Previous IPE

SOLVED PAPERS

MARCH -2020 (AP)

PREVIOUS PAPERS

IPE: MARCH-2020(AP)

Time : 3 Hours

MATHS-2B

Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **$10 \times 2 = 20$**

1. Find the other end of the diameter of the circle $x^2 + y^2 - 8x - 8y + 27 = 0$ if one end of it is (2, 3).
2. Define chord of contact and find the chord of contact (1,1) with respect to the circle $x^2 + y^2 = 9$
3. Find *k* if the pair of circles $x^2 + y^2 - 5x - 14y - 34 = 0$, $x^2 + y^2 + 2x + 4y + k = 0$ are orthogonal.
4. Find the equation of the parabola whose vertex is (3, –2) ,focus is(3, 1)
5. Find the value of k if $3x - 4y + k = 0$ is a tangent to the hyperbola $x^2 - 4y^2 = 5$.
6. Evaluate $\int \frac{\cos x}{(1+\sin x)^2} dx$
7. Evaluate $\int x \log x \, dx$
8. Evaluate $\underset{n\to\infty}{Lt} \frac{1^4 + 2^4 + 3^4 + .. + n^4}{n^5}$
9. Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x \, dx$
10. Solve $y(1 + x)dx + x(1 + y)dy = 0$

SECTION-B

II. Answer any FIVE of the following SAQs: **$5 \times 4 = 20$**

11. Find the area of the triangle formed by the normal at $P(x_1,y_1)$ to the circle $x^2+y^2=a^2$ with the coordinate axes where $x,y \neq 0$
12. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$, $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then show that $f'g = fg'$.
13. S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.
14. Find the condition for the line $x\cos\alpha + y\sin\alpha = p$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
15. Find the centre, eccentricity, foci, length of latus rectum and equations of the directrices of the Hyperbola $x^2 - 4y^2 = 4$
16. Find the area of the region bounded by the parabolas $y^2=4x$ and $x^2=4y$
17. Solve $(x^2 + y^2)dx = 2xydy$

SECTION-C

III. Answer any FIVE of the following LAQs: **$5 \times 7 = 35$**

18. Find the equation of the circle passing through (4, 1) (6, 5) and having the centre on the line $4x + 3y - 24 = 0$
19. Find the equation of the circle which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at (5, 5) with radius 5.
20. From an external point P, tangents are drawn to the parabola $y^2 = 4ax$ and these tangents make angles θ_1, θ_2 with its axis, such that $\tan\theta_1 + \tan\theta_2$ is a constant b. Then show that P lies on the line $y = bx$.
21. Evaluate $\int \frac{1}{1+\sin x + \cos x} dx$
22. If $I_n = \int \cos^n x dx$, then show that $I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n} I_{n-2}$
23. Show that $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$
24. Solve $x\log x \frac{dy}{dx} + y = 2\log x$

IPE AP MARCH-2020
SOLUTIONS

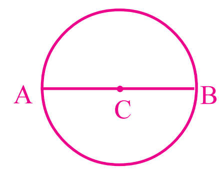
SECTION-A

1. Find the other end of the diameter of the circle $x^2 + y^2 - 8x - 8y + 27 = 0$ if one end of it is (2, 3).

A: Given circle $x^2 + y^2 - 8x - 8y + 27 = 0$

Compare the given circle with $x^2 + y^2 + 2gx + 2fy + c = 0$; Centre = $(-g, -f) = (4,4)$

Let the other end of the diameter be (α,β)



Centre = Mid point of (α,β) and $(2,3)$

$(4,4) = \left(\frac{\alpha+2}{2}, \frac{\beta+3}{2}\right) \Rightarrow \alpha + 2 = 8; \beta + 2 = 8 \Rightarrow \alpha=8-2=6;\ \alpha=8-3=5$

$\therefore$ Other end of the diameter = (6,5)

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2. Define chord of contact and find the chord of contact (1,1) with respect to the circle $x^2 + y^2 = 9$

Sol: Chord of contact of a point w.r.t a circle: If the tangents through an external point $P(x_1,y_1)$ to the circle S=0 touch the circle at A and B then the line segment $\overline{AB}$ is called the chord of contact of P w.r.to the circle S=0

The equation of the chord of contact of P(1,1) w.r.t the circle $S = x^2+y^2-9=0$ is $S_1=0$

$\Rightarrow x_1x+y_1y - r^2 = 0 \Rightarrow (1)x+(1)y - 9 = 0 \Rightarrow x+y-9=0$

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3. Find *k* if the pair of circles $x^2+y^2-5x-14y-34=0$, $x^2+y^2+2x+4y+k=0$ are orthogonal.

Sol: From the given circles, we get $g = -5/2$, $f = -7$, $c = -34$ and

$g' = 1, f' = 2, c' = k$

Orthogonal condition: $2gg'+2ff'=c+c'$

$\Rightarrow \cancel{2}\left(\frac{-5}{\cancel{2}}\right)(1) + 2(-7)(2) = -34 + k$

$\Rightarrow -5 - 28 = -34 + k \Rightarrow k = -33 + 34 = 1$

4. Find the equation of the parabola whose vertex is (3,–2) ,focus is(3,1)

Sol: Given Vertex A=(3, –2), Focus S=(3,1)

Here, the x-coordinates of A, S are equal

$\therefore$ The axis is parallel to the y-axis.

Also, the parabola is vertically upward. ($\because$ focus S lies above the vertex A)

Now, $a = AS = \sqrt{(3-3)^2+(1+2)^2} = \sqrt{9} = 3$

$\therefore$ Parabola with vertex (h,k)=(3,–2)is

$(x-h)^2 = 4a(y-k) \Rightarrow (x-3)^2=4(3)(y-(-2)) \Rightarrow (x-3)^2=12(y+2)$

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5. Find the value of k if 3x–4y+k=0 is a tangent to the hyperbola $x^2-4y^2=5$.

Sol: Given hyperbola $x^2-4y^2=5 \Rightarrow \frac{x^2}{5}-\frac{4y^2}{5}=1 \Rightarrow \frac{x^2}{5}-\frac{y^2}{5/4}=1 \Rightarrow a^2=5$ and $b^2=5/4$

Comparing 3x–4y+k=0 with $lx+my+n=0$, we get $l=3$, m= –4, n=k

Tangential condition: $n^2=a^2l^2-b^2m^2$,

$\Rightarrow (k)^2 = 5(3^2)-\frac{5}{4}(-4)^2 = 45-20=25$

$\therefore k^2=25 \Rightarrow k=\pm 5$

Try this: Reduce 3x–4y+k=0 into the form y=mx+c and apply the condition $c^2=a^2m^2-b^2$

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6. Evaluate $\int \frac{\cos x}{(1+\sin x)^2}\,dx$

Sol: Put 1+sinx=t $\Rightarrow$ cosx dx = dt

$$\therefore I = \int \frac{\cos x\,dx}{(1+\sin x)^2} = \int \frac{dt}{t^2} = -\frac{1}{t}+c = \frac{-1}{1+\sin x}+c$$

7. Evaluate $\int x\log x\,dx$

Sol : We take the first function u=logx and second function v = x

From the "By parts rule", we have

$$I=\int \log x.x dx=(\log x)\left(\frac{x^2}{2}\right)-\int\frac{1}{x}.\frac{x^2}{2}dx \quad =(\log x)\left(\frac{x^2}{2}\right)-\int\frac{x}{2}dx$$

$$=(\log x)\left(\frac{x^2}{2}\right)-\frac{1}{2}.\frac{x^2}{2}+c=(\log x)\left(\frac{x^2}{2}\right)-\frac{x^2}{4}+c$$

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8. Evaluate $\underset{n\to\infty}{Lt}\frac{1^4+2^4+3^4+..+n^4}{n^5}$

Sol: $$\underset{n\to\infty}{Lt}\frac{1^4+2^4+3^4+..+n^4}{n^5}=\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\frac{r^4}{n^5}=\underset{n\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^4=\int_0^1 x^4dx=\left[\frac{x^5}{5}\right]_0^1=\frac{1}{5}$$

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9. Evaluate $\int_{-\pi/2}^{\pi/2}\sin^2x\cos^4x\,dx$

Sol : Let $f(x)=\sin^2x\cos^4x$ $\Rightarrow f(-x)=\sin^2(-x)\cos^4(-x)=(-\sin x)^2(\cos x)^4=\sin^2x\cos^4x=f(x)$

$\therefore$ f(x) is an even function

$$\therefore \int_{-\pi/2}^{\pi/2}\sin^2x\cos^4x\,dx=2\int_0^{\pi/2}\sin^2x.\cos^4x\,dx \quad =2\frac{[(1)][(3)(1)]}{(6)(4)(2)}\frac{\pi}{2}=\frac{\pi}{16}$$

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10. Solve y(1 + x) dx + x(1 + y) dy = 0

Sol: Given D.E is y(1 + x) dx + x(1 + y) dy = 0 $\Rightarrow$ x(1 + y) dy = – y(1 + x) dx

$$\Rightarrow\frac{(1+y)dy}{y}=-\frac{(1+x)dx}{x}\Rightarrow\int\frac{(1+y)dy}{y}=-\int\frac{(1+x)dx}{x}\Rightarrow\int\left(\frac{1}{y}+1\right)dy=-\int\left(\frac{1}{x}+1\right)dx$$

$\Rightarrow \log y + y = -\log x - x + c \Rightarrow$ y+x+logy+logx = c

$\therefore$ The solution is y+x+log(yx)= c

SECTION-B

11. Find the area of the triangle formed by the tangent drawn at (x_1,y_1) on the circle $x^2+y^2-a^2=0$ with the coordinate axes.

$O=(0,0), A(x_1,y_1), B(x_2,y_2)$

$\text{Area} = \frac{1}{2}|(x_1y_2 - x_2y_1)|$

Sol : Given point $P(x_1,y_1)$ and circle $S=x^2+y^2-a^2=0$.

The equation of the tangent at (x_1,y_1) on $S=0$ is $S_1=0 \Rightarrow xx_1+yy_1-a^2=0$

$\therefore$ the area of the triangle formed by above line with the coordinate axes is

$$\Delta = \frac{1}{2}|(x-\text{intercept})(y-\text{intercept})| = \frac{1}{2}\left|\frac{a^2}{x_1}\cdot\frac{a^2}{y_1}\right| = \frac{a^4}{2|x_1y_1|} \text{ sq.units}$$

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12. If the two circles $x^2+y^2+2gx+2fy=0$, $x^2+y^2+2g'x+2f'y=0$ touch each other, then show that $f'g=fg'$.

Sol: Both the given equations of circles do not contain the constant term.

$\therefore$ the circles pass through O(0,0).

If the circles touch each other, then O(0,0) and the centres

$C_1=(-g,-f)$, $C_2=(-g',-f')$ become collinear

$\Rightarrow$ Area of $\Delta\ OC_1C_2=0$

$\Rightarrow \frac{1}{2}[(-g)(-f')+(f)(-g')]=0 \Rightarrow gf'-fg'=0 \Rightarrow fg' = f'g.$

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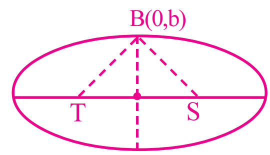
13. S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.

Sol: Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b)$

The two foci are S(ae,0), T=(–ae,0) and B(0,b) be the one end of the minor axis.

The distance between two foci is ST=2ae



Now ΔSTB is equilateral $\Rightarrow$ SB=ST=TB.

Now SB=ST $\Rightarrow (SB)^2=(ST)^2 \Rightarrow (ae)^2+b^2=(2ae)^2=4a^2e^2$

$\Rightarrow a^2e^2+a^2(1-e^2)=4a^2e^2\ [\because b^2=a^2(1-e^2)] \Rightarrow e^2+(1-e^2)=4e^2 \Rightarrow 4e^2=1$

$\Rightarrow e^2=\frac{1}{4}\Rightarrow e=\frac{1}{2}$

$\therefore$ Eccentricity of the ellipse is e=1/2

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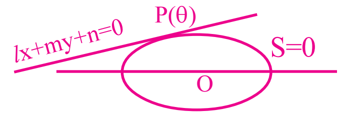
14. Find the condition for the line $lx+my+n=0$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol: Let $lx+my+n=0$ be the tangent at P(θ) =(acosθ, bsinθ) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore$ Equation of the tangent at P(θ) is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

Comparing the above equation with $lx+my=-n$, we get



$\frac{\cos\theta}{al}=\frac{\sin\theta}{bm}=\frac{-1}{n}\Rightarrow \cos\theta=-\frac{al}{n},\ \sin\theta=\frac{-bm}{n}$

Now $\cos^2\theta+\sin^2\theta=1 \Rightarrow \frac{a^2l^2}{n^2}+\frac{b^2m^2}{n^2}=1\Rightarrow a^2l^2+b^2m^2=n^2$

15. Find the centre, eccentricity, foci, length of latus rectum and equations of the directrices of the Hyperbola $x^2-4y^2=4$

Sol: Given hyperbola is $x^2-4y^2=4$

$\Rightarrow \frac{x^2}{4}-\frac{y^2}{1}=1$. Here $a^2=4$, $b^2=1$

(i) Centre C= (0,0)

(ii) Eccentricity $e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{4+1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$

(iii) Foci $=(\pm ae,0)=\left(\pm 2\left(\frac{\sqrt{5}}{2}\right),0\right)=(\pm\sqrt{5},0)$

(iv) Equation of the directrices is $x=\pm\frac{a}{e}\Rightarrow x=\pm\frac{2}{\frac{\sqrt{5}}{2}}\Rightarrow x=\pm\frac{4}{\sqrt{5}}$

(v) Length of latusrectum $=\frac{2b^2}{a}=\frac{2(1)}{2}=1$

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16. Find the area of the region bounded by the parabolas $y^2=4x$ and $x^2=4y$

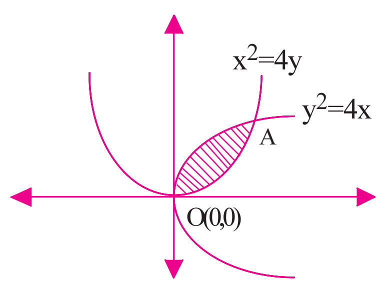
Sol: The given curves are $y^2=4x$ ....(1), $x^2=4y$ .....(2)

Solving (1), (2) we get $x^2=4y \Rightarrow x^4=16y^2=16(4x)=4^3x$.

$\Rightarrow x=0$ (or) $x^3=4^3 \Rightarrow x=4$

The upper boundary curve is $y^2=4x \Rightarrow y=2\sqrt{x}$

The lower boundary curve is $x^2=4y \Rightarrow y=x^2/4$



$\therefore$ Required area A $=\int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx=\left[2\times\frac{2}{3}x^{3/2}-\frac{x^3}{12}\right]_0^4=\frac{32}{3}-\frac{16}{3}=\frac{16}{3}$ sq.units

17. Solve $(x^2+y^2)dx=2xydy$

Sol: Given D.E is $(x^2+y^2)dx = 2xydy \Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ .....(1). This is a homogeneous D.E

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (1), $v + x\frac{dv}{dx} = \frac{x^2+(vx)^2}{2x(vx)} = \frac{x^2+v^2x^2}{2x^2v} = \frac{x^2(1+v^2)}{2x^2v} = \frac{1+v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$

$\therefore x\frac{dv}{dx} = \frac{1-v^2}{2v} \Rightarrow \frac{2vdv}{1-v^2} = \frac{dx}{x} \Rightarrow \int\frac{2vdv}{1-v^2} = \int\frac{dx}{x} \Rightarrow -\int\frac{-2vdv}{1-v^2} = \int\frac{dx}{x}$

$\Rightarrow -\log(1-v^2) = \log x + \log c \Rightarrow \log x + \log(1-v^2) = \log c$

$\Rightarrow \log(x(1-v^2)) = \log c \Rightarrow x(1-v^2) = c \Rightarrow x\left(1-\frac{y^2}{x^2}\right) = c; \quad \left(\because v = \frac{y}{x}\right)$

$\Rightarrow x\left(\frac{x^2-y^2}{x^2}\right) = c \Rightarrow \frac{x^2-y^2}{x} = c \Rightarrow x^2-y^2 = cx$

$\therefore$ The solution is $x^2-y^2 = cx$

☞ 'c' is modified accordingly.

SECTION-C

18. Find the equation of the circle passing through (4,1) (6,5) and having the centre on the line 4x+3y–24=0

Sol: Let A=(4,1), B=(6,5).

We take $S(x_1,y_1)$ as the centre of the circle $\Rightarrow SA=SB \Rightarrow SA^2=SB^2$.

$\Rightarrow (x_1-4)^2+(y_1-1)^2=(x_1-6)^2+(y_1-5)^2$

$\Rightarrow (\cancel{x_1^2}-8x_1+16)+(\cancel{y_1^2}-2y_1+1) \quad = (\cancel{x_1^2}-12x_1+36)+(\cancel{y_1^2}-10y_1+25)$

$\Rightarrow 17-8x_1-2y_1=61-12x_1-10y_1$

$\Rightarrow 17-8x_1-2y_1-61+12x_1+10y_1=0$

$\Rightarrow 4x_1+8y_1-44=0$..... (1)

But centre (x_1,y_1) lies on $4x+3y-24=0$

$\Rightarrow 4x_1+3y_1-24=0$ .... (2)

(2)–(1) $\Rightarrow -5y_1+20=0 \Rightarrow 5y_1=20 \Rightarrow y_1=4$

From (2), $4x_1+3(4)-24=0 \Rightarrow 4x_1-12=0 \quad \Rightarrow 4x_1=12 \Rightarrow x_1=3$

$\therefore$ Centre of the circle $S(x_1,y_1)$ = (3,4). Also, we have A=(4,1)

So, radius $r=SA \Rightarrow r^2=SA^2$.

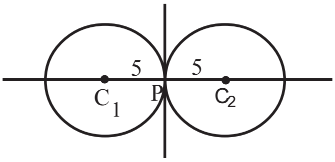
$\therefore r^2=(3-4)^2+(4-1)^2=1+9=10$

$\therefore$ Circle with centre (3,4) and $r^2=10$ is $(x-3)^2+(y-4)^2=10$

$\Rightarrow (x^2+9-6x)+(y^2+16-8y)=10 \quad \Rightarrow x^2+y^2-6x-8y+15=0$

19. Find the equation of the circle which touches the circle $x^2+y^2-2x-4y-20=0$ externally at (5,5) with radius 5.

Sol: For the circle $x^2+y^2-2x-4y-20=0$ centre $C_1 = (1, 2)$



radius $r_1 = \sqrt{1+4+20} = \sqrt{25} = 5$

Let C_2 be the centre and r_2 be the radius of the required circle. Here, $r_2 = 5$

The point of contact P=(5,5)

The two circles touch each other externally.

Since Radii are equal, P(5,5) is mid point of C_1C_2 where $C_1 = (1, 2)$ and $C_2=(x_1,y_1)$

$$\Rightarrow\left(\frac{x_1+1}{2},\frac{y_1+2}{2}\right)=(5,5)\Rightarrow\frac{x_1+1}{2}=5,\ \frac{y_1+2}{2}=5\Rightarrow x_1+1=10, y_1+2=10$$

$\Rightarrow x_1=9, y_1=8 \quad \therefore C_2=(9,8)$

The equation of the required circle is $(x-9)^2+(y-8)^2=5^2 \Rightarrow x^2+y^2-18x-16y+120=0$

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20. From an external point P, tangents are drawn to the parabola $y^2=4ax$ and these tangents make angles θ_1,θ_2 with its axis, such that $\tan\theta_1+\tan\theta_2$ is a constant b. Then show that P lies on the line y=bx.

Sol: Let the external point $P=(x_1,y_1)$

The equation of the tangent with slope m to the parabola $y^2=4ax$ is $y = mx + \frac{a}{m}$

If this tangent passes through $P(x_1,y_1)$ then $y_1 = mx_1 + \frac{a}{m} \Rightarrow m^2x_1 - my_1 + a = 0$

The above equation is a quadratic in m and its roots be taken as m_1, m_2.

Here, $m_1 = \tan\theta_1$ and $m_2=\tan\theta_2$.

Then $m_1 + m_2 = \frac{y_1}{x_1} \Rightarrow \tan\theta_1 + \tan\theta_2 = \frac{y_1}{x_1} \Rightarrow b = \frac{y_1}{x_1} (\because \tan\theta_1 + \tan\theta_2 = b)$

$\Rightarrow y_1 = bx_1$

Hence the point $P(x_1,y_1)$ lies on the line y=bx

21. Evaluate $\int \frac{1}{1+\sin x+\cos x}dx.$

Sol: Put $\tan\frac{x}{2}=t$ then $\sin x=\frac{2t}{1+t^2}$; $\cos x=\frac{1-t^2}{1+t^2}$ and $dx=\frac{2dt}{1+t^2}$

$$\therefore I=\int\frac{1}{1+\sin x+\cos x}dx=\int\frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\left(\frac{2dt}{1+t^2}\right)=\int\frac{1}{\frac{(1+t^2)+2t+(1-t^2)}{\cancel{1+t^2}}}\cdot\frac{2dt}{\cancel{1+t^2}}$$

$$=\int\frac{2dt}{2+2t}=\frac{\cancel{2}}{\cancel{2}}\int\frac{dt}{1+t}=\log|1+t|+c=\log\left|1+\tan\left(\frac{x}{2}\right)\right|+c$$

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22. Obtain the reduction formula for $I_n=\int \csc^n x dx$ **, and hence find** $\int \csc^5 x dx$

Sol: $I_n=\int \csc^n x dx=\int \csc^{n-2}x \csc^2 x dx$.

We take first function $u=\csc^{n-2}x$

Second function $v=\csc^2 x \Rightarrow \int v=-\cot x$

From Byparts Rule, we have $I_n=-\csc^{n-2}x\cot x-\int(n-2)\csc^{n-3}x(-\csc x\cot x)(-\cot x)dx$

$=-\csc^{n-2}x\cot x-(n-2)\int\csc^{n-2}x\cot^2 x dx$

$=-\csc^{n-2}x\cot x-(n-2)\int\csc^{n-2}x(\csc^2 x-1)dx$

$=-\csc^{n-2}x\cot x-(n-2)\int\csc^n x dx+(n-2)\int\csc^{n-2}x dx$

$I_n=-\csc^{n-2}x\cot x-(n-2)I_n+(n-2)I_{n-2}$

$\Rightarrow I_n+(n-2)I_n=\csc^{n-2}x\cot x+(n-2)I_{n-2}$ $\Rightarrow I_n(1+n-2)=-\csc^{n-2}x\cot x+(n-2)I_{n-2}$

$\Rightarrow I_n(n-1)=-\csc^{n-2}x\cot x+(n-2)I_{n-2}$ $\Rightarrow I_n=-\frac{\csc^{n-2}x\cot x}{n-1}+\frac{n-2}{n-1}I_{n-2}$

Put n=5,3,1 successively in (1), we get

$$I_5=\int\csc^5 x dx=\frac{-\csc^3 x\cot x}{4}+\frac{3}{4}I_3=-\frac{\csc^3 x\cot x}{4}+\frac{3}{4}\left[\frac{-\csc x+\cot x}{2}+\frac{1}{2}I_1\right]$$

$$=-\frac{\csc^3 x\cot x}{4}-\frac{3\csc x\cot x}{8}+\frac{3}{8}\int\csc x dx$$

$$=-\frac{\csc^3 x\cot x}{4}-\frac{3\csc x\cot x}{8}+\frac{3}{8}\log|\csc x-\cot x|+c$$

$$\therefore I_5=\frac{\csc^3 x.\cot x}{4}-\frac{3}{8}\csc x\cot x+\frac{3}{8}\log\left|\tan\frac{x}{2}\right|+c$$

23. Show that $\int_0^{\pi/2}\frac{x}{\sin x+\cos x}dx=\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

Sol: We know $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$$\therefore I=\int_0^{\pi/2}\frac{x}{\sin x+\cos x}dx=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)}{\cos x+\sin x}dx=\frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}-\int_0^{\pi/2}\frac{x}{\sin x+\cos x}dx=\frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}-I$$

$$\Rightarrow I+I=2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}\Rightarrow I=\frac{\pi}{4}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$

$$=\frac{\pi}{4}\int_0^{\pi/2}\frac{dx}{\sqrt{2}\left(\sin x\frac{1}{\sqrt{2}}+\cos x\frac{1}{\sqrt{2}}\right)}=\frac{\pi}{4\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\left[\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right]}$$

$$=\frac{\pi}{4\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\sin\left[x+\frac{\pi}{4}\right]}=\frac{\pi}{4\sqrt{2}}\int_0^{\pi/2}\operatorname{cosec}\left[x+\frac{\pi}{4}\right]dx=\frac{\pi}{4\sqrt{2}}\log\left[\tan\left(\frac{x+\frac{\pi}{4}}{2}\right)\right]_0^{\pi/2}\quad\left[\because\int\csc x\,dx=\log\tan\frac{x}{2}\right]$$

$$=\frac{\pi}{4\sqrt{2}}\log\left[\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right]_0^{\pi/2}=\frac{\pi}{4\sqrt{2}}\left[\log\left(\tan\frac{3\pi}{8}\right)-\log\left(\tan\frac{\pi}{8}+0\right)\right]$$

$$=\frac{1}{2\sqrt{2}}\left[\log\left(\tan\frac{3\pi}{8}\right)-\log\left(\tan\frac{\pi}{8}\right)\right]\quad\left[\because\frac{\pi}{8}+\frac{\pi}{4}=\frac{\pi+2\pi}{8}=\frac{3\pi}{8}\right]$$

$$=\frac{\pi}{4\sqrt{2}}\left[\log(\sqrt{2}+1)-\log(\sqrt{2}-1)\right]\quad\left[\because\tan\frac{3\pi}{8}=\sqrt{2}+1;\ \tan\frac{\pi}{8}=\sqrt{2}-1\right]$$

$$=\frac{\pi}{4\sqrt{2}}\left[\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]\quad\left[\because\log a-\log b=\log\frac{a}{b}\right]$$

$$=\frac{\pi}{4\sqrt{2}}\log\left[\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)\right]=\frac{\pi}{4\sqrt{2}}\log\left[\frac{(\sqrt{2}+1)^2}{2-1}\right]$$

$$=\frac{\pi}{4\sqrt{2}}.2\log(\sqrt{2}+1)=\frac{\pi}{2\sqrt{2}}.\log(\sqrt{2}+1)$$

24. Solve $x\log x\frac{dy}{dx}+y=2\log x$

Sol: Given D.E is $x\log x\frac{dy}{dx}+y=2\log x \Rightarrow \frac{dy}{dx}+y\left(\frac{1}{x\log x}\right)=\frac{2}{x}$.

The above equation is in the form $\frac{dy}{dx}+yP(x)=Q(x)$, which is a linear D.E is y

Here $P(x)=\frac{1}{x\log x} \Rightarrow \int P(x)dx=\int\frac{1}{x\log x}dx=\log(\log x)$ $\left[\because \int\frac{f'(x)}{f(x)}=\log f(x)\right]$

$\therefore I.F=e^{\int P(x)dx}=e^{\log(\log x)}=\log x.$

$\therefore$ The solution is $y\,(I.F)=\int(I.F)Q(x)\,dx$

$\Rightarrow y(\log x)=\int\log x\left(\frac{2}{x}\right)dx=2\int\log x\left(\frac{dx}{x}\right)$

Put $\log x=t \Rightarrow \frac{1}{x}dx=dt$

$\therefore y(t)=2\int tdt=\cancel{2}\left(\frac{t^2}{\cancel{2}}\right)+c \Rightarrow y(t)=t^2+c \Rightarrow y(\log x)=(\log x)^2+c$