 2\sqrt{r^2 - p^2} = 2\sqrt{\frac{9}{2} - \frac{1}{2}} = 2\sqrt{4} = 2 \times 2 = 4$ units.

13. Find the eccentricity, coordinates of foci of the ellipse $3x^2+y^2-6x-2y-5=0$

Sol: Given that $3x^2 + y^2 - 6x - 2y - 5 = 0$

$\Rightarrow 3x^2 + y^2 - 6x - 2y = 5$

$\Rightarrow 3(x^2 - 2x) + (y^2 - 2y) = 5$

$\Rightarrow 3(x^2 - 2x + 1) + (y^2 - 2y + 1) = 5 + 3 + 1$

$\Rightarrow 3(x-1)^2 + (y-1)^2 = 9$

$$\Rightarrow \frac{(x-1)^2}{3} + \frac{(y-1)^2}{9} = 1$$

Comparing this with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get $a^2=3 \Rightarrow a = \sqrt{3}$, $b^2=9 \Rightarrow b=3$

Here $a<b$

$\therefore$ the ellipse is vertical

Also $(h, k) = (1, 1)$

(i) Eccentricity $e = \sqrt{\frac{b^2-a^2}{b^2}} = \sqrt{\frac{9-3}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}$

(ii) Foci $= (h, k \pm be)$ $= \left(1, 1 \pm \cancel{3}\left(\frac{\sqrt{2}}{\cancel{\sqrt{3}}}\right)\right) = (1, 1 \pm \sqrt{3}(\sqrt{2})) = (1, 1 \pm \sqrt{6})$

14. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentricity e, passes through one end of the minor axis, then show that $e^4+e^2=1$

Sol: The equation of the normal at (x_1,y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

Let $L=(ae,b^2/a)$ be the one end of the latus rectum

Hence equation of the normal at L is $\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2 \Rightarrow \frac{ax}{e}-ay=a^2-b^2$.....(1)

But (1) passes through the one end B'(0,–b) of minor axis

$$\Rightarrow \frac{a(0)}{e}-a(-b)=a^2-b^2 \Rightarrow ab=a^2-a^2(1-e^2) \Rightarrow ab=a^2e^2 \Rightarrow e^2=\frac{b}{a}$$

$$\therefore e^4=\frac{b^2}{a^2}=\frac{a^2(1-e^2)}{a^2}=1-e^2 \Rightarrow e^4+e^2=1$$

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15. Find the equations of the tangents to the hyperbola $3x^2-4y^2=12$ which are (a) Parallel (b) Perpendicular to the line $y=x-7$

Sol: Given hyperbola is $3x^2-4y^2=12$ $\Rightarrow \frac{x^2}{4}-\frac{y^2}{3}=1 \Rightarrow a^2=4, b^2=3$

Slope of the given line $y=x-7$ is $m=1$

$\Rightarrow$ Slope of its perpendicular is -1

Formula:

Tangent with slope m is $y=mx\pm\sqrt{a^2m^2-b^2}$

(i) Parallel tangent with slope m=1 is $y=1.x\pm\sqrt{4(1)^2-3}=x\pm1 \Rightarrow x-y\pm1=0$

(ii)Perpendicular tangent with slope m=–1 is $y=(-1)x\pm\sqrt{4(1)^2-3}=-x\pm1$

$\Rightarrow x+y\pm1=0$

16. Evaluate $\int_0^{\pi/2} \frac{dx}{4+5\cos x}$

Sol: Put $\tan\frac{x}{2} = t$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$.

Also, $x = 0 \Rightarrow t = 0,\ x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore I = \int_0^1 \frac{(2dt)/(1+t^2)}{4+5\left(\frac{1-t^2}{1+t^2}\right)} = \int_0^1 \frac{(2dt)/\cancel{(1+t^2)}}{\frac{4(1+t^2)+5(1-t^2)}{\cancel{1+t^2}}} = 2\int_0^2 \frac{dt}{9-t^2} = 2\int_0^2 \frac{dt}{3^2-t^2} = 2.\frac{1}{2(3)}\log\left[\frac{3+t}{3-t}\right]_0^1 = \frac{1}{3}\log\frac{4}{2} = \frac{1}{3}\log 2$$

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17. Solve $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$

Sol: Given D.E is $\frac{dy}{dx} = e^{x-y} + x^2e^{-y} = \frac{e^x}{e^y} + \frac{x^2}{e^y} = \frac{e^x + x^2}{e^y}$ $\quad \therefore \frac{dy}{dx} = \frac{e^x + x^2}{e^y}$

$$\Rightarrow e^y dy = (e^x + x^2)dx \Rightarrow \int e^y dy = \int(e^x + x^2)dx \Rightarrow e^y = e^x + \frac{x^3}{3} + c$$

SECTION-C

18. Find the equation of the circle passing through (4, 1) ,(6, 5) and having the centre on the line $4x + y - 16 = 0$.

Sol: **Method II:** Let the equation of the required circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$

A(4, 1) lies on S=0 $\Rightarrow 16 + 1 + 8g + 2f + c = 0 \Rightarrow 8g + 2f + c = -17$ ....(1)

B(6, 5) lies on S=0 $\Rightarrow 36 + 25 + 12g + 10f + c = 0 \Rightarrow 12g + 10f + c = -61$.....(2)

The centre $(-g, -f)$ lies on $4x + y - 16 = 0 \Rightarrow -4g - f - 16 = 0 \Rightarrow 4g + f = -16$....(3)

Now (2) – (1)$\Rightarrow 4g + 8f = -44$...........(4); Now (4) – (3)$\Rightarrow 7f = -28 \Rightarrow f = -4$

From (3), $4g - 4 = -16 \Rightarrow 4g = -12 \Rightarrow g = -3$

From (1), $8(-3) + 2(-4) + c = -17 \Rightarrow c = -17 + 24 + 8 = 15$

Substituting $g = -3$, $f = -4$, $c = 15$ in S=0 we have

Equation of the required circle is $x^2 + y^2 - 6x - 8y + 15 = 0$

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19. Find the equation of the circle which touches the circle $x^2+y^2-2x-4y-20=0$ externally at (5,5) with radius 5.

Sol: For the circle $x^2+y^2-2x-4y-20=0$ centre $C_1 = (1, 2)$

radius $r_1 = \sqrt{1+4+20} = \sqrt{25} = 5$

Let C_2 be the centre and r_2 be the radius of the required circle. Here, $r_2 = 5$

The point of contact P=(5,5)

The two circles touch each other externally.

Since Radii are equal, P(5,5) is mid point of C_1C_2 where $C_1 = (1, 2)$ and $C_2 = (x_1, y_1)$

$$\Rightarrow \left(\frac{x_1+1}{2}, \frac{y_1+2}{2}\right) = (5,5) \Rightarrow \frac{x_1+1}{2} = 5, \frac{y_1+2}{2} = 5 \Rightarrow x_1+1 = 10, y_1+2 = 10$$

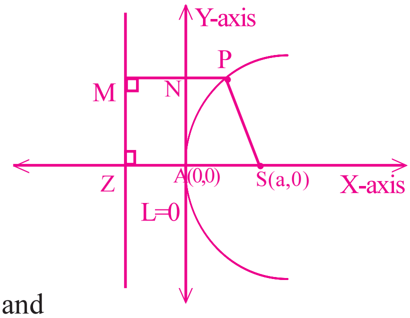
$\Rightarrow x_1 = 9, y_1 = 8 \quad \therefore C_2 = (9,8)$

The equation of the required circle is $(x-9)^2+(y-8)^2=5^2 \Rightarrow x^2+y^2-18x-16y+120=0$

20. Show that the equation of the parabola in the standard form is $y^2 = 4ax$.

Sol: Let S be the focus and L=0 be the directrix of the parabola.

Let Z be the projection of S on to the directrix



Let A be the mid point of SZ

$\Rightarrow SA=AZ \Rightarrow \frac{SA}{AZ}=1$

$\Rightarrow$ A is a point on the parabola.

Take AS, the principle axis of the parabola as X-axis and

the line perpendicular to AS through A as the Y-axis $\Rightarrow A=(0,0)$

Let $AS=a \Rightarrow S=(a,0),\ Z=(-a,0)$

$\Rightarrow$ the equation of the directrix is $x=-a \Rightarrow x+a=0$

Let $P(x_1,y_1)$ be any point on the parabola.

N be the projection of P on to the Y-axis.

M be the projection of P on to the directrix.

Here $PM=PN+NM=x_1+a$ $(\because PN = x-\text{coordinate of } P \text{ and } NM = AZ = AS = a)$

Now, by the focus directrix property of the parabola,

we have $\frac{SP}{PM}=1 \Rightarrow SP = PM \Rightarrow SP^2 = PM^2$

$\Rightarrow (x_1-a)^2+(y_1-0)^2=(x_1+a)^2$

$\Rightarrow y_1^2=(x_1+a)^2-(x_1-a)^2$

$\Rightarrow y_1^2=4ax_1$ $[\because (a+b)^2-(a-b)^2=4ab]$

$\therefore$ The equation of locus of $P(x_1,y_1)$ is $y^2=4ax$

21. Evaluate $\int \frac{\cos x+3\sin x+7}{\cos x+\sin x+1}dx$

Sol: Let $\cos x+3\sin x+7=A\frac{d}{dx}(\cos x+\sin x+1)+B(\cos x+\sin x+1)+C$

$\Rightarrow \cos x+3\sin x+7=A(-\sin x+\cos x)+B(\cos x+\sin x+1)+C$ ....(I)

$\Rightarrow \cos x+3\sin x+7=\cos x(A+B)+\sin x(-A+B)+(B+C)$

Equating the coefficients of cosx, we have A+B=1 ....(1)

Equating the coefficients of sinx, we have –A+B=3 ....(2)

Equating the constant terms, we have B+C=7 ....(3)

Now (1) + (2) $\Rightarrow$ 2B=4 $\Rightarrow$ B=2

From (1) , A=1–B =1–2= –1

From (3) , C=7–B =7–2= 5

Putting A= –1, B=2, C= 5 in (I) we get Nr.

$\cos x+3\sin x+7=-1(-\sin x+\cos x)+2(\cos x+\sin x+1)+5$

$\therefore I=\int\frac{\cos x+3\sin x+7}{\cos x+\sin x+1}dx=\int\frac{-1(-\sin x+\cos x)+2(\cos x+\sin x+1)+5}{\cos x+\sin x+1}dx$

$=-\int\frac{-\sin x+\cos x}{\cos x+\sin x+1}dx+2\int\frac{\cos x+\sin x+1}{\cos x+\sin x+1}dx+5\int\frac{1}{\cos x+\sin x+1}dx$

$=-\log|\cos x+\sin x+1|+2x+5\int\frac{1}{\cos x+\sin x+1}dx$ ......(II) $\left(\because \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right)$

Now, we find $I_1=\int\frac{1}{\cos x+\sin x+1}dx$

Put $\tan\frac{x}{2}=t \Rightarrow \sin x=\frac{2t}{1+t^2},\cos x=\frac{1-t^2}{1+t^2}$ and $dx=\frac{2dt}{1+t^2}$

$\therefore I_1=\int\frac{1}{\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}+1}\left(\frac{2dt}{1+t^2}\right)=\int\frac{1}{\frac{1-t^2+2t+1+t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}$

$=\int\frac{2dt}{2+2t}=\int\frac{2dt}{2(1+t)}=\int\frac{dt}{1+t}=\log|1+t|+c=\log\left|1+\tan\frac{x}{2}\right|+c$

From (II), $I=-\log|\cos x+\sin x+1|+2x+5\log\left|1+\tan\frac{x}{2}\right|+c$

22. Evaluate the reduction formula for $I_n=\int\cos^n x\,dx$ and hence find $\int\cos^4 x\,dx$

Sol: Given $I_n=\int\cos^n x\,dx=\int\cos^{n-1}x(\cos x)dx$.

We apply the "By parts rule"

We take First function $u=\cos^{n-1}x$ and Second function $v=\cos x \Rightarrow \int v=\sin x$

From By parts rule, we have

$I_n=\cos^{n-1}x\,(\sin x)-\int(n-1)\cos^{n-2}x\,(-\sin x)\sin x\,dx$

$=\cos^{n-1}x(\sin x)+(n-1)\int\cos^{n-2}x\,(\sin x)\sin x\,dx$

$=\cos^{n-1}x\,(\sin x)+(n-1)\int\cos^{n-2}x\,(\sin^2 x)\,dx$ $=\cos^{n-1}x\,(\sin x)+(n-1)\int\cos^{n-2}x\,(1-\cos^2 x)dx$

$=\cos^{n-1}x\,(\sin x)+(n-1)\int\cos^{n-2}x\,dx-(n-1)\int\cos^n x\,dx$

$=\cos^{n-1}x\,(\sin x)+(n-1)I_{n-2}-(n-1)I_n$

$\therefore I_n=\cos^{n-1}x\,(\sin x)+(n-1)I_{n-2}-nI_n+I_n$

$\Rightarrow nI_n=\cos^{n-1}x(\sin x)+(n-1)I_{n-2}+\cancel{I_n}-\cancel{I_n}$

$\therefore I_n=\dfrac{\cos^{n-1}x(\sin x)}{n}+\dfrac{(n-1)}{n}I_{n-2}\ ...(1)$

Put n=4, 2,0 successively in (1), we get

$I_4=\dfrac{\cos^3 x\sin x}{4}+\dfrac{3}{4}I_2$

$\Rightarrow \dfrac{\cos^3 x\sin x}{4}+\dfrac{3}{4}\left[\dfrac{\cos x\sin x}{2}+\dfrac{1}{2}I_0\right]=\dfrac{\cos^3 x\sin x}{4}+\dfrac{3}{8}\cos x\sin x+\dfrac{3}{8}I_0$

$=\dfrac{\cos^3 x\sin x}{4}+\dfrac{3}{8}\cos x\sin x+\dfrac{3}{8}x+c$ $[\because I_0=x]$

23. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Sol: Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$ Also, $x=0\Rightarrow\theta=0; x=1\Rightarrow\theta=\frac{\pi}{4}$

And $1+x^2=1+\tan^2\theta=\sec^2\theta$

$$\therefore \int_0^1 \frac{\log(1+x)}{1+x^2}dx = \int_0^{\pi/4}\frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta d\theta = \int_0^{\pi/4}\log(1+\tan\theta)d\theta$$

$$\therefore I=\int_0^{\pi/4}\log[1+\tan\theta]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[\frac{(1+\tan\theta)+(1-\tan\theta)}{1+\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$=\int_0^{\pi/4}[\log 2-\log(1+\tan\theta)]\,d\theta$$

$$=\log 2\int_0^{\pi/4}1\,d\theta-\int_0^{\pi/4}\log(1+\tan\theta)d\theta=\log 2\int_0^{\pi/4}1\,d\theta-I$$

$$=\log 2[\theta]_0^{\pi/4}-I$$

$$\Rightarrow I+I=(\log 2)\left(\frac{\pi}{4}\right)\Rightarrow 2I=\left(\frac{\pi}{4}\right)(\log 2)$$

$$\Rightarrow I=\left(\frac{\pi}{8}\right)(\log 2)$$

24. Solve $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$

Sol: Given D.E is $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0 \Rightarrow \frac{dy}{dx}+y\left(\frac{2x}{1+x^2}\right)=\frac{4x^2}{1+x^2}$

It is in the form $\frac{dy}{dx}+yP(x)=Q(x)$ where $P(x)=\frac{2x}{1+x^2}$ and $Q(x)=\frac{4x^2}{1+x^2}$

Here, $P(x)=\frac{2x}{1+x^2} \Rightarrow \int P(x)dx=\int\frac{2x}{1+x^2}dx=\log(1+x^2)$ $\left(\because \int\frac{f'(x)}{f(x)}dx=\log f(x)\right)$

Now, $IF=e^{\int P(x)dx}=e^{\log(1+x^2)}=1+x^2$

$\therefore$ The solution is y(I.F)= $\int$(I.F)Q(x)dx

$\Rightarrow y(1+x^2)=\int(1+x^2)\left(\frac{4x^2}{1+x^2}\right)dx=\int 4x^2dx=\frac{4x^3}{3}+c$

$\therefore$ The solution is $y(1+x^2)=\frac{4x^3}{3}+c$