Previous IPE

SOLVED PAPERS

MARCH -2023 (AP)

PREVIOUS PAPERS

IPE: MARCH-2023(AP)

MATHS-1B

Time : 3 Hours | Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **$10 \times 2 = 20$**

1. Find the distance between parallel lines $5x - 3y - 4 = 0$ and $10x - 6y - 9 = 0$.
2. Show that the points A(3, 2, –4), B(5, 4, –6) and C(9, 8, –10) are collinear.
3. Write the equation of the plane $4x - 4y + 2z + 5 = 0$ in the intercept form.
4. If the increase in the side of a square is 4% then find the approximate percentage of increase in the area of the square.
5. Transform the equation $\sqrt{3}x + y = 4$ into (i) slope intercept form (ii) intercept form
6. Find the second order derivative of $y = \mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$.
7. Find the derivative of $e^{2x}.\log(3x+4)\left(x > \frac{-4}{3}\right)$. **8.** Verify Rolle's theorem for the function $x^2 - 1$ on $[-1, 1]$.
9. Compute $\underset{x\to 2^+}{\mathrm{Lt}}([x]+x)$ and $\underset{x\to 2-}{\mathrm{Lt}}([x]+x)$. **10.** Find $\underset{x\to 0}{\mathrm{Lt}}\left(\frac{3^x - 1}{\sqrt{1+x}-1}\right)$

SECTION-B

II. Answer any FIVE of the following SAQs: **$5 \times 4 = 20$**

11. If f is given by $f(x)=\begin{cases} k^2x - k & \text{if } x \geq 1 \\ 2 & \text{if } x<1 \end{cases}$ is a continuous function on R, then find k.
12. Find the derivative of sec3x using first principle.
13. The volume of a cube is increasing at a rate of 8 cubic centimeters per second. How fast is the surface area increasing when length of edge is 12 cm?
14. Find the lengths of normal and subnormal at a point on the curve $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$
15. Find the locus of P(x,y) which moves such that its distances from A(5,–4), B(7,6) are in the ratio 2:3.
16. When the axes are rotated through an angle $\pi/4$, find the transformed equation of $3x^2 + 10xy + 3y^2 = 9$.
17. Find the point on the straight lines $3x + y + 4 = 0$ which is equidistant from the points (–5, 6),(3, 2).

SECTION-C

III. Answer any FIVE of the following LAQs: **$5 \times 7 = 35$**

18. Find orthocentre of triangle whose sides are $x + 2y = 0$, $4x + 3y - 5 = 0$ and $3x + y = 0$.
19. Find the angle between the lines joining the origin to the points of intersection of the curve $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ and the line $3x - y + 1 = 0$.
20. S.T the area of triangle formed by the pair of lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is $\frac{n^2\sqrt{h^2-ab}}{\left|am^2 - 2hlm + bl^2\right|}$
21. Find the angle between two non-parallel lines whose direction cosines satisfy the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.
22. If $x^y + y^x = a^b$ then show that $\frac{dy}{dx} = -\left(\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$
23. If the tangent at a point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the coordinate axes in A,B then show that the length AB is a constant
24. The profit function p(x) of a company, selling x items per day is given by p(x)=(150–x)x–1000. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.

IPE AP MARCH-2023
SOLUTIONS

SECTION-A

1. Find the distance between the parallel lines 5x – 3y – 4 = 0, 10x – 6y – 9 = 0.

A:
- We write the first line 5x – 3y – 4 = 0 as 10x – 6y – 8 = 0 ....(1)
- Second line is 10x – 6y – 9 = 0 ....(2)
- $\therefore$ The distance between (1) & (2) is $\dfrac{|c_1 - c_2|}{\sqrt{a^2+b^2}} = \dfrac{|-8+9|}{\sqrt{10^2+6^2}} = \dfrac{|1|}{\sqrt{100+36}} = \dfrac{1}{\sqrt{136}}$

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2. Show that the points A(3, 2, –4), B(5, 4, –6) and C(9, 8, –10) are collinear and find the ratio in which B divides $\overline{AC}$.

A: A(3, 2, –4), B(5, 4, –6) and C(9, 8, –10) are the given points

$$AB = \sqrt{(3-5)^2 + (2-4)^2 + (-4+6)^2}$$

$$= \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$BC = \sqrt{(5-9)^2 + (4-8)^2 + (-6+10)^2}$$

$$= \sqrt{16+16+16} = \sqrt{48} = 4\sqrt{3}$$

$$CA = \sqrt{(9-3)^2 + (8-2)^2 + (-10+4)^2}$$

$$= \sqrt{36+36+36} = \sqrt{108} = 6\sqrt{3}$$

Now, $AB + BC = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3} = CA$

$\therefore$ A,B,C are collinear

Ratio in which B divides AC is AB : BC $= 2\sqrt{3} : 4\sqrt{3} = 2:4 = 1:2$

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3. Write the equation of the plane 4x – 4y + 2z + 5 = 0 in the intercept form.

A: The given equation of the plane is 4x – 4y + 2z + 5 = 0 $\Rightarrow$ 4x – 4y + 2z = –5

$\Rightarrow \dfrac{4x}{-5} + \dfrac{-4y}{-5} + \dfrac{2z}{-5} = 1 \Rightarrow \dfrac{x}{\left(\frac{-5}{4}\right)} + \dfrac{y}{\left(\frac{5}{4}\right)} + \dfrac{z}{\left(\frac{-5}{2}\right)} = 1$ which is in the intercept form $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

4. If the increase in the side of a square is 4% then find the approximate percentage of increase in the area of the square.

A:
- We take x as side of the square.
- ★ Given $\frac{dx}{x}\times 100 = 4$
- Area of the square A = x^2
- ★ $\Rightarrow$ dA=2xdx
- ★ $\Rightarrow \frac{dA}{A} = \frac{2\cancel{x}\,dx}{\cancel{x^2}} \Rightarrow \frac{dA}{A} = 2\left(\frac{dx}{x}\right)$
- ★ $\therefore \frac{dA}{A}\times 100 = 2\left(\frac{dx}{x}\right)\times 100 = 2(4) = 8$

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5. Transform the equation $\sqrt{3}x + y = 4$ into (i) slope intercept form(ii) intercept form

A: (i) Slope intercept form is y = mx + c

$\therefore \sqrt{3}x + y = 4 \Rightarrow y = -\sqrt{3}x + 4$

(ii) Intercept form is $\frac{x}{a} + \frac{y}{b} = 1$

$\therefore \sqrt{3}x + y = 4 \Rightarrow \frac{\sqrt{3}x}{4} + \frac{y}{4} = 1 \quad \Rightarrow \frac{x}{4/\sqrt{3}} + \frac{y}{4} = 1$

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6. Find the derivative of $\text{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$

A: We take $x = \tan\theta$, then $\theta = \text{Tan}^{-1}x$

$$\therefore \text{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) = \text{Tan}^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \text{Tan}^{-1}(\tan 2\theta) = 2\theta = 2(\text{Tan}^{-1}x) \qquad [\because x = \tan\theta \Rightarrow \theta = \text{Tan}^{-1}x]$$

$$\therefore \frac{d}{dx}(2\text{Tan}^{-1}x) = 2\frac{d}{dx}\text{Tan}^{-1}x \quad = 2\left(\frac{1}{1+x^2}\right) = \frac{2}{1+x^2}$$

7. If $y = e^{2x}.\log(3x+4)$ then find $\frac{dy}{dx}$

A: Formula: $\frac{d}{dx}(uv) = u.\frac{d}{dx}(v) + v.\frac{d}{dx}(u)$

Given $y = e^{2x}.\log(3x+4)$, then $\frac{dy}{dx} = e^{2x}\frac{d}{dx}\log(3x+4) + \log(3x+4)\frac{d}{dx}e^{2x}$

$$= e^{2x}\left(\frac{1}{3x+4}(3)\right) + \log(3x+4)e^{2x}.2$$

$$= e^{2x}\left(\frac{3}{3x+4} + 2\log(3x+4)\right)$$

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8. Verify Rolle's theorem for the function $x^2 - 1$ on $[-1, 1]$

A: Given $f(x) = x^2 - 1 \Rightarrow f'(x) = 2x$

f(x) is (i) continuous on $[-1, 1]$ and (ii) differentiable in $(-1, 1)$

(iii) $f(-1) = (-1)^2 - 1 = 1 - 1 = 0$; $f(1) = 1^2 - 1 = 1 - 1 = 0$

$\Rightarrow f(-1) = f(1)$

So, from Rolle's theorem, $f'(c) = 0 \Rightarrow 2c = 0 \Rightarrow c = 0$

$\therefore c = 0 \in (-1, 1)$ Hence, Rolle's theorem is verified.

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9. Compute $\underset{x\to 2+}{\text{Lt}}([x]+x)$ and $\underset{x\to 2-}{\text{Lt}}([x]+x)$

A: When $x \to 2+$ then $[x] = 2$ Now R.H.L $= \underset{x\to 2+}{\text{Lt}}\left([x]+x\right) = 2+2 = 4$

when $x \to 2-$ then $[x] = 1$ Now L.H.L $= \underset{x\to 2-}{\text{Lt}}\left([x]+x\right) = 1+2 = 3$

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10. Find $\underset{x\to 0}{\text{Lt}}\frac{3^x - 1}{\sqrt{1+x} - 1}$

A: $$\underset{x\to 0}{\text{Lt}}\frac{3^x - 1}{\sqrt{1+x} - 1} = \underset{x\to 0}{\text{Lt}}\left(\frac{3^x - 1}{x}\right)\left(\frac{x}{\sqrt{1+x} - 1}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\frac{3^x - 1}{x}.\underset{x\to 0}{\text{Lt}}\frac{x}{\sqrt{1+x} - 1} = \log 3.\underset{x\to 0}{\text{Lt}}\frac{x}{\sqrt{1+x} - 1}$$

$$= \log 3.\underset{x\to 0}{\text{Lt}}\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} = \log 3.\underset{x\to 0}{\text{Lt}}\frac{x(\sqrt{1+x}+1)}{1+x-1} = \log 3.\underset{x\to 0}{\text{Lt}}\frac{x(\sqrt{1+x}+1)}{x}$$

$$= \log 3.\underset{x\to 0}{\text{Lt}}(\sqrt{1+x}+1) = \log 3.(\sqrt{1+0}+1) = (\log 3)(1+1) = 2\log 3$$

SECTION-B

11. If f is given by $f(x)=\begin{cases}k^2x-k & \text{if } x\ge 1\\ 2 & \text{if } x<1\end{cases}$ is a continuous function on R, then find k.

A: (a) When x<1, $\quad L.H.L=\underset{x\to 1-}{Lt} f(x)=\underset{x\to 1-}{Lt} 2=2$ ....(1)

(b) When x>1, $\quad R.H.L=\underset{x\to 1+}{Lt} f(x)=\underset{x\to 1+}{Lt} k^2x-k=k^2(1)-k=k^2-k$ ....(2)

From (1) & (2), L.H.L= R.H.L $\quad$ [∵ f(x) is continuous at x=1as it is continuous on R]

So, $k^2-k=2 \Rightarrow k^2-k-2=0 \Rightarrow (k-2)(k+1)=0 \Rightarrow k=2$ (or) -1

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12. Find the derivative of sec3x using first principle.

A: We take f(x) = sec3x, then f(x + h) = sec3(x + h) = sec(3x + 3h)

From the first principle, $f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\frac{\sec(3x+3h)-\sec 3x}{h}$

$$=\underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{1}{\cos(3x+3h)}-\frac{1}{\cos 3x}\right)=\underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{\cos 3x-\cos(3x+3h)}{\cos(3x+3h)\cos 3x}\right)$$

$$=\underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{2\sin\left(\frac{3x+(3h+3x)}{2}\right)\sin\left(\frac{(\cancel{3x}+3h)-\cancel{3x}}{2}\right)}{\cos(3x+3h)\cos 3x}\right)\quad \left(\because \cos C-\cos D=2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)\right)$$

$$=\underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{2\sin\left(\frac{\cancel{6}x}{\cancel{2}}+\frac{3h}{2}\right)\sin\left(\frac{3h}{2}\right)}{\cos(3x+3h)\cos 3x}\right)$$

$$=2\underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{\sin\left(3x+\frac{3h}{2}\right)\sin\left(\frac{3h}{2}\right)}{\cos(3x+3h)\cos 3x}\right)=2\underset{h\to 0}{Lt}\frac{\sin\left(3x+\frac{3h}{2}\right)}{\cos(3x+3h)\cos 3x}.\underset{h\to 0}{Lt}\frac{\sin\left(\frac{3h}{2}\right)}{h}$$

$$=\cancel{2}.\frac{\sin(3x+0)}{\cos(3x+0)\cos 3x}.\frac{3}{\cancel{2}}\quad =3\frac{\sin 3x}{\cos 3x\cos 3x}=3\frac{1}{\cos 3x}.\left(\frac{\sin 3x}{\cos 3x}\right)=3\sec 3x.\tan 3x$$

13. The volume of a cube is increasing at a rate of 8 cubic centimeters per second. How fast is the surface area increasing when length of edge is 12cm?

A: For the cube, we take

length of the edge = x , Volume = V and

Surface area = S

Given $\frac{dV}{dt}=8$ and x = 12 cm

Volume of the cube $V = x^3$

On diff. w.r.t 't', we get $\frac{dV}{dt}=3x^2\frac{dx}{dt}$

$\Rightarrow 8=3x^2\frac{dx}{dt}\Rightarrow\frac{dx}{dt}=\frac{8}{3x^2}$

Surface area $S=6x^2$

On diff. w.r.t 't', we get $\frac{dS}{dt}=12x\frac{dx}{dt}$ $=12x\left(\frac{8}{3x^2}\right)=\frac{32}{x}=\frac{32}{12}=\frac{8}{3}\ cm^2/sec$

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14. Find the lengths of normal and subnormal at a point on the curve $y=\frac{a}{2}\left(e^{x/a}+e^{-x/a}\right)$

A: Given that $y=\frac{a}{2}\left(e^{x/a}+e^{-x/a}\right)=a\left(\frac{e^{x/a}+e^{-x/a}}{2}\right)=a\cosh\left(\frac{x}{a}\right)$

$\Rightarrow\frac{dy}{dx}=a\sinh\left(\frac{x}{a}\right)\left(\frac{1}{a}\right)=\sinh\left(\frac{x}{a}\right)\Rightarrow m=\sinh\left(\frac{x}{a}\right)$

(i) Length of the normal at (x,y) $=|y.\sqrt{1+(m)^2}|=|a\cosh\left(\frac{x}{a}\right).\sqrt{1+\sinh^2\left(\frac{x}{a}\right)}|$

$=|a\cosh\left(\frac{x}{a}\right).\cosh\left(\frac{x}{a}\right)|=|a\cosh^2\frac{x}{a}|.$

(ii) Length of the subnormal at (x,y) $=|ym|=|a\cosh\frac{x}{a}\sinh\frac{x}{a}|=|\frac{a}{2}.2\sinh\frac{x}{a}\cosh\frac{x}{a}|=|\frac{a}{2}.\sinh\frac{2x}{a}|$

15. Find the equation of Locus of P, if the ratio of the distances from P to A (5, –4) and B(7, 6) is 2 : 3.

A: Let P = (x, y) and A = (5, –4). B = (7, 6)

Given condition is $\frac{PA}{PB}=\frac{2}{3}$ $\Rightarrow 3PA = 2PB$ $\Rightarrow 9PA^2 = 4PB^2$

$\Rightarrow 9[(x-5)^2 + (y+4)^2] = 4[(x-7)^2 + (y-6)^2]$

$\Rightarrow 9[(x^2 - 10x + 25 + y^2 + 8y + 16] = 4[x^2 - 14x + 49 + y^2 - 12y + 36]$

$\Rightarrow 9x^2 - 90x + 225 + 9y^2 + 72y + 144 = 4x^2 - 56x + 196 + 4y^2 - 48y + 144$

$\Rightarrow 5x^2 + 5y^2 - 34x + 120y + 29 = 0$

$\therefore$ $5x^2 + 5y^2 - 34x + 120y + 29 = 0$ is the equation of locus.

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16. Find the transformed equation of $3x^2+10xy+3y^2=9$ when the axes are rotated through an angle $\pi/4$

A: Given equation (original) is

$3x^2+10xy+3y^2=9$ ......(1)

Angle of rotation $\theta=\pi/4=45^\circ$, then

$x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos 45^\circ - Y\sin 45^\circ$

$= X\left(\frac{1}{\sqrt{2}}\right) - Y\left(\frac{1}{\sqrt{2}}\right) \Rightarrow x = \frac{X-Y}{\sqrt{2}}$

$y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos 45^\circ + X\sin 45^\circ$

$= Y\left(\frac{1}{\sqrt{2}}\right) + X\left(\frac{1}{\sqrt{2}}\right) \Rightarrow y = \frac{X+Y}{\sqrt{2}}$

From (1), transformed equation is $3\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 3\left(\frac{X+Y}{\sqrt{2}}\right)^2 = 9$

$\Rightarrow 3\left(\frac{X^2+Y^2-2XY}{2}\right) + 10\left(\frac{X^2-Y^2}{2}\right) + 3\left(\frac{X^2+Y^2+2XY}{2}\right) = 9$

$\Rightarrow \frac{3X^2+3y^2\cancel{-6XY}+10X^2-10Y^2+3X^2+3Y^2\cancel{+6XY}}{2} = 9$

$\Rightarrow 16X^2 - 4Y^2 = 2(9)$

$\Rightarrow \cancel{2}(8X^2 - 2Y^2) = \cancel{2}(9) \Rightarrow 8X^2 - 2Y^2 = 9$

17. Find the point on the straight lines 3x + y + 4 = 0 which is equidistant from the points (–5, 6) and (3, 2).

A: Let $P(x_1, y_1)$ be the required point on the line $3x + y + 4 = 0 \Rightarrow 3x_1 + y_1 + 4 = 0$ ....(1)

$P(x_1, y_1)$ is equidistant from $A(-5, 6)$, $B(3, 2) \Rightarrow PA = PB \Rightarrow PA^2 = PB^2$.

$$(x_1 + 5)^2 + (y_1 - 6)^2 = (x_1 - 3)^2 + (y_1 - 2)^2$$

$$\Rightarrow (x_1^2 + 10x_1 + 25) + (y_1^2 - 12y_1 + 36) = (x_1^2 - 6x_1 + 9) + (y_1^2 - 4y_1 + 4)$$

$$\Rightarrow 16x_1 - 8y_1 + 48 = 0 \Rightarrow 8(2x_1 - y_1 + 6) = 0 \Rightarrow 2x_1 - y_1 + 6 = 0 \text{ ....(2)}$$

Solving (1), (2), we get (x_1, y_1)

$(1) + (2) \Rightarrow 5x_1 + 10 = 0 \Rightarrow 5x_1 = -10 \Rightarrow x_1 = -2$

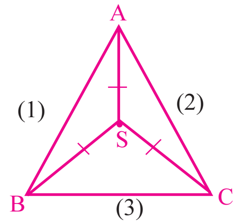
From (2), $y_1 = 2x_1 + 6 = 2(-2) + 6 = -4 + 6 = 2 \Rightarrow y_1 = 2$

$\therefore (x_1, y_1) = (-2, 2)$

SECTION-C

18. Find the orthocentre of the triangle whose sides are $x + 2y = 0$, $4x + 3y - 5 = 0$, $3x + y = 0$

A: Given lines are $x + 2y = 0$ ....(1), $4x + 3y - 5 = 0$ ....(2), $3x + y = 0$ .....(3)



To find altitude through A:

Solving (1) and (2), we get A; $x + 2y = 0$

$$4x + 3y - 5 = 0$$

$$\Rightarrow \frac{x}{-10-0} = \frac{y}{0+5} = \frac{1}{3-8} \Rightarrow \frac{x}{-10} = \frac{y}{5} = \frac{1}{-5}$$

$\Rightarrow x = 2, y = -1 \quad \therefore A = (2, -1)$

From (3), slope of $3x + y = 0$ is $m = \frac{-a}{b} = \frac{-3}{1} = -3$

The slope of the opposite side BC, $3x + y = 0$ is -3

So slope of its perpendicular is $1/3$ $[\because m_1 m_2 = -1]$

Equation of the altitude passing through A(2, –1) and with slope 1/3 is

$y + 1 = (1/3)(x - 2) \Rightarrow 3y + 3 = x - 2 \Rightarrow x - 3y - 5 = 0$ .... (4)

To find altitude through B:

By solving (1) & (3) we get B(0, 0)

The slope of the opposite side AC, $4x + 3y - 5 = 0$ is $-4/3$

So slope of its perpendicular is $3/4$ $[\because m_1 m_2 = -1]$

Equation of the altitude through B(0,0) with slope $\frac{3}{4}$ is $y - 0 = \frac{3}{4}(x - 0) \Rightarrow 3x - 4y = 0$ .....(5)

Solving (4) & (5) we get orthocentre O; From (4), $x = 3y + 5$ ....(6)

From (5), $3(3y + 5) - 4y = 0 \Rightarrow 9y + 15 - 4y = 0 \Rightarrow 5y = -15 \Rightarrow y = -3$

From (6), $x = 3(-3) + 5 \Rightarrow x = -9 + 5 \Rightarrow x = -4$

$\therefore$ Orthocentre $= (-4, -3)$

19. Find the angle between the lines joining the origin to the points of intersection of the curve $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ and the line $3x - y + 1 = 0$.

A:
- Given line is $3x - y + 1 = 0 \Rightarrow 3x - y = -1 \Rightarrow \frac{3x-y}{-1} = 1$ ...(1)
- Given curve is $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$............(2)
- Homogenising (1) & (2), we get

★ $x^2 + 2xy + y^2 + 2x(1) + 2y(1) - 5(1)^2 = 0$

★ $\Rightarrow x^2 + 2xy + y^2 + 2x\left(\frac{3x-y}{-1}\right) + 2y\left(\frac{3x-y}{-1}\right) - 5\frac{(3x-y)^2}{1} = 0$

★ $\Rightarrow x^2 + 2xy + y^2 - 2x(3x-y) - 2y(3x-y) - 5(3x-y)^2 = 0$

- $\Rightarrow x^2 + 2xy + y^2 - 6x^2 + 2xy - 6xy + 2y^2 - 5(9x^2 + y^2 - 6xy) = 0$
- $\Rightarrow x^2 + 2xy + y^2 - 6x^2 + 2xy - 6xy + 2y^2 - 45x^2 - 5y^2 + 30xy = 0$

★ $\Rightarrow 50x^2 + 2y^2 - 28xy = 0$

- $\Rightarrow \not{2}(25x^2 + y^2 - 14xy) = 0$

★ $\Rightarrow 25x^2 + y^2 - 14xy = 0$

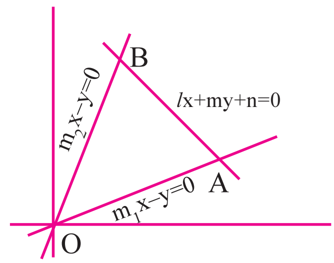
★ On comparing with $ax^2 + by^2 + 2hxy = 0$, we get $a = 25,\ b = 1, 2h = -14$

★ $\therefore \cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2 + (2h)^2}}$, where θ is the angle between the required lines

- $= \frac{|25+1|}{\sqrt{(25-1)^2 + (-14)^2}} = \frac{26}{\sqrt{576+196}} = \frac{26}{\sqrt{772}} = \frac{26}{\sqrt{4\times193}} = \frac{\not{26}}{\not{2}\sqrt{193}} = \frac{13}{\sqrt{193}}$
- $\therefore \cos\theta = \frac{13}{\sqrt{193}} \Rightarrow \theta = \text{Cos}^{-1}\frac{13}{\sqrt{193}}$

Hence angle between the lines $\boxed{\theta = \text{Cos}^{-1}\frac{13}{\sqrt{193}}}$

20. Prove that the area of the triangle formed by the pair of lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is $\dfrac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$

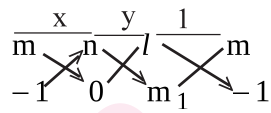


A: ★ Let $ax^2+2hxy+by^2 \equiv (m_1x-y)(m_2x-y)$

- On equating like term coeff., we get

★ $m_1+m_2=-\dfrac{2h}{b}, \quad m_1m_2=\dfrac{a}{b}$ ....(1)

- By solving $lx+my+n=0,\ m_1x-y=0$ we get A



$lx+my+n=0$

$m_1x-y+0=0$

★ $\Rightarrow \dfrac{x}{m(0)-(-1)(n)}=\dfrac{y}{n(m_1)-l(0)}=\dfrac{1}{l(-1)-mm_1}$

- $\Rightarrow \dfrac{x}{n}=\dfrac{y}{nm_1}=\dfrac{1}{-l-mm_1} \Rightarrow A=\left(\dfrac{-n}{l+mm_1},\dfrac{-nm_1}{l+mm_1}\right)$

★ Similarly, we get $B=\left(\dfrac{-n}{l+mm_2},\dfrac{-nm_2}{l+mm_2}\right)$

★ The area of the triangle with vertices, $O(0,0), A(x_1,y_1), B(x_2,y_2)$ is $\Delta=\dfrac{1}{2}|x_1y_2-x_2y_1|$

★ $\therefore$ Area of $\Delta OAB=\dfrac{1}{2}\left|\left(\dfrac{-n}{l+mm_1}\right)\left(\dfrac{-nm_2}{l+mm_2}\right)-\left(\dfrac{-n}{l+mm_2}\right)\left(\dfrac{-nm_1}{l+mm_1}\right)\right|$

- $=\dfrac{1}{2}\left|\dfrac{n^2m_2-n^2m_1}{(l+mm_1)(l+mm_2)}\right|=\dfrac{1}{2}\left|\dfrac{n^2(m_2-m_1)}{l^2+lmm_2+lmm_1+m^2m_1m_2}\right|$

★ $=\dfrac{1}{2}\dfrac{n^2\sqrt{(m_1+m_2)^2-4m_1m_2}}{|l^2+lm(m_1+m_2)+m^2(m_1m_2)|}$ $\quad\left[\because (a-b)=\sqrt{(a+b)^2-4ab}\right]$

★ $=\dfrac{1}{2}\dfrac{n^2\sqrt{\left(-\dfrac{2h}{b}\right)^2-4\left(\dfrac{a}{b}\right)}}{\left|l^2+lm\left(\dfrac{-2h}{b}\right)+m^2\left(\dfrac{a}{b}\right)\right|}=\dfrac{1}{2}\dfrac{n^2\sqrt{\dfrac{4h^2}{b^2}-4\dfrac{a}{b}}}{\left|l^2-\left(\dfrac{2hlm}{b}\right)+\left(\dfrac{am^2}{b}\right)\right|}=\dfrac{1}{2}\dfrac{n^2\sqrt{\dfrac{4h^2-4ab}{\not{b}^2}}}{\left|\dfrac{bl^2-2hlm+am^2}{\not{b}}\right|}$ [from (1)]

★ $=\dfrac{1}{2}\dfrac{n^2\sqrt{4h^2-4ab}}{|am^2-2hlm+bl^2|}=\dfrac{1}{\not{2}}\dfrac{\not{2}\left(n^2\sqrt{h^2-ab}\right)}{|am^2-2hlm+bl^2|}=\dfrac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$ sq. units

21. Find the angle between the lines whose dc's are related by $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.

A: Given $3l + m + 5n = 0 \Rightarrow m = -3l - 5n$ .....(1) , $6mn - 2nl + 5lm = 0$ .....(2)

Solving (1) & (2) we get

$6n(-3l - 5n) - 2nl + 5l(-3l - 5n) = 0$

$\Rightarrow -18ln - 30n^2 - 2ln - 15l^2 - 25ln = 0$

$\Rightarrow -15l^2 - 45ln - 30n^2 = 0$

$\Rightarrow -15(l^2 + 3ln + 2n^2) = 0 \Rightarrow l^2 + 3ln + 2n^2 = 0$

$\Rightarrow l^2 + ln + 2ln + 2n^2 = 0 \Rightarrow l(l + n) + 2n(l + n) = 0$

$\Rightarrow (l + n)(l + 2n) = 0 \Rightarrow l = -n$ or $l = -2n$

Case (i): Put $l = -n$ in (1), then

$m = -3(-n) - 5n = 3n - 5n = -2n$

$\therefore m = -2n$

Now, $l : m : n = -\cancel{n} : -2\cancel{n} : \cancel{n}$

$= -1 : -2 : 1 = 1 : 2 : -1$

So, d.r's of $L_1 = (a_1, b_1, c_1) = (1, 2, -1)$.....(3)

Case (ii): Put $l = -2n$ in (1), then

$m = -3(-2n) - 5n = 6n - 5n = n \quad \therefore m = n$

Now, $l : m : n = -2\cancel{n} : \cancel{n} : \cancel{n}$

$= -2 : 1 : 1 = 2 : -1 : -1$

So, d.r's of $L_2 = (a_2, b_2, c_2) = (2, -1, -1)$....(4)

If θ is the angle between the lines then from (3), (4), we get

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}} = \frac{|1(2) + 2(-1) + (-1)(-1)|}{\sqrt{(1^2 + 2^2 + (-1)^2)(2^2 + (-1)^2 + (-1)^2)}}$$

$$= \frac{|\cancel{2} - \cancel{2} + 1|}{\sqrt{(6)(6)}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

$$\therefore \cos\theta = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\frac{1}{6}$$

Hence angle between the lines is $\cos^{-1}\frac{1}{6}$

22. If $x^y+y^x=a^b$ then show that $\frac{dy}{dx}=-\left(\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}\right)$

A: Let $u=x^y$, $v=y^x$

Now $u=x^y \Rightarrow \log u=\log x^y \Rightarrow \log u=y\log x$ Differentiating w.r.t x we have

$$\frac{1}{u}\frac{du}{dx}=y\left(\frac{1}{x}\right)+\log x\frac{dy}{dx}\Rightarrow\frac{1}{u}\frac{du}{dx}=\frac{y}{x}+\log x\frac{dy}{dx}$$

$$\frac{du}{dx}=u\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)\quad\Rightarrow\frac{du}{dx}=x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)..........(1)$$

Also, $v=y^x \Rightarrow \log v=\log y^x \Rightarrow \log v=x\log y$ Differentiating w.r.t x, we have

$$\frac{1}{v}\frac{dv}{dx}=x.\frac{1}{y}.\frac{dy}{dx}+\log y(1)\Rightarrow\frac{1}{v}\frac{dv}{dx}=\frac{x}{y}\frac{dy}{dx}+\log y$$

$$\frac{dv}{dx}=v\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)\Rightarrow\frac{dv}{dx}=y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)..........(2)$$

$$\text{From(1) \& (2)},\frac{du}{dx}+\frac{dv}{dx}=0\Rightarrow x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow\frac{dy}{dx}\left(x^y\log x+y^x\left(\frac{x}{y}\right)\right)=-\left(x^y\left(\frac{y}{x}\right)+y^x\log y\right)\Rightarrow\frac{dy}{dx}=-\left(\frac{x^{y-1}y+y^x\log y}{x^y\log x+y^{x-1}x}\right)$$

23. If the tangent at a point on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ intersects the coordinate axes in A,B then show that the length AB is a constant.

A: ★ The point on the curve taken as $P(a\cos^3\theta, a\sin^3\theta)$

- $x = a\cos^3\theta$ and $y = a\sin^3\theta$

★ $\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(a\sin^3\theta)}{\frac{d}{d\theta}(a\cos^3\theta)} = \frac{a.3\sin^2\theta(\cos\theta)}{a.3\cos^2\theta(-\sin\theta)} = -\frac{\sin\theta}{\cos\theta}$

- So, slope of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ is $m = -\frac{\sin\theta}{\cos\theta}$

★ $\therefore$ Equation of the tangent at $P(a\cos^3\theta, a\sin^3\theta)$ having slope $-\frac{\sin\theta}{\cos\theta}$ is $y-y_1=m(x-x_1)$

★ $\Rightarrow y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$

- $\Rightarrow \frac{y - a\sin^3\theta}{\sin\theta} = -\frac{(x - a\cos^3\theta)}{\cos\theta}$

- $\Rightarrow \frac{y}{\sin\theta} - \frac{a\sin^3\theta}{\sin\theta} = -\frac{x}{\cos\theta} + \frac{a\cos^3\theta}{\cos\theta}$

- $\Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a\cos^2\theta + a\sin^2\theta = a(\cos^2\theta + \sin^2\theta) = a(1)$

- $\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$

★ $\therefore A=(a\cos\theta, 0), B=(0, a\sin\theta)$

- $\therefore AB = \sqrt{(a\cos\theta - 0)^2 + (0 - a\sin\theta)^2}$

$= \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = \sqrt{a^2(\cos^2\theta + \sin^2\theta)} = \sqrt{a^2(1)} = a$

$\therefore$ Hence proved that AB is a constant.

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24. The profit function p(x) of a company, selling x items per day is given by p(x) = (150 – x) x – 1000. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.

A: Given profit function is $p(x) = (150 - x)\,x - 1000$

$p'(x) = (150 - x)\,1 + x(-1) = 150 - 2x$. Also $p''(x) = -2$

For Maximum profit, $p'(x) = 0 \Rightarrow 150 - 2x = 0 \Rightarrow x = 75$

$\therefore$ Profit p(x) is maximum when $x = 75$

$\therefore$ Maximum Profit $= (150 - 75)75 - 1000 = 75(75) - 1000 = 5625 - 1000 = 4625$ units.