Previous IPE

SOLVED PAPERS

MARCH -2023(AP)

PREVIOUS PAPERS

IPE: MARCH-2023(AP)

MATHS-1A

Time : 3 Hours Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **10 × 2 = 20**

1. If $f(x) = 2x - 1$, $g(x)=\frac{x+1}{2}$ for all $x \in R$, find $(gof)(x)$
2. Find the domain of the real function $f(x) = \sqrt{x^2 - 25}$
3. Define symmetric matrix & give an example.
4. If $A=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{bmatrix}$, det A = 45 then find x.
5. If $\overline{OA} = \bar{i} + \bar{j} + \bar{k}$, $\overline{AB} = 3\bar{i} - 2\bar{j} + \bar{k}$, $\overline{BC} = \bar{i} + 2\bar{j} - 2\bar{k}$, $\overline{CD} = 2\bar{i} + \bar{j} + 3\bar{k}$ then find vector $\overline{OD}$
6. Find the vector equation of the plane passing through $\bar{i} - 2\bar{j} + 5\bar{k}$, $-5\bar{j} - \bar{k}$, $-3\bar{i} + 5\bar{j}$
7. If $|\bar{a} + \bar{b}| = |\bar{a} - \bar{b}|$ then find the angle between $\bar{a}$ and $\bar{b}$
8. Find $\sin 330^\circ . \cos 120^\circ + \cos 210^\circ . \sin 300^\circ$
9. If $A - B = \frac{3\pi}{4}$, then show that $(1-\tan A)(1+\tan B)=2$
10. Show that $Tanh^{-1}\frac{1}{2} = \frac{1}{2}\log_e 3$.

SECTION-B

II. Answer any FIVE of the following SAQs: **5 × 4 = 20**

11. If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is a non-singular matrix, then prove that A is invertible and $A^{-1} = \frac{AdjA}{detA}$
12. If the points whose position vectors are $3\bar{i} - 2\bar{j} - \bar{k}$, $2\bar{i} + 3\bar{j} - 4\bar{k}$, $-\bar{i} + \bar{j} + 2\bar{k}$, $4\bar{i} + 5\bar{j} + \lambda\bar{k}$ are co-planar, then show that $\lambda = -146/17$
13. If $\bar{a} = 2\bar{i} + \bar{j} - \bar{k}$, $\bar{b} = -\bar{i} + 2\bar{j} - 4\bar{k}$, $\bar{c} = \bar{i} + \bar{j} + \bar{k}$ then find $(\bar{a} \times \bar{b}).(\bar{b} \times \bar{c})$
14. S.T $\cos A \cos\left(\frac{\pi}{3} + A\right)\cos\left(\frac{\pi}{3} - A\right) = \frac{1}{4}\cos 3A$. Hence deduce that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{16}$
15. If $0 < x < \frac{\pi}{2}$ then solve $\cot^2 x - (\sqrt{3} + 1)\cot x + \sqrt{3} = 0$
16. Prove that $Tan^{-1}\frac{1}{2} + Tan^{-1}\frac{1}{5} + Tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.
17. In ΔABC, show that $\frac{b^2 - c^2}{a^2} = \frac{\sin(B - C)}{\sin(B + C)}$

SECTION-C

III. Answer any FIVE of the following LAQs: **5 × 7 = 35**

18. If $f:A \rightarrow B$ is a bijective function then prove that (i) $fof^{-1} = I_B$ (ii) $f^{-1}of = I_A$
19. By using Mathematical Induction, to prove $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}, \forall n \in N$
20. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$
21. Solve $2x - y + 3z = 8$, $-x + 2y + z = 4$, $3x + y - 4z = 0$ by using Matrix inversion method.
22. Find the shortest distance between the skew lines $\bar{r} = (6\bar{i} + 2\bar{j} + 2\bar{k}) + t(\bar{i} - 2\bar{j} + 2\bar{k})$ and $\bar{r} = (-4\bar{i} - \bar{k}) + s(3\bar{i} - 2\bar{j} - 2\bar{k})$.
23. If $A+B+C=\frac{\pi}{2}$ then prove that $\cos 2A + \cos 2B + \cos 2C = 1 + 4 \sin A \sin B \sin C$.
24. Prove that $r + r_1 + r_2 - r_3 = 4R\cos C$.

IPE AP MARCH-2023
SOLUTIONS

SECTION-A

1. If f(x)=2x–1, $g(x)=\frac{x+1}{2}$ for all x∈R, find (i) (gof)(x) (ii) (fog)(x)

A: (i) (gof)(x)=g(f(x))=g(2x–1) $=\frac{(2x-1)+1}{2}=\frac{2x}{2}=x$

(ii) (fog)(x)=f(g(x))= $f\left(\frac{x+1}{2}\right)=\cancel{2}\left(\frac{x+1}{\cancel{2}}\right)-1=x+1-1=x$

---

2. Find the domain of the real function $\sqrt{x^2-25}$

A: Given f(x) is defined when $x^2-25\geq 0$

$\Rightarrow (x-5)(x+5)\geq 0$

$\Rightarrow x\in(-\infty,-5]\cup[5,\infty)$

$\therefore$ Domain = $(-\infty,-5]\cup[5,\infty)$

---

3. Define symmetric matrix & give an example.

A: SYMMETRIC MATRIX: A square matrix A is said to be a Symmetric matrix if $A^T=A$.

Ex: $A=\begin{bmatrix}1&2\\2&3\end{bmatrix}$, $B=\begin{bmatrix}1&2&0\\2&-3&-1\\0&-1&4\end{bmatrix}$, $C=\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}$

---

4. If $A=\begin{bmatrix}1&0&0\\2&3&4\\5&-6&x\end{bmatrix}$ and det A= 45 then find x.

A: Given $\begin{vmatrix}1&0&0\\2&3&4\\5&-6&x\end{vmatrix}=45$

$\Rightarrow 1(3x+24)+0+0=45 \Rightarrow 3x=45-24 \Rightarrow \cancel{3}x=\cancel{21} \Rightarrow x=7$ $\therefore$ x=7

5. If $\overline{OA}=\overline{i}+\overline{j}+\overline{k}$, $\overline{AB}=3\overline{i}-2\overline{j}+\overline{k}$, $\overline{BC}=\overline{i}+2\overline{j}-2\overline{k}$, $\overline{CD}=2\overline{i}+\overline{j}+3\overline{k}$ then find the vector $\overline{OD}$

A: Consider $\overline{OA}+\overline{AB}+\overline{BC}+\overline{CD}=(\overline{OA}+\overline{AB})+\overline{BC}+\overline{CD}=(\overline{OB}+\overline{BC})+\overline{CD}=\overline{OC}+\overline{CD}=\overline{OD}$

Hence $\overline{OD}=\overline{OA}+\overline{AB}+\overline{BC}+\overline{CD}=(\overline{i}+\overline{j}+\overline{k})+(3\overline{i}-2\overline{j}+\overline{k})+(\overline{i}+2\overline{j}-2\overline{k})+(2\overline{i}+\overline{j}+3\overline{k})$

$=7\overline{i}+2\overline{j}+3\overline{k}$

---

6. Find the vector equation of the plane passing through the points $\overline{i}-2\overline{j}+5\overline{k}$, $-5\overline{j}-\overline{k}$, $-3\overline{i}+5\overline{j}$

A: Given $A(\overline{a})=\overline{i}-2\overline{j}+5\overline{k}$,

$B(\overline{b})=-5\overline{j}-\overline{k}$, $C(\overline{c})=-3\overline{i}+5\overline{j}$

Vector equation of the plane is $\overline{r}=(1-s-t)\overline{a}+s\overline{b}+t\overline{c}$, $s, t\in R$

$\therefore\ \overline{r}=(1-s-t)(\overline{i}-2\overline{j}+5\overline{k})+s(-5\overline{j}-\overline{k})+t(-3\overline{i}+5\overline{j})$, $s, t\in R$

---

7. If $|\overline{a}+\overline{b}|=|\overline{a}-\overline{b}|$ then find the angle between $\overline{a}$ and $\overline{b}$

A: Given that $|\overline{a}+\overline{b}|=|\overline{a}-\overline{b}|$

$\Rightarrow|\overline{a}+\overline{b}|^2=|\overline{a}-\overline{b}|^2\Rightarrow\overline{a}^2+\overline{b}^2+2\overline{a}.\overline{b}=\overline{a}^2+\overline{b}^2-2\overline{a}.\overline{b}$

$\Rightarrow 4\overline{a}.\overline{b}=0\Rightarrow\overline{a}.\overline{b}=0\Rightarrow\overline{a}\perp\overline{b}$

$\therefore$ Angle between $\overline{a}$ and $\overline{b}$ is 90°

---

8. Find sin330°.cos120°+cos210°.sin300°

A: $\sin330^\circ=\sin(360^\circ-30^\circ)=-\sin30^\circ=-1/2$

$\cos120^\circ=\cos(180^\circ-60^\circ)=-\cos60^\circ=-1/2$

$\cos210^\circ=\cos(180^\circ+30^\circ)=-\cos30^\circ=-\sqrt{3}/2$

$\sin300^\circ=\sin(360^\circ-60^\circ)=-\sin60^\circ=-\sqrt{3}/2$

$\therefore$ G.E$=\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{4}+\frac{3}{4}=\frac{4}{4}=1$

---

9. If $A-B=\frac{3\pi}{4}$, then show that (1–tanA)(1+tanB)=2

A: Given $A-B=\frac{3\pi}{4} \Rightarrow \tan(A-B)=\tan\frac{3\pi}{4} \Rightarrow \frac{\tan A-\tan B}{1+\tan A\tan B}=-1$

$\Rightarrow$tan A–tanB= –1 –tan A tan B = –tan A + tan B – tan A tan B=1 Adding 1 on both sides

$\Rightarrow$ 1 – tan A + tan B – tan A tan B=1+1

$\Rightarrow$ (1 – tan A) – tanB(1–tanA) = 2 $\Rightarrow$ (1 – tan A) (1 – tan B) = 2

---

10. Show that $\text{Tanh}^{-1}\frac{1}{2}=\frac{1}{2}\log_e 3$

A: We know $\text{Tanh}^{-1}x=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

$$\therefore \text{Tanh}^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)=\frac{1}{2}\log_e\left(\frac{\frac{3}{\not 2}}{\frac{1}{\not 2}}\right) \quad =\frac{1}{2}\log_e(3)$$

SECTION-B

11. If A is a non-singular matrix then prove that $A^{-1}=\dfrac{\text{Adj A}}{\det A}$

A: We take $A=\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}$

We take cofactors of $a_1, b_1, c_1,$ .....as A_1, B_1, C_1 ......

$$\therefore \text{Adj } A=\begin{bmatrix}A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\\ A_3 & B_3 & C_3\end{bmatrix}^T \Rightarrow \text{Adj}A=\begin{bmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{bmatrix}$$

$$A.(\text{Adj}A)=\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{bmatrix}$$

$$=\begin{bmatrix}a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3\\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3\\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3\end{bmatrix}$$

$$=\begin{bmatrix}\det A & 0 & 0\\ 0 & \det A & 0\\ 0 & 0 & \det A\end{bmatrix} \text{(From properties of determinants)}$$

$$=(\det A)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=(\det A).I$$

$\therefore A(\text{Adj } A)=(\det A)\,I$; Similarly, we can prove that $(\text{Adj } A)A=(\det A)\,I$

$\therefore A\left(\dfrac{\text{Adj } A}{\det A}\right)=I$ $\quad(\because \det A\neq 0,$ as A is non-singular$)$

$\therefore A^{-1}=\dfrac{\text{Adj } A}{\det A}$ $\quad[\because AB=I \Rightarrow A^{-1}=B]$

12. If the points whose position vectors are $3\bar{i}-2\bar{j}-\bar{k}, 2\bar{i}+3\bar{j}-4\bar{k}, -\bar{i}+\bar{j}+2\bar{k}, 4\bar{i}+5\bar{j}+\lambda\bar{k}$ are coplanar, then show that $\lambda=-146/17$

A: We take

$\overline{OP}=3\bar{i}-2\bar{j}-\bar{k}, \overline{OQ}=2\bar{i}+3\bar{j}-4\bar{k}$,

$\overline{OR}=-\bar{i}+\bar{j}+2\bar{k}, \overline{OS}=4\bar{i}+5\bar{j}+\lambda\bar{k}$,

where 'O' is the origin.

$\therefore \overline{PQ}=\overline{OQ}-\overline{OP}$

$=(2\bar{i}+3\bar{j}-4\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})$

$=-\bar{i}+5\bar{j}-3\bar{k}$

$\overline{PR}=\overline{OR}-\overline{OP}$

$=(-\bar{i}+\bar{j}+2\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})$

$=-4\bar{i}+3\bar{j}+3\bar{k}$

$\overline{PS}=\overline{OS}-\overline{OP}$

$=(4\bar{i}+5\bar{j}+\lambda\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})$

$=\bar{i}+7\bar{j}+(\lambda+1)\bar{k}$

But, $[\overline{PQ}\ \overline{PR}\ \overline{PS}]=0$ [Since P,Q,R,S are coplanar]

$$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix}=0$$

$\Rightarrow (-1)[3(\lambda+1)-21]-5[-4(\lambda+1)-3]$
$-3[(-28)-3]=0$

$\Rightarrow -1(3\lambda-18)-5(-4\lambda-7)-3(-31)=0$

$\Rightarrow -3\lambda+18+20\lambda+35+93=0$

$\Rightarrow -3\lambda+20\lambda+35+93+18=0$

$\Rightarrow 17\lambda+146=0$

$\Rightarrow 17\lambda=-146$

$\Rightarrow \lambda=-146/17$

13. **If $\bar{a} = 2\bar{i} + \bar{j} - \bar{k}, \bar{b} = -\bar{i} + 2\bar{j} - 4\bar{k}, \bar{c} = \bar{i} + \bar{j} + \bar{k}$ then find $(\bar{a}\times\bar{b}).(\bar{b}\times\bar{c})$**

A: Given that $\bar{a} = 2\bar{i} + \bar{j} - \bar{k}, \bar{b} = -\bar{i} + 2\bar{j} - 4\bar{k}$ and $\bar{c} = \bar{i} + \bar{j} + \bar{k}$

Now $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -1 \\ -1 & 2 & -4 \end{vmatrix} = \bar{i}(-4+2) - \bar{j}(-8-1) + \bar{k}(4+1) = -2\bar{i} + 9\bar{j} + 5\bar{k}$

Also, $\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \bar{i}(2+4) - \bar{j}(-1+4) + \bar{k}(-1-2) = 6\bar{i} - 3\bar{j} - 3\bar{k}$

Now $(\bar{a}\times\bar{b}).(\bar{b}\times\bar{c}) = (-2\bar{i} + 9\bar{j} + 5\bar{k}).(6\bar{i} - 3\bar{j} - 3\bar{k}) = (-2)(6)+(9)(-3)+5(-3) = -12-27-15 = -54$

$\therefore (\bar{a}\times\bar{b}).(\bar{b}\times\bar{c}) = 54$

---

14. **Show that** $\cos A\cos\left(\frac{\pi}{3}+A\right)\cos\left(\frac{\pi}{3}-A\right) = \frac{1}{4}\cos 3A$ **Hence deduce that** $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{16}$

A: $\text{L.H.S} = \cos A\cos\left(\frac{\pi}{3}+A\right)\cos\left(\frac{\pi}{3}-A\right) = \cos A\left(\cos^2\frac{\pi}{3} - \sin^2 A\right)$

$= \cos A\left[\left(\frac{1}{2}\right)^2 - \left(1-\cos^2 A\right)\right] = \cos A\left[\frac{1}{4} - 1 + \cos^2 A\right]$

$= \cos A\left(\frac{1-4+4\cos^2 A}{4}\right) = \frac{1}{4}\cos A(4\cos^2 A - 3) = \frac{1}{4}[4\cos^3 A - 3\cos A] = \frac{1}{4}\cos 3A = \text{R.H.S}$

$\therefore \cos A\cos(60^\circ + A)\cos(60^\circ - A) = \frac{1}{4}\cos 3A$

Put $A = 20^\circ$ then $\cos 20^\circ\cos 40^\circ\cos 80^\circ = \frac{1}{4}\cos 60^\circ$

On multiplying both sides by $\cos 60^\circ$, we get $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ = \frac{1}{4}\cos^2 60^\circ$

$\Rightarrow \cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{3\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) = \frac{1}{4}\left(\frac{1}{2}\right)^2 = \frac{1}{16}$

15. If $0 < x < \frac{\pi}{2}$ then solve $\cot^2 x - (\sqrt{3}+1)\cot x + \sqrt{3} = 0$

A: Given that $\cot^2 x - (\sqrt{3}+1)\cot x + \sqrt{3} = 0 \Rightarrow \cot^2 x - \sqrt{3}\cot x - \cot x + \sqrt{3} = 0$

$\Rightarrow \cot x(\cot x - \sqrt{3}) - (\cot x - \sqrt{3}) = 0 \Rightarrow (\cot x - 1)(\cot x - \sqrt{3}) = 0$

$\Rightarrow \cot x - 1 = 0$ (or) $\cot x - \sqrt{3} = 0 \Rightarrow \cot x = 1$ (or) $\cot x = \sqrt{3}$

$\Rightarrow \tan x = 1$ (or) $\tan x = \frac{1}{\sqrt{3}}$

$\tan x = 1 = \tan\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$; $\left[\because 0 < x < \frac{\pi}{2}\right]$

$\tan x = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$

$\therefore$ The solutions are $x = \frac{\pi}{4}, \frac{\pi}{6}$

---

16. Prove that $\mathrm{Tan}^{-1}\frac{1}{2} + \mathrm{Tan}^{-1}\frac{1}{5} + \mathrm{Tan}^{-1}\frac{1}{8} = \frac{\pi}{4}$

A: We know, $\mathrm{Tan}^{-1}x + \mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\therefore \mathrm{Tan}^{-1}\frac{1}{2} + \mathrm{Tan}^{-1}\frac{1}{5} = \mathrm{Tan}^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}\right) = \mathrm{Tan}^{-1}\left(\frac{\frac{5+2}{10}}{\frac{10-1}{10}}\right) = \mathrm{Tan}^{-1}\frac{7}{9}$$

$$\therefore \text{L.H.S} = \mathrm{Tan}^{-1}\frac{7}{9} + \mathrm{Tan}^{-1}\frac{1}{8} = \mathrm{Tan}^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}\right) = \mathrm{Tan}^{-1}\left(\frac{\frac{56+9}{72}}{\frac{72-7}{72}}\right)$$

$$= \mathrm{Tan}^{-1}\left(\frac{65}{65}\right) = \mathrm{Tan}^{-1}1 = \frac{\pi}{4} = \text{R.H.S}$$

---

17. In ΔABC, show that $\frac{b^2 - c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$

A: $$\text{L.H.S} = \frac{b^2-c^2}{a^2} = \frac{(2R\sin B)^2 - (2R\sin C)^2}{(2R\sin A)^2} = \frac{4R^2(\sin^2 B - \sin^2 C)}{4R^2\sin^2 A}$$

$$= \frac{\sin^2 B - \sin^2 C}{\sin^2 A} = \frac{\sin(B+C)\sin(B-C)}{\sin^2(B+C)} = \frac{\sin(B-C)}{\sin(B+C)} = \text{R.H.S}$$

SECTION-C

18. If $f:A\to B$ is a bijective function then prove that (i) $fof^{-1}=I_B$ (ii) $f^{-1}of=I_A$

A: (i) **To prove that $fof^{-1}=I_B$**

Part–1 : Given $f:A\to B$ is a bijective function, then $f^{-1}: B\to A$ is also a bijection

$\therefore fof^{-1}: B\to B$

We know, $I_B: B\to B$

So, fof^{-1} and I_B, both have same domain B

Part–2: For $b\in B$, $(fof^{-1})(b)=f[f^{-1}(b)]$

$=f(a)$ [$\because f:A\to B$ is bijection $\Rightarrow f(a)=b\Rightarrow f^{-1}(b)=a$, for $a\in A$]

$=b=I_B(b)$ [$\because I_B(b)=b$, for $b\in B$]

Hence we proved that $fof^{-1}=I_B$

(ii) **To prove that $f^{-1}of=I_A$**

Part–1: Given $f:A\to B$ is a bijective function, then $f^{-1}:B\to A$ is also a bijection

$\therefore f^{-1}of: A\to A$

We know $I_A: A\to A$

So, $f^{-1}of$ and I_A, both have same domain A

Part–2: for $a\in A$, $(f^{-1}of)(a)=f^{-1}[f(a)]$

$=f^{-1}(b)=a$ [$\because f:A\to B$ is a bijection $\Rightarrow f(a)=b\Rightarrow f^{-1}(b)=a$]

$=I_A(a)$[$\because I_A(a)=a$, for $a\in A$]

Hence we proved that $f^{-1}of=I_A$

19. By using Mathematical Induction,Show that $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.....+n$ terms $=\frac{n}{2n+1}$

A: To find n^{th} term:

1,3,5... are in A.P with a=1, d=2

$\therefore T_n = a+(n-1)d \Rightarrow T_n = 1+(n-1)2 = 1+2n-2 = 2n-1$

3,5,7... are in A.P with a=3, d=2

$\therefore T_n = 3+(n-1)2 = 3+2n-2 = 2n+1$

$\therefore n^{th}$ term is $T_n = \frac{1}{(2n-1)(2n+1)}$

$$\text{Let } S(n) : \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.....+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Step 1: L.H.S of S(1) $= \frac{1}{1.3} = \frac{1}{3}$;

R.H.S of S(1) $= \frac{1}{2.1+1} = \frac{1}{3}$

$\therefore$ L.H.S = R.H.S.

So, S(1) is true

Step 2: Assume that S(k) is true ,for $k \in N$

$$S(k) : \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.....+\frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \qquad .....(1)$$

Step 3: We show that S(k+1) is true

$$(k+1)^{th} \text{ term} = \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{1}{(2k+1)(2k+3)}$$

On adding $(k+1)^{th}$ term to both sides of (1), we get

$$L.H.S = \left[\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.....+\frac{1}{(2k-1)(2k+1)}\right]+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(k+1)\cancel{(2k+1)}}{\cancel{(2k+1)}(2k+3)} = \frac{k+1}{2k+3} = \frac{k+1}{2k+2+1} = \frac{k+1}{2(k+1)+1} = R.H.S$$

$\therefore$ L.H.S =R.H.S.

So, S(k+1) is true whenever S(k) is true

Hence, by P.M.I the given statement is true, for all $n \in N$

20. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$

A: We take $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= a(bc-a^2) - b(b^2-ac) + c(ab-c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= -(a^3+b^3+c^3-3abc)$$

$$\Rightarrow \Delta^2 = (a^3+b^3+c^3-3abc)^2 \text{...............(1)}$$

Again $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

On applying C_{23} on the first determinent, we get

$$= -\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+cb+bc & \cancel{-ab}+c^2\cancel{+ab} & \cancel{-ac}+\cancel{ac}+b^2 \\ \cancel{-ab}\,\cancel{+ab}+c^2 & -b^2+ac+ca & \cancel{-bc}+a^2+\cancel{cb} \\ \cancel{-ca}+b^2\cancel{+ac} & \cancel{-cb}\,\cancel{+bc}+a^2 & -c^2+ba+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \text{................(2)}$$

From (1) and (2), the given result is proved.

21. By using Matrix inversion method, 2x–y+3z=8, –x+2y+z=4, 3x+y–4z=0

A: The matrix equation corresponding to the given system of equations be AX=D, where

$$A=\begin{bmatrix}2&-1&3\\-1&2&1\\3&1&-4\end{bmatrix}; X=\begin{bmatrix}x\\y\\z\end{bmatrix}, D=\begin{bmatrix}8\\4\\0\end{bmatrix}$$

$\therefore$ The solution of AX=D is $X=A^{-1}D$

First we find A^{-1}

$$|A|=\begin{vmatrix}2&-1&3\\-1&2&1\\3&1&-4\end{vmatrix}=2(-8-1)+1(4-3)+3(-1-6)=2(-9)+1(1)+3(-7)=-18+1-21=-38\neq 0$$

The co-factor matrix of A is

$$\begin{bmatrix}+\begin{vmatrix}2&1\\1&-4\end{vmatrix}&-\begin{vmatrix}-1&1\\3&-4\end{vmatrix}&+\begin{vmatrix}-1&2\\3&1\end{vmatrix}\\-\begin{vmatrix}-1&3\\1&-4\end{vmatrix}&+\begin{vmatrix}2&3\\3&-4\end{vmatrix}&-\begin{vmatrix}2&-1\\3&1\end{vmatrix}\\+\begin{vmatrix}-1&3\\2&1\end{vmatrix}&-\begin{vmatrix}2&3\\-1&1\end{vmatrix}&+\begin{vmatrix}2&-1\\-1&2\end{vmatrix}\end{bmatrix}=\begin{bmatrix}(-8-1)&-(4-3)&(-1-6)\\-(4-3)&(-8-9)&-(2+3)\\(-1-6)&-(2+3)&(4-1)\end{bmatrix}=\begin{bmatrix}-9&-1&-7\\-1&-17&-5\\-7&-5&3\end{bmatrix}$$

$$\therefore \text{Adj } A=\begin{bmatrix}-9&-1&-7\\-1&-17&-5\\-7&-5&3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj}A)=\frac{1}{-38}\begin{bmatrix}-9&-1&-7\\-1&-17&-5\\-7&-5&3\end{bmatrix}$$

$$\text{Now, } X=A^{-1}D=\frac{-1}{38}\begin{bmatrix}-9&-1&-7\\-1&-17&-5\\-7&-5&3\end{bmatrix}\begin{bmatrix}8\\4\\0\end{bmatrix}=\frac{-1}{38}\begin{bmatrix}-72-4-0\\-8-68-0\\-56-20-0\end{bmatrix}=\frac{-1}{38}\begin{bmatrix}-76\\-76\\-76\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$$

$$\therefore X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$$

Hence the solution is x=2, y=2, z=2

22. Find the shortest distance between the skew lines $\bar{r} = (6\bar{i} + 2\bar{j} + 2\bar{k}) + t(\bar{i} - 2\bar{j} + 2\bar{k})$ and $\bar{r} = (-4\bar{i} - \bar{k}) + s(3\bar{i} - 2\bar{j} - 2\bar{k})$

A: Given skew lines $\bar{r} = (6\bar{i} + 2\bar{j} + 2\bar{k}) + t(\bar{i} - 2\bar{j} + 2\bar{k})$; $\bar{r} = (-4\bar{i} - \bar{k}) + s(3\bar{i} - 2\bar{j} - 2\bar{k})$

Formula: For the skew lines $\bar{r} = \bar{a} + t\bar{b}$, $\bar{r} = \bar{c} + s\bar{d}$ shortest distance(SD) $= \dfrac{|(\bar{a} - \bar{c}).(\bar{b} \times \bar{d})|}{|\bar{b} \times \bar{d}|}$

On comparing the given skew lines with $\bar{r} = \bar{a} + t\bar{b}$; $\bar{r} = \bar{c} + s\bar{d}$ we get

$\bar{a} = 6\bar{i} + 2\bar{j} + 2\bar{k}$ and $\bar{b} = \bar{i} - 2\bar{j} + 2\bar{k}$

Also, $\bar{c} = -4\bar{i} - \bar{k}$ and $\bar{d} = 3\bar{i} - 2\bar{j} - 2\bar{k}$

So, $\bar{a} - \bar{c} = (6\bar{i} + 2\bar{j} + 2\bar{k}) - (-4\bar{i} - \bar{k}) = 10\bar{i} + 2\bar{j} + 3\bar{k}$

$$\bar{b} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = \bar{i}(4+4) - \bar{j}(-2-6) + \bar{k}(-2+6) = 8\bar{i} + 8\bar{j} + 4\bar{k}$$

$\therefore (\bar{a} - \bar{c}).(\bar{b} \times \bar{d}) = (10\bar{i} + 2\bar{j} + 3\bar{k}).(8\bar{i} + 8\bar{j} + 4\bar{k}) = 80+16+12 = 108$

Also, $|\bar{b} \times \bar{d}| = \sqrt{8^2 + 8^2 + 4^2} = \sqrt{64+64+16} = \sqrt{144} = 12$

$\therefore$ Shortest distance(SD) $= \dfrac{|(\bar{a} - \bar{c}).(\bar{b} \times \bar{d})|}{|\bar{b} \times \bar{d}|} = \dfrac{108}{12} = 9$

---

23. If $A+B+C=\frac{\pi}{2}$ then prove that cos2A+cos2B+cos2C=1+4sinAsinBsinC.

A: L.H.S $= \cos 2A + \cos 2B + \cos 2C = 2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + \cos 2C = 2\cos\left(\frac{\pi}{2} - C\right)\cos(A-B) + \cos 2C$

$= 2\sin C\cos(A-B) + (1 - 2\sin^2 C)$ $[\because \cos 2\theta = 1 - 2\sin^2\theta]$

$= 1 + 2\sin C[\cos(A-B) - \sin C] = 1 + 2\sin C[\cos(A-B) - \sin\left(\frac{\pi}{2} - (A+B)\right)]$

$= 1 + 2\sin C[\cos(A-B) - \cos(A+B)] = 1 + 2\sin C[2\sin A\sin B]$

$= 1 + 4\sin A\sin B\sin C$ = R.H.S

24. Prove that $r + r_1 + r_2 - r_3 = 4R\cos C$

A: L.H.S $= r + r_1 + r_2 - r_3 = (r + r_1) + (r_2 - r_3)$

$$=\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)+\left(4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}-4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=4R\sin\frac{A}{2}\left(\sin\frac{B}{2}\sin\frac{C}{2}+\cos\frac{B}{2}\cos\frac{C}{2}\right)+4R\cos\frac{A}{2}\left(\sin\frac{B}{2}\cos\frac{C}{2}-\cos\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=4R\sin\frac{A}{2}\cos\left(\frac{B}{2}-\frac{C}{2}\right)+4R\cos\frac{A}{2}\sin\left(\frac{B}{2}-\frac{C}{2}\right)$$

$$=4R\left[\sin\frac{A}{2}\cos\left(\frac{B}{2}-\frac{C}{2}\right)+\cos\frac{A}{2}\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right]$$

$$=4R\sin\left[\frac{A}{2}+\frac{B}{2}-\frac{C}{2}\right] \qquad \left[\because A+B+C=180^\circ \Rightarrow \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90^\circ\right]$$

$$=4R\sin(90^\circ - C) = 4R\cos C = \text{R.H.S}$$