Previous IPE

SOLVED PAPERS

MARCH-2020 (AP)

PREVIOUS PAPERS

IPE: MARCH-2020(AP)

MATHS-1B

Time : 3 Hours Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: $10 \times 2 = 20$

1. Find the equation of the straight line passing through (–4, 5) and cutting off equal intercepts on the coordinate axes.
2. Find the area of the triangle formed by $x - 4y + 2 = 0$ with the coordinate axes.
3. Find the coordinates of the vertex 'C' of ΔABC if its centroid is the origin & the vertices A, B are (1, 1, 1) and (–2, 4, 1) respectively.
4. Find the equation of the plane passing through the point (–2, 1, 3) and having (3, –5, 4) as d.r.'s of its normal.
5. Compute $\lim_{x \to 0} \frac{\sin ax}{x \cos x}$ **6.** Evaluate $\underset{x \to \infty}{\text{Lt}} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$ **7.** Find the derivative of $\frac{1 - \cos 2x}{1 + \cos 2x}$
8. Define the derivative of a function **9.** If $y = x^2 + 3x + 6$ then find Δy and dy when $x = 10$, $\Delta x = 0.01$.
10. State Lagrange's Mean value theorem.

SECTION-B

II. Answer any FIVE of the following SAQs: $5 \times 4 = 20$

11. Find the equation of locus of a point the difference of whose distances from (–5, 0) and (5, 0) is 8 units.
12. When the axes are rotated through an angle α, find the transformed equation of $x\cos\alpha + y\sin\alpha = p$
13. Transform the equation $4x - 3y + 12 = 0$ into (i) slope intercept form (ii) intercept form (iii) Normal form
14. Is f given by $f(x) = \begin{cases} \frac{x^2 - 9}{x^2 - 2x - 3} & \text{if } 0 < x < 5 \text{ and } x \neq 3 \\ 1.5 & \text{if } x = 3 \end{cases}$, continuous at the point 3.
15. Find the derivative of $f(x) = \sin 2x$ using the first principle.
16. Find the equations of the tangent and the normal to the curve $y = x^3 + 4x^2$ at (–1, 3)
17. A container is in the shape of an inverted cone has height 8m and radius 6m at the top. If it is filled with water at the rate of $2m^3$/minute, how fast is the height of water changing when the level is 4m?

SECTION-C

III. Answer any FIVE of the following LAQs: $5 \times 7 = 35$

18. (a) If Q(h, k) is the foot of the perpendicular of $P(x_1, y_1)$ on the line $ax + by + c = 0$ then prove that $(h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$.
(b) Find the foot of the perpendicular drawn from (4, 1) on the line $3x - 4y + 12 = 0$.
19. (a) If θ is the angle between the pair of lines $ax^2 + 2hxy + by^2 = 0$ then prove that $\cos\theta = \frac{a + b}{\sqrt{(a - b)^2 + 4h^2}}$
(b) Find the angle between the pair of lines represented by the equation $x^2 - 7xy + 12y^2 = 0$
20. Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and the straight line $x - y - \sqrt{2} = 0$ are mutually perpendicular.
21. Find the angle between two diagonals of a cube.
22. If $y = \text{Tan}^{-1}\left(\frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}\right)$ then find $\frac{dy}{dx}$ **23.** Find the angle between the curves $y^2 = 4x$ and $x^2 + y^2 = 5$
24. The profit function p(x) of a company, selling x items per day is given by $p(x) = (150 - x)x - 1600$. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.

IPE AP MARCH-2020

SOLUTIONS

SECTION-A

1. Find the equation of the straight line passing through (–4, 5) and cutting off equal intercepts on the coordinate axes.

A:
- Given that, intercepts are equal.
- ★ So, we take a,a as the intercepts.
- ★ Intercept form: $\frac{x}{a}+\frac{y}{b}=1$
- $\therefore \frac{x}{a}+\frac{y}{a}=1$
- $\Rightarrow x+y=a$ ....(1)
- ★ But (–4, 5) lies on (1)
- $\Rightarrow -4+5=a \Rightarrow a=1$
- ★ $\therefore x+y=1$
- $\Rightarrow x+y-1=0$

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2. Find the area of the triangle formed by x – 4y + 2 = 0 with the coordinate axes.

A: X-intercept of the given line $x-4y+2=0$ is $-c/a = -2/1 = -2$

Y-intercept of the line is $-c/b = -2/-4 = 1/2$

Area of the triangle made by the line with the coordinate axes is

$$\Delta = \frac{1}{2}|(X-\text{intercept})(Y-\text{intercept})|$$

$$= \frac{1}{2}\left|-2\left(\frac{1}{2}\right)\right| = \frac{1}{2}(1) = \frac{1}{2}\text{ sq.units}$$

3. Find the coordinates of the vertex 'C' of ΔABC if its centroid is the origin and the vertices A,B are (1,1,1) and (–2,4,1) respectively.

A:
- We take A = (1, 1, 1), B = (–2, 4, 1) and
- Third vertex C = (x_3, y_3, z_3)
- Given Centroid G = (0, 0, 0).
- ★ Centroid of ΔABC
- ★ $G=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$
- ★ $\Rightarrow\left(\frac{1-2+x_3}{3},\frac{1+4+y_3}{3},\frac{1+1+z_3}{3}\right)=(0,0,0)$
- ★ $\Rightarrow\left(\frac{x_3-1}{3},\frac{y_3+5}{3},\frac{z_3+2}{3}\right)=(0,0,0)$
- $\Rightarrow x_3-1=0\Rightarrow x_3=1;$
- $y_3+5=0\Rightarrow y_3=-5;$
- $z_3+2=0\Rightarrow z_3=-2$
- Hence, third vertex C=(1, –5, –2)

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4. Find the equation of the plane passing through the point (–2, 1, 3) and having (3, –5, 4) as d.r.'s of its normal.

A: The equation of the plane with a,b,c as normal d.r's passing through the point (x_1, y_1, z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

Here (x_1, y_1, z_1) = (–2, 1, 3) and a = 3, b = –5, c = 4

∴ The required equation is 3(x + 2) – 5(y – 1) + 4(z – 3) = 0 ⇒ 3x + 6 – 5y + 5 + 4z – 12 = 0

⇒3x – 5y + 4z – 1 = 0.

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5. Compute $\underset{x\to 0}{\text{Lt}}\frac{\sin ax}{x\cos x}$

A: $\underset{x\to 0}{\text{Lt}}\frac{\sin ax}{x\cos x}=\underset{x\to 0}{\text{Lt}}\left(\frac{\sin ax}{x}\right)\frac{1}{(\cos x)}=a\frac{1}{(\cos 0)}=a\frac{1}{1}=a$ $\left(\because \lim_{x\to 0}\frac{\sin kx}{x}=k\right)$

6. Evaluate $\underset{x\to\infty}{Lt}\dfrac{11x^3-3x+4}{13x^3-5x^2-7}$

A: Taking x^3, the highest power of x as common factor in the numerator and denominator.

$$\underset{x\to\infty}{Lt}\frac{11x^3-3x+4}{13x^3-5x^2-7}=\underset{x\to\infty}{Lt}\frac{\cancel{x^3}\left(11-\frac{3}{x^2}+\frac{4}{x^3}\right)}{\cancel{x^3}\left(13-\frac{5}{x}-\frac{7}{x^3}\right)}=\underset{\frac{1}{x}\to 0}{Lt}\frac{11-\frac{3}{x^2}+\frac{4}{x^3}}{13-\frac{5}{x}-\frac{7}{x^3}}=\frac{11-0+0}{13-0-0}=\frac{11}{13}$$

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7. Find the derivative of $\frac{1-\cos 2x}{1+\cos 2x}$

A: $y=\dfrac{1-\cos 2x}{1+\cos 2x}=\dfrac{2\sin^2 x}{2\cos^2 x}=\tan^2 x$

Diff. w.r.t x $\dfrac{dy}{dx}=2\tan(\sec^2 x)$

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8. Define the derivative of a function

A: The derivative of a function y f(x) is defined as the instantaneous rate of change of a function at a specific point. The derivative gives the exact slope along the curve at a specified point. The derivative of the function is represented as dy/dx, which means the derivative of 'y' with respect to variable 'x'.

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9. If $y=x^2+3x+6$ then find Δy and dy when $x=10,\ \Delta x=0.01$.

A: We take $y=f(x)=x^2+3x+6,\ x=10,\ \Delta x=0.01$

(i) $\Delta y = f(x+\Delta x)-f(x)$

$=[(x+\Delta x)^2+3(x+\Delta x)+6]-(x^2+3x+6)$

$=[\cancel{x^2}+(\Delta x)^2+2x\Delta x]\cancel{+3x}+3\Delta x\cancel{+6}-\cancel{x^2}\cancel{-3x}\cancel{-6}$

$=(\Delta x)^2+2x\Delta x+3\Delta x$

$=\Delta x[\Delta x+2x+3]$

$=(0.01)[0.01+2(10)+3]$

$=(0.01)[0.01+23]$

$=0.01(23.01)=0.2301$

(ii) $dy = f'(x)\Delta x=(2x+3)(\Delta x)$

$=[2(10)+3](0.01)$

$=(23)(0.01)=0.23$

10. State Lagrange's Mean value theorem.

A: Lagrange's Mean value theorem:

If a function f(x) is (i) continuous on [a, b] and (ii) derivable on (a, b) then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

SECTION-B

11. Find the equation of locus of a point the difference of whose distances from (–5, 0) and (5, 0) is 8 units.

A :
- Let the locus point P = (x, y)
- Given points A = (–5, 0), B = (5, 0)
- ★ **Given condition:** $|PA - PB| = 8$
- ★ $\Rightarrow PA - PB = \pm 8$
- ★ $\Rightarrow PA = \pm 8 + PB \Rightarrow PA^2 = (\pm 8 + PB)^2$
- ★ $\Rightarrow PA^2 = 64 + PB^2 \pm 16PB$
- ★ $\Rightarrow \pm 16PB = 64 + PB^2 - PA^2$
- ★ $\Rightarrow \pm 16PB = 64 + [(x-5)^2 + \cancel{(y-0)^2}]$
  $-[(x+5)^2 + \cancel{(y-0)^2}]$
- $\Rightarrow \pm 16PB = 64 + (x-5)^2 - (x+5)^2$
- $\Rightarrow \pm 16PB = 64 - 4(x)(5)$ $[\because (a-b)^2 - (a+b)^2 = -4ab]$
- $\Rightarrow \pm 16PB = 64 - 20x$
- $\Rightarrow \pm \cancel{16}PB = \cancel{4}(16 - 5x)$
- $\Rightarrow \pm 4PB = 16 - 5x$, Squaring on both sides
- $\Rightarrow 16PB^2 = (16 - 5x)^2$
- $\Rightarrow 16[(x-5)^2 + (y-0)^2] = 256 + 25x^2 - 160x$
- ★ $\Rightarrow 16(x^2 + 25 - 10x + y^2) = 256 - 160x + 25x^2$
- $\Rightarrow 16x^2 + 400 \cancel{-160x} + 16y^2$.
  $= 256 \cancel{-160x} + 25x^2$
- ★ $\Rightarrow (25x^2 - 16x^2) - 16y^2 = 400 - 256$
- $\Rightarrow 9x^2 - 16y^2 = 144$
- Hence, locus of P is $9x^2 - 16y^2 = 144$.

12. When the axes are rotated through an angle α, find the transformed equation of $x\cos\alpha + y\sin\alpha = p$

A:
- Given original equation is $x\cos\alpha + y\sin\alpha = p$........(1)
- Angle of rotation $\theta = \alpha$, then

★ $x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos\alpha - Y\sin\alpha$

$y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos\alpha + X\sin\alpha$

- From (1), transformed equation is
- $(X\cos\alpha - Y\sin\alpha)\cos\alpha + (Y\cos\alpha + X\sin\alpha)\sin\alpha = p$
- $\Rightarrow X\cos^2\alpha \cancel{- Y\sin\alpha\cos\alpha} \cancel{+ Y\cos\alpha\sin\alpha} + X\sin^2\alpha = p$
- $\Rightarrow X(\cos^2\alpha + \sin^2\alpha) = p \Rightarrow X(1) = p \Rightarrow X = p$

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13. Transform the equation $4x - 3y + 12 = 0$ into (i) slope intercept form (ii) intercept form

A: ★ **(i) Slope intercept form is $y = mx + c$**

- $\therefore 4x - 3y + 12 = 0 \Rightarrow 3y = 4x + 12$

★ $\Rightarrow y = \frac{4}{3}x + \frac{\cancel{12}}{\cancel{3}} \Rightarrow y = \frac{4}{3}x + 4$

★ **(ii) Intercept form is $\frac{x}{a} + \frac{y}{b} = 1$**

- $\therefore 4x - 3y + 12 = 0 \Rightarrow 4x - 3y = -12$

★ $\Rightarrow \frac{\cancel{4}x}{-\cancel{12}} - \frac{\cancel{3}y}{-\cancel{12}} = 1$

- $\Rightarrow \frac{x}{-3} + \frac{y}{4} = 1$

14. Is f given by $f(x)=\begin{cases}\dfrac{x^2-9}{x^2-2x-3} & \text{if } 0<x<5, x\neq 3\\ 1.5 & \text{if } x=3\end{cases}$, continuous at the point 3.

A: (a) Given f(3) = 1.5 ....(1)

(b) $\underset{x\to 3}{\text{Lt}}\, f(x)=\underset{x\to 3}{\text{Lt}}\dfrac{x^2-9}{x^2-2x-3}=\underset{x\to 3}{\text{Lt}}\dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+1)}$

$=\underset{x\to 3}{\text{Lt}}\dfrac{x+3}{x+1}=\dfrac{3+3}{3+1}=\dfrac{6}{4}=\dfrac{3}{2}=1.5$ ....(2)

From (1) & (2), $\underset{x\to 3}{\text{Lt}}\, f(x)=f(3)$. Hence proved that f(x) is continuous at x = 3

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15. Find the derivative of sin2x from the first principle.

A:
- We take f(x) = sin2x, then
- ★ f(x + h) = sin2(x + h) = sin (2x + 2h)
- From the first principle,
- ★ $f'(x)=\underset{h\to 0}{\text{Lt}}\dfrac{f(x+h)-f(x)}{h}$
- ★ $=\underset{h\to 0}{\text{Lt}}\dfrac{\sin(2x+2h)-\sin 2x}{h}$
- ★ $=\underset{h\to 0}{\text{Lt}}\dfrac{1}{h}\left(2\cos\left(\dfrac{(2x+2h)+2x}{2}\right)\sin\left(\dfrac{(\cancel{2x}+2h)-\cancel{2x}}{2}\right)\right)$ $\left[\because \sin C-\sin D=2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)\right]$
- ★ $=2\underset{h\to 0}{\text{Lt}}\dfrac{1}{h}\cos\left(\dfrac{4x+2h}{2}\right)\sin\left(\dfrac{\cancel{2}h}{\cancel{2}}\right)$
- ★ $=2\underset{h\to 0}{\text{Lt}}\dfrac{1}{h}\cos\left(\dfrac{\cancel{2}(2x+h)}{\cancel{2}}\right)\sin(h)$
- $=2\underset{h\to 0}{\text{Lt}}\dfrac{1}{h}\cos(2x+h)\sin(h)$
- ★ $=2\underset{h\to 0}{\text{Lt}}\cos(2x+h).\underset{h\to 0}{\text{Lt}}\dfrac{\sinh}{h}$
- ★ $=2\cos(2x+0)(1)=2\cos 2x$

16. Find the equations of the tangent and the normal to the curve $y=x^3+4x^2$ at (–1,3)

A:
- Given curve is $y = x^3 + 4x^2$,
- $\Rightarrow \frac{dy}{dx} = 3x^2 + 8x$
- $\therefore$ Slope of the tangent at P(–1, 3) is
- ★ $m = 3(-1)^2 + 8(-1) = 3 - 8 = -5$
- (i) Tangent at (–1, 3) with slope –5 is
- ★ $y - y_1 = m(x - x_1)$
- $\Rightarrow y-3 = -5(x+1) = -5x-5 \Rightarrow 5x+y+2=0$
- ★ (ii) Slope of the normal is $\frac{-1}{m} = \frac{-1}{-5} = \frac{1}{5}$
- Normal at (–1,3) with slope $\frac{1}{5}$ is
- ★ $y - y_1 = -\frac{1}{m}(x - x_1)$
- ★ $\Rightarrow y - 3 = \frac{1}{5}(x+1)$
- $\Rightarrow 5y - 15 = x + 1 \Rightarrow x - 5y + 16 = 0$

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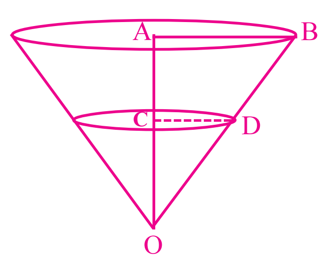
17. A container in the shape of an inverted cone has height 8 m and radius 6 m at the top. If it is filled with water at the rate of $2m^3$/min, what is the rate of change in the height of water level when the tank is filled 4 m?

A: Let OC be height of water level at t sec.
Let OC = h, CD = r and volume = V. Given that AB = 6, OA= 8 , $\frac{dV}{dt} = 2$

The triangles OAB and OCD are similar triangles. $\therefore \frac{CD}{AB} = \frac{OC}{OA} \Rightarrow \frac{r}{6} = \frac{h}{8} \Rightarrow r = \frac{3h}{4}$ ....(1)

Volume of the cone V is given by $V = \frac{\pi r^2 h}{3}$ ....(2)



From (1), we have $V = \frac{\pi}{3}\left(\frac{3h}{4}\right)^2 \times h = \frac{9\pi h^3}{48}$ ....(3)

Differentiating (3) w.r.to t, we get $\frac{dV}{dt} = \frac{9\pi}{48}.3h^2 \frac{dh}{dt} = \frac{9\pi}{16} h^2 \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{16}{9\pi h^2}\frac{dV}{dt} = \frac{16}{9\pi 4^2}(2) = \frac{2}{9\pi} = \left(\frac{1}{\pi}\right)\frac{4}{8^2}(12) = \frac{3}{4\pi} cm / sec$

SECTION-C

18. (a) If Q(h, k) is the foot of the perpendicular of $P(x_1, y_1)$ on the line $ax + by + c = 0$ then prove that $(h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$.

A:
- Given $P = (x_1, y_1)$, $Q = (h, k)$
- Slope of $\overline{PQ}$ is $m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k - y_1}{h - x_1}$
- Slope of the line $ax + by + c = 0$ is $m_2 = -\frac{a}{b}$
- Two lines are perpendicular $\Rightarrow m_1 m_2 = -1$

★ $\Rightarrow \left(\frac{k-y_1}{h-x_1}\right)\left(-\frac{a}{b}\right) = -1 \Rightarrow \left(\frac{k-y_1}{h-x_1}\right)\left(\frac{a}{b}\right) = 1$

- $\Rightarrow \frac{k - y_1}{h - x_1} = \frac{b}{a} \Rightarrow \frac{k - y_1}{b} = \frac{h - x_1}{a}$

★ We take $\frac{h - x_1}{a} = \frac{k - y_1}{b} = r$ ....(1)

★ $\therefore \frac{h - x_1}{a} = r \Rightarrow h - x_1 = ar \Rightarrow h = x_1 + ar$

★ $\frac{k - y_1}{b} = r \Rightarrow k - y_1 = br \Rightarrow k = y_1 + br$

★ But Q(h, k) lies on $ax + by + c = 0$

- $\Rightarrow ah + bk + c = 0$
- $\Rightarrow a(x_1 + ar) + b(y_1 + br) + c = 0$
- $\Rightarrow ax_1 + a^2r + by_1 + b^2r + c = 0$
- $\Rightarrow a^2r + b^2r + ax_1 + by_1 + c = 0$

★ $\Rightarrow r(a^2 + b^2) = -(ax_1 + by_1 + c)$

★ $\Rightarrow r = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$ ....(2)

★ From (1) & (2), $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$

(b) Find the foot of the perpendicular drawn from (4,1) on the line 3x – 4y + 12 = 0.

A: Let (h, k) be the foot of the perpendicular from (4, 1) on the line 3x – 4y + 12 = 0

Here $(x_1, y_1) = (4, 1)$, a = 3, b = –4, c = 12.

$$\therefore \frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-(ax_1+by_1+c)}{a^2+b^2}$$

$$\Rightarrow \frac{h-4}{3} = \frac{k-1}{-4} = \frac{-[3(4)-4(1)+12]}{3^2+4^2} = \frac{-(12-4+12)}{9+16} = \frac{-20}{25} = \frac{-4}{5}$$

Now, $\frac{h-4}{3} = \frac{-4}{5} \Rightarrow 5h-20 = -12 \Rightarrow 5h = 20-12 = 8 \Rightarrow h = \frac{8}{5}$

Also $\frac{k-1}{-4} = \frac{-4}{5} \Rightarrow 5k-5 = 16 \Rightarrow 5k = 16+5 = 21 \Rightarrow k = \frac{21}{5}$ $\therefore (h,k) = \left(\frac{8}{5}, \frac{21}{5}\right)$

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19. (a)If θ is the angle between the pair of lines $ax^2+2hxy+by^2=0$ then $\cos\theta = \frac{a+b}{\sqrt{(a-b)^2+4h^2}}$

Proof: Let the separate equations of $ax^2+2hxy+by^2=0$ be $l_1x+m_1y=0$ .....(1) and $l_2x+m_2y=0$ ......(2)

$$\therefore ax^2+2hxy+by^2 \equiv (l_1x+m_1y)(l_2x+m_2y)$$

Comparing both sides, we get $l_1l_2 = a$, $l_1m_2+l_2m_1 = 2h$, $m_1m_2 = b$.

If θ is an angle between the lines (1) and (2) then

$$\cos\theta = \frac{l_1l_2+m_1m_2}{\sqrt{(l_1^2+m_1^2).(l_2^2+m_2^2)}} = \frac{l_1l_2+m_1m_2}{\sqrt{l_1^2l_2^2+m_1^2m_2^2+l_1^2m_2^2+l_2^2m_1^2}}$$

$$= \frac{l_1l_2+m_1m_2}{\sqrt{(l_1l_2-m_1m_2)^2 + \cancel{2l_1l_2m_1m_2} + (l_1m_2+l_2m_1)^2 - \cancel{2l_1l_2m_1m_2}}}$$

$$= \frac{a+b}{\sqrt{(a-b)^2+(2h)^2}} = \frac{a+b}{\sqrt{(a-b)^2+4h^2}}$$

$$\left[\begin{array}{l} \because a^2+b^2 = (a-b)^2+2ab \\ a^2+b^2 = (a+b)^2-2ab \end{array}\right]$$

(b) Find the angle between the pair of lines represented by the equation $x^2 - 7xy + 12y^2 = 0$

A: Comparing the given equation with $ax^2+2hxy+by^2=0$, we get a=1, 2h=–7, b=12

$$\therefore \cos\theta = \frac{a+b}{\sqrt{(a-b)^2+(2h)^2}} = \frac{1+12}{\sqrt{(1-12)^2+(-7)^2}} = \frac{13}{\sqrt{121+49}} = \frac{13}{\sqrt{170}} \Rightarrow \theta = \cos^{-1}\left(\frac{13}{\sqrt{170}}\right)$$

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20. Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and the straight line $x - y - \sqrt{2} = 0$ are mutually perpendicular.

A:
- Given line is $x - y = \sqrt{2} \Rightarrow \frac{x-y}{\sqrt{2}} = 1$ ...(1)
- Given curve is $x^2 - xy + y^2 + 3x + 3y - 2 = 0$............(2)
- Homogenising (1) & (2), we get

$$x^2 - xy + y^2 + 3x(1) + 3y(1) - 2(1)^2 = 0$$

$$\Rightarrow x^2 - xy + y^2 + 3x\left(\frac{x-y}{\sqrt{2}}\right) + 3y\left(\frac{x-y}{\sqrt{2}}\right) - 2\frac{(x-y)^2}{2} = 0$$

$$\Rightarrow \frac{\sqrt{2}x^2 - \sqrt{2}xy + \sqrt{2}y^2 + 3x(x-y) + 3y(x-y) - \sqrt{2}(x-y)^2}{\sqrt{2}} = 0$$

$$\Rightarrow \sqrt{2}x^2 - \sqrt{2}xy + \sqrt{2}y^2 + 3x^2 - 3xy + 3yx - 3y^2 - \sqrt{2}(x^2 + y^2 - 2xy) = 0$$

$$\Rightarrow \sqrt{2}x^2 - \sqrt{2}xy + \sqrt{2}y^2 + 3x^2 - 3y^2 - \sqrt{2}x^2 - \sqrt{2}y^2 + 2\sqrt{2}xy = 0$$

$$\Rightarrow 3x^2 - 3y^2 + \sqrt{2}xy = 0$$

- Here, coeff. of x^2 + coeff. of y^2 is 3–3=0

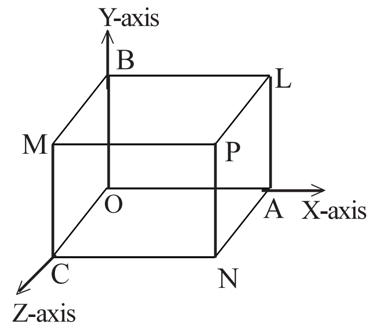
$\therefore$ The pair of lines are perpendicular

21. Find the angle between two diagonals of a cube

A: ★ Consider a cube of side 'a' with vertices O, A, B, C, L, M, N, P where O = (0, 0, 0)

★ Take A,B,C are on the X-axis, Y-axis, Z-axis, then

A = (a, 0, 0), B = (0, a, 0), C = (0, 0, a)



★ Take L,M,N on the XY-plane, YZ-plane, ZX-plane, then

L = (a, a, 0), M = (0, a, a), N = (a, 0, a)

★ Take P in the XYZ space, then P = (a, a, a)

★ Take the 2 diagonals $\overline{OP}$, $\overline{CL}$.

• d.r's of $\overline{OP}$ = (a – 0, a – 0, a – 0) = (a, a, a) = (a_1, b_1, c_1)

• d.r's of $\overline{CL}$ =(a – 0, a – 0, 0 – a) = (a, a, – a) = (a_2, b_2, c_2)

• So, angle between the two diagonals is given by

★ $$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}$$

$$= \frac{|a(a) + a(a) + a(-a)|}{\sqrt{(a^2 + a^2 + a^2)(a^2 + a^2 + a^2)}}$$

$$= \frac{a^2}{\sqrt{(3a^2)(3a^2)}} = \frac{a^2}{3a^2} = \frac{1}{3}$$

★ $$\therefore \cos\theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$$

So, that angle between diagonals of a cube is $\cos^{-1}\frac{1}{3}$

Double Dhamaka!!

☺ Do you remember?
I was in
Vectors of Maths-1A
Almost Same 2 Same

☺ You know!
Angle between diagonals for Square is 90°
Cube is $\cos^{-1}\frac{1}{3}$

22. If $y = \mathrm{Tan}^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ then find $\frac{dy}{dx}$

A: • Given $y=\mathrm{Tan}^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

★ We take $x^2=\cos 2\theta$, then

• $y = \mathrm{Tan}^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right) = \mathrm{Tan}^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$

• $= \mathrm{Tan}^{-1}\left(\frac{\sqrt{2}[\cos\theta+\sin\theta]}{\sqrt{2}[\cos\theta-\sin\theta]}\right) = \mathrm{Tan}^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$

★ $= \mathrm{Tan}^{-1}\left(\frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}\right) = \mathrm{Tan}^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

★ $= \mathrm{Tan}^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right] = \frac{\pi}{4}+\theta$

★ $\therefore y = \frac{\pi}{4}+\theta$

★ But $\cos 2\theta = x^2 \Rightarrow 2\theta = \mathrm{Cos}^{-1}(x^2) \Rightarrow \theta = \frac{1}{2}\mathrm{Cos}^{-1}(x^2)$

★ So, $y=\frac{\pi}{4}+\frac{1}{2}\mathrm{Cos}^{-1}(x^2)$

★ On diff. w.r.t x, we get

• $\frac{dy}{dx} = 0+\frac{1}{2}\left(\frac{-1}{\sqrt{1-(x^2)^2}}(2x)\right),$ $\quad \left[\because \frac{d}{dx}\mathrm{Cos}^{-1}f(x) = \frac{-1}{\sqrt{1-(f(x))^2}}\cdot\frac{d}{dx}f(x)\right]$

$$\therefore \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^4}}$$

23. Find the angle between the curves $y^2 = 4x$ and $x^2 + y^2 = 5$.

A: 1) Finding points of intersection:

- Given $y^2=4x$ ....(1), $x^2+y^2 = 5$ ....(2)

From (1) & (2),

- $x^2 + 4x = 5 \Rightarrow x^2 + 4x - 5 = 0$
- ★ $\Rightarrow (x-1)(x+5) = 0 \Rightarrow x=1$ or -5
- If x=1 then $y^2 = 4(1) = 4 = 2^2 \Rightarrow y = \pm 2$
- ★ $\therefore$ Points of intersection P=(1,2),Q=(1,–2)

2) Finding derivatives:

- ★ $y^2 = 4x \Rightarrow 2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$
- $x^2 + y^2 = 5 \Rightarrow 2x + 2y\frac{dy}{dx} = 0$
- $\Rightarrow \not{2}y\frac{dy}{dx} = -\not{2}x \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

3) Finding Slopes at P(1,2):

- ★ $m_1 = \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{2}{y} = \frac{2}{2} = 1$;
- ★ $m_2 = \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{-x}{y} = \frac{-1}{2}$

4) Finding angle at P: If θ is the angle between the curves at P then

- ★ $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{1-\left(-\frac{1}{2}\right)}{1+\left(1\times\left(-\frac{1}{2}\right)\right)}\right| = \left|\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$ $\therefore \theta = \mathrm{Tan}^{-1}3$

5) Finding slopes at Q(1,–2):

- $m_1 = \left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{2}{y} = \frac{2}{-2} = -1$;
- $m_2 = \left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{-x}{y} = \frac{-1}{-2} = \frac{1}{2}$

6) Finding angle at Q: If θ is the angle between the curves at Q then

- ★ $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{-1-\frac{1}{2}}{1+\left(-1\times\frac{1}{2}\right)}\right| = \left|\frac{\frac{-3}{2}}{\frac{1}{2}}\right| = |-3| = 3$ $\therefore \theta = \mathrm{Tan}^{-1}3$

24. The profit function p(x) of a company, selling x items per day is given by $p(x) = (150 - x)x - 1600$. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.

A: Given profit function is $p(x) = (150 - x)x - 1600$

$p'(x) = (150-x)\,1 + x(-1) = 150 - 2x$. Also $p''(x) = -2$

For Maximum profit, $p'(x) = 0 \Rightarrow 150 - 2x = 0 \Rightarrow x = 75$

$\therefore$ Profit p(x) is maximum when $x = 75$

$\therefore$ Maximum Profit $= (150 - 75)\,75 - 1600 = 75(75) - 1600 = 5625 - 1600 = 4025$ units.