Previous IPE

SOLVED PAPERS

MARCH-2019 (AP)

PREVIOUS PAPERS

IPE: MARCH-2019(AP)

Time : 3 Hours **MATHS-2B** Max.Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **10 × 2 = 20**

1. Find the equation of the circle for which the points (4,2), (1,5) are the end points of a diameter.
2. Find the value of k if the points (4,2), (k,–3) are conjugate w.r.to the circle $x^2+y^2-5x+8y+6=0$
3. Find the equation of the radical axis of the circles $x^2+y^2+4x+6y-7=0$, $4(x^2+y^2)+8x+12y-9=0$.
4. Find the equation of the tangent to the parabola $x^2-4x-8y+12=0$ at $\left(4,\frac{3}{2}\right)$
5. Find the product of lengths of the perpendiculars from any point on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ to its asymptotes
6. Evaluate $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$
7. Evaluate $\int \frac{dx}{(x+1)(x+2)}$
8. Evaluate $\int_0^1 \frac{dx}{\sqrt{3-2x}}$
9. Evaluate $\int_0^{\pi/2} \sin^6 x\cos^4 x\,dx$
10. Form the D.E corresponding to $y=cx-2c^2$, where c is a parameter.

SECTION-B

II. Answer any FIVE of the following SAQs: **5 × 4 = 20**

11. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+18y+26=0$ are in the ratio 2:3 then find the equation of the locus of P.
12. Find the equation of the circle passing through the points of intersection of the circles $x^2+y^2-8x-6y+21=0$ and $x^2+y^2-2x-15=0$ and (1,2).
13. Find the equation of the ellipse referred to its major, minor axes as the coordinate axes X,Y-respectively with latus rectum of length 4 and distance between foci $4\sqrt{2}$.
14. Show that the locus of the feet of the perpendiculars drawn from either of the foci to any tangent to the ellipse is the auxiliary circle.
15. Find the equation of the tangents to the hyperbola $x^2-4y^2=4$ which are (i) parallel to and (ii) perpendicular to the line $x+2y=0$
16. Find the area of one of the curvilinear traingles bounded by $y=\sin x$, $y=\cos x$ and X - axis.
17. Solve $x(x-1)\frac{dy}{dx}-y=x^3(x-1)^3$

SECTION-C

III. Answer any FIVE of the following Long Answer Questions: **5 × 7 = 35**

18. Show that (1,1), (–6,0), (–2,2), (–2,–8) are concyclic and find the equation of the circle on which they lie.
19. Find the equation to the pair of transverse common tangents to the circles $x^2+y^2-4x-10y+28=0$ and $x^2+y^2+4x-6y+4=0$
20. Find the equation of the parabola whose focus is S(3, 5) and vertex is A(1,3).
21. Evaluate $\int \frac{3\sin x+\cos x+7}{\sin x+\cos x+1}dx$
22. Obtain the reduction formula for $I_n=\int \operatorname{cosec}^n x\,dx$,n being a positive integer, $n\ge 2$ and deduce that the value of $\int \operatorname{cosec}^5 x\,dx$
23. Evaluate $\int_0^{\pi/4} \log(1+\tan x)\,dx$
24. Solve $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-xy}$

IPE AP MARCH-2019
SOLUTIONS

SECTION-A

1. Find the equation of the circle with (4, 2), (1, 5) as ends of a diameter.

Sol: Let $A(x_1,y_1) = (4,2)$ and $B(x_2,y_2) = (1, 5)$

The equation of the circle with A(4,2) , B(1, 5) as ends of a diameter is given by

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2)=0 \Rightarrow (x-4)(x-1) + (y-2)(y-5) = 0$

$\Rightarrow x^2 - x - 4x + 4 + y^2 - 5y - 2y + 10 = 0 \Rightarrow x^2 + y^2 - 5x - 7y + 14 = 0$

---

2. Find the value of k if the points (4,2), (k,–3) are conjugate with respect to the circle $x^2+y^2-5x+8y+6=0$

Sol: The points (4,2), (k,–3) are conjugate w.r.t the circle $S = x^2+y^2-5x+8y+6=0$

$\Rightarrow S_{12} = 0$

$\Rightarrow x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0$

$\Rightarrow 4(k) + 2(-3) - \frac{5}{2}(4+k) + 4(2-3) + 6 = 0$

$\Rightarrow 4k - \frac{5k}{2} - 14 = 0 \Rightarrow \frac{3k}{2} = 14 \Rightarrow k = \frac{28}{3}$

---

3. Find the equation of the radical axis of $x^2+y^2+4x+6y-7=0$, $4(x^2+y^2)+8x+12y-9=0$

Sol: Let $S \equiv x^2 + y^2 + 4x + 6y - 7 = 0$ and $S' \equiv x^2 + y^2 + 2x + 3y - \frac{9}{4} = 0$

Radical axis is $S - S' = 0$

$\Rightarrow (4-2)x + (6-3)y + \left(-7 + \frac{9}{4}\right) = 0 \Rightarrow 2x + 3y - \frac{19}{4} = 0 \Rightarrow 8x + 12y - 19 = 0$

4. Find the equation of the tangent to the parabola $x^2-4x-8y+12=0$ at $\left(4,\frac{3}{2}\right)$

Sol: Equation of the tangent to the parabola $S= lx^2+mx+ny+c=0$ is

$$S_1= l(x_1x)+\frac{m}{2}(x+x_1)+\frac{n}{2}(y+y_1)+c=0$$

$\therefore$ Equation of the tangent to the parabola $x^2-4x-8y+12=0$ is

$$1(4x)-2(x+4)-4\left(y+\frac{3}{2}\right)+12=0 \Rightarrow 4x-2x-8-4\left(\frac{2y+3}{2}\right)+12=0$$

$$\Rightarrow 4x-2x-8-2(2y+3)+12=0 \Rightarrow 4x-2x-8-4y-6+12=0$$

$$\Rightarrow 2x-4y-2=0 \Rightarrow 2(x-2y-1)=0 \Rightarrow x-2y-1=0 \qquad \text{(or)}$$

[**Hint:** Differentiating the given equation w.r.t x we get $\frac{dy}{dx}=\frac{x-2}{4}$

Hence slope of the tangent at (4,3/2) is 1/2.]

---

5. Find the product of lengths of the perpendiculars from any point on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ to its asymptotes.

Sol: Given hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1 \Rightarrow a^2=16, b^2=9$

Product of the perpendiculars from any point on the hyperbola to its asymptotes

$$=\frac{a^2b^2}{a^2+b^2}=\frac{16\times 9}{16+9}=\frac{144}{25}$$

---

6. Evaluate $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Sol: Put $xe^x = t \Rightarrow [xe^x+e^x(1)]dx=dt \quad \Rightarrow e^x(1+x)dx=dt$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(xe^x)}dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt = \tan t + c = \tan(xe^x)+c$$

---

7. Evaluate $\int \frac{dx}{(x+1)(x+2)}$

Sol: $\int \frac{dx}{(x+1)(x+2)} = \int \frac{(x+2)-(x+1)}{(x+1)(x+2)}dx = \int\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx = \log|x+1|-\log|x+2|+c$

8. Evaluate $\int_0^1 \frac{dx}{\sqrt{3-2x}}$

Sol: $I = \int_0^1 \frac{dx}{\sqrt{3-2x}} = \frac{-1}{2}\left[2\sqrt{3-2x}\right]_0^1 = -\left[\sqrt{3-2x}\right]_0^1 = -(\sqrt{3-2(1)} - \sqrt{3-2(0)})$

$= -(1-\sqrt{3}) = \sqrt{3}-1$

---

9. Evaluate $\int_0^{\pi/2} \sin^6 x\cos^4 x dx$

Sol : $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx \quad = \frac{[(5)(3)(1)][(3)(1)]}{(10)(8)(6)(4)(2)} \frac{\pi}{2} = \frac{3\pi}{512}$

☞ Note the factor $\frac{\pi}{2}$ ($\because$ m,n are even)

---

10. Form the D.E corresponding to $y=cx-2c^2$ where c is a parameter.

Sol: Given that $y=cx-2c^2$ .....(1)

Differentiating w.r.to x, we get $y_1=c$

$\therefore (1) \Rightarrow y = y_1 x - 2y_1^2 \Rightarrow 2y_1^2 - xy_1 + y = 0 \Rightarrow 2\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$

SECTION-B

11. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+18y+26=0$ are in the ratio 2:3 then find the equation of the locus of P.

Sol: Let $P(x_1,y_1)$ be a point on the locus.

Let PT_1 = length of the tangent P to $S= x^2+y^2-4x-6y-12=0$

and PT_2 = length of the tangent P to $S' = x^2+y^2+6x+18y+26=0$

$PT_1 = \sqrt{S_{11}}$

$PT_2 = \sqrt{S'_{11}}$

Given that $PT_1 : PT_2 = 2:3$

$$\Rightarrow \frac{PT_1}{PT_2} = \frac{2}{3} \Rightarrow 3PT_1 = 2PT_2 \Rightarrow 9(PT_1)^2 = 4(PT_2)^2$$

$$\Rightarrow 9[x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12] = 4[x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26]$$

$$\Rightarrow 9x_1^2 + 9y_1^2 - 36x_1 - 54y_1 - 108 = 4x_1^2 + 4y_1^2 + 24x_1 + 72y_1 + 104$$

$$\Rightarrow 5x_1^2 + 5y_1^2 - 60x_1 - 126y_1 - 212 = 0$$

$\therefore$ Equation of locus of $P(x_1,y_1)$ is $5x^2+5y^2-60x-126y-212 = 0$.

---

12. Find the equation of the circle passing through the points of intersection of the circles $x^2+y^2-8x-6y+21=0$, $x^2+y^2-2x-15=0$ and (1,2)

Sol: Given circles are $S=x^2+y^2-8x-6y+21=0$, $S'=x^2+y^2-2x-15=0$

Radical axis of the circles is $L=S-S'=0$

$\Rightarrow -8x+2x-6y+21+15=0 \Rightarrow -6x-6y+36=0 \Rightarrow -6(x+y-6)=0 \Rightarrow x+y-6=0$

Equation of any circle passing through the points of intersection of $S'=0$, $L=0$ is $S'+\lambda L=0$

$\Rightarrow (x^2+y^2-2x-15) + \lambda(x + y- 6) =0$ .....(1)

(1) passes through the point (1,2)

$\Rightarrow (1+4-2-15)+\lambda(1+2-6)=0 \Rightarrow -12 + \lambda(-3)=0 \Rightarrow 3\lambda= -12 \Rightarrow \lambda= -4$

Put $\lambda= -4$ in (1) then $(x^2+y^2-2x-15) -4(x + y- 6) =0$

$\Rightarrow x^2+y^2-2x-15-4x +4y+24 =0 \Rightarrow x^2+y^2-6x-4y+9=0$

13. Find the equation of the ellipse referred to its major, minor axes as the coordinate axes X,Y-respectively with latus rectum of length 4 and distance between foci $4\sqrt{2}$.

Sol: Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b)$

Length of latus rectum is $4 \Rightarrow \frac{2b^2}{a}=4 \Rightarrow b^2=2a$

Distance between foci S=(ae,0) and S'=(–ae,0) is $4\sqrt{2}$

$\Rightarrow 2ae=4\sqrt{2} \Rightarrow ae=2\sqrt{2} \Rightarrow (ae)^2=(2\sqrt{2})^2=8$

Now, $b^2=a^2(1-e^2) \Rightarrow 2a=a^2-(ae)^2=a^2-8$

$\Rightarrow a^2-2a-8=0 \Rightarrow (a-4)(a+2)=0 \Rightarrow a=4$ ($\because$ 'a' cannot be negative)

$\therefore\ b^2=2a=2(4)=8$

$\therefore$ the equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{8}=1 \Rightarrow x^2+2y^2=16$

---

14. Show that the locus of the feet of the perpendiculars drawn from either of the foci to any tangent to the ellipse is the auxiliary circle.

Sol: Let the equation of the ellipse be $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

Let $P(x_1,y_1)$ be a point on the locus.

Tangent with slope m, is $y=mx\pm\sqrt{a^2m^2+b^2} \Rightarrow y-mx=\pm\sqrt{a^2m^2+b^2}$ .....(1)

The equation of any line perpendicular to the above tangent has slope –1/m.

If it passes through foci (±ae,0) then its equation is

$y-0=-\frac{1}{m}(x\pm ae) \Rightarrow my=-(x\pm ae) \Rightarrow my+x=\pm ae$......(2)

But, $P(x_1,y_1)$ is the point of intersection of (1) and (2).

$\Rightarrow y_1-mx_1=\pm\sqrt{a^2m^2+b^2}$ and $my_1+x_1=\pm ae$

Squaring and adding the above 2 equations we get

$\Rightarrow (y_1-mx_1)^2+(my_1+x_1)^2=(a^2m^2+b^2)+(ae)^2$

$\Rightarrow (y_1^2+m^2x_1^2-2mx_1y_1)+(m^2y_1^2+x_1^2+2mx_1y_1)=a^2m^2+b^2+a^2e^2$

$\Rightarrow x_1^2(1+m^2)+y_1^2(1+m^2)=a^2m^2+[a^2(1-e^2)]+a^2e^2.$

$\Rightarrow (x_1^2+y_1^2)(1+m^2)=a^2(m^2+1) \Rightarrow x_1^2+y_1^2=a^2$

$\therefore$ The locus of $P(x_1,y_1)$ is $x^2+y^2=a^2$. This is called **auxiliary circle.**

15. Find the equation of the tangents to the hyperbola $x^2-4y^2=4$ which are (i) parallel (ii) perpendicular to the line $x+2y=0$

Sol: Given hyperbola is $x^2-4y^2=4 \Rightarrow \frac{x^2}{4}-\frac{y^2}{1}=1 \Rightarrow a^2=4, b^2=1$

Slope of the given line $x+2y=0$ is $m=-1/2 \Rightarrow$ Slope of its perpendicular is 2

Formula:

Tangent with slope m is $y = mx \pm \sqrt{a^2m^2-b^2}$

(i) Parallel tangent with slope $\frac{-1}{2}$ is $y=-\frac{1}{2}x \pm \sqrt{4\left(\frac{1}{4}\right)-1} \Rightarrow y=\frac{-x}{2} \Rightarrow x+2y=0$

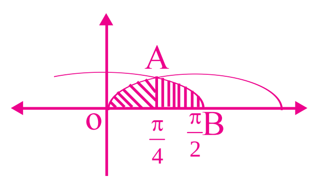
(ii) Perpendicular tangent with slope 2 is $y=2x \pm \sqrt{4(2^2)-1} \Rightarrow y=2x \pm \sqrt{15}$

---

16. Find the area of one of the curvilinear traingles bounded by y=sinx, y=cosx and X-axis.

Sol: OAB is one of the curvilinear triangles bounded by y=sinx, y=cosx and the X-axis.

The area of this curvilinear triangle as shown in figure.



We have $\cos x \geq \sin x$ for $x \in [0, \pi/4]$ and $\cos x \leq \sin x$ for $x \in [\pi/4, \pi/2]$

$$\therefore A = \int_0^{\pi/4} \sin x dx + \int_{\pi/4}^{\pi/2} \cos x dx = [-\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}$$

$$=\left(1-\frac{1}{\sqrt{2}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)=(2-\sqrt{2}) \text{ sq.units}$$

17. Solve $x(x-1)\frac{dy}{dx}-y=x^3(x-1)^3$

Sol: Given D.E is $x(x-1)\frac{dy}{dx}-y=x^3(x-1)^3 \Rightarrow \frac{dy}{dx}-\frac{y}{x(x-1)}=\frac{x^3(x-1)^3}{x(x-1)}=x^2(x-1)^2$

$\Rightarrow \frac{dy}{dx}+y\left(\frac{-1}{x(x-1)}\right)=x^2(x-1)^2$ This is a linear D.E in y.

It is in the form $\frac{dy}{dx}+yP(x)=Q(x)$ where $P(x)=\frac{-1}{x(x-1)}$ and $Q(x)=x^2(x-1)^2$

Here, $P(x)=\frac{-1}{x(x-1)} \Rightarrow \int P(x)dx=-\int\frac{dx}{x(x-1)}=\int\left(\frac{1}{x}-\frac{1}{x-1}\right)dx=\log x-\log(x-1)=\log\left(\frac{x}{x-1}\right)$

Now, $I.F=e^{\int P(x)dx}=e^{\log\left(\frac{x}{x-1}\right)}=\frac{x}{x-1}$

$\therefore$ The solution is $y(I.F)=\int(I.F)Q(x)dx$

$\Rightarrow y\left(\frac{x}{x-1}\right)=\int\left(\frac{x}{x-1}\right)x^2(x-1)^2dx=\int x^3(x-1)dx=\int(x^4-x^3)dx=\frac{x^5}{5}-\frac{x^4}{4}+c$

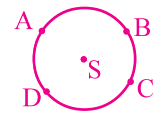
$\therefore$ The solution is $y\left(\frac{x}{x-1}\right)=\frac{x^5}{5}-\frac{x^4}{4}+c$

SECTION-C

18. Show that the points (1, 1), (–6, 0), (–2, 2) and (–2, –8) are concyclic.

Sol: Let A=(1,1), B=(–6,0), C=(–2,2), D=(–2,–8)

We take $S(x_1,y_1)$ as the centre of the circle $\Rightarrow$ SA=SB=SC



Now, $SA = SB \Rightarrow SA^2 = SB^2 \Rightarrow (x_1-1)^2+(y_1-1)^2 = (x_1+6)^2+(y_1-0)^2$

$\Rightarrow (\cancel{x_1^2}-2x_1+1)+(\cancel{y_1^2}-2y_1+1) = (\cancel{x_1^2}+12x_1+36)+\cancel{y_1^2}$

$\Rightarrow 14x_1+2y_1+34=0 \Rightarrow 2(7x_1+y_1+17)=0 \Rightarrow 7x_1+y_1+17=0$ ....(1)

Also, $SB = SC \Rightarrow SB^2 = SC^2 \Rightarrow (x_1+6)^2+(y_1-0)^2 = (x_1+2)^2+(y_1-2)^2$

$\Rightarrow (\cancel{x_1^2}+12x_1+36)+\cancel{y_1^2} = (\cancel{x_1^2}+4x_1+4)+(\cancel{y_1^2}-4y_1+4)$

$\Rightarrow 8x_1+4y_1+28=0 \Rightarrow 4(2x_1+y_1+7)=0 \Rightarrow 2x_1+y_1+7=0$ ......(2)

Solving (1) & (2) we get the centre $S(x_1,y_1)$

$(1)-(2) \Rightarrow 5x_1+10=0 \Rightarrow 5x_1=-10 \Rightarrow x_1=-2$

From (1), $7(-2)+y_1+17=0 \Rightarrow y_1+3=0 \Rightarrow y_1=-3$

$\therefore$ Centre of the circle is $S(x_1,y_1)=(-2,-3)$

Also, we have A=(1,1) $\Rightarrow$ r=SA$\Rightarrow$ $r^2=SA^2$

$\therefore r^2 = SA^2 = (1+2)^2+(1+3)^2 = 9+16 = 25$

$\therefore$ Equation of the circle with centre (–2,–3) and $r^2=25$ is

$(x+2)^2+(y+3)^2=25 \Rightarrow (x^2+4x+4)+(y^2+6y+9)=25$

$\Rightarrow x^2+y^2+4x+6y-12=0$

Now, put D(–2,–8) in the above equation, then we get

$(-2)^2+(-8)^2+4(-2)+6(-8)-12=4+64-8-48-12=68-68=0$

$\therefore$ D(–2,–8) lies on the circle

$\therefore$ the given 4 points are concyclic.

19. Find the equation to the pair of transverse common tangents to the circles $x^2+y^2-4x-10y+28=0$ and $x^2+y^2+4x-6y+4=0$.

Sol: For the circle $x^2+y^2-4x-10y+28=0$, centre $C_1=(2,5)$,

radius $r_1=\sqrt{(-2)^2+(-5)^2-28}=\sqrt{1}=1$

For the circle $x^2+y^2+4x-6y+4=0$, centre $C_2=(-2,3)$, radius $r_2=\sqrt{2^2+(-3)^2-4}=\sqrt{9}=3$

The internal centre of similitude, I divides C_1C_2 internally in the ratio $r_1:r_2=1:3$

$$\therefore I=\left(\frac{1(-2)+3(2)}{1+3},\frac{1(3)+3(5)}{1+3}\right)=\left(\frac{4}{4},\frac{18}{4}\right)=\left(1,\frac{9}{2}\right)$$

The equation to the pair of transverse common tangents is $S_1^2=S_{11}(S)$

$$\Rightarrow\left[x+\frac{9}{2}y-2(x+1)-5\left(y+\frac{9}{2}\right)+28\right]^2=\left(1+\frac{81}{4}-4-45+28\right)(x^2+y^2-4x-10y+28)$$

$$\Rightarrow\left(-x-\frac{y}{2}+\frac{7}{2}\right)^2=\frac{1}{4}(x^2+y^2-4x-10y+28)\Rightarrow\frac{(2x-y+7)^2}{4}=\frac{1}{4}(x^2+y^2-4x-10y+28)$$

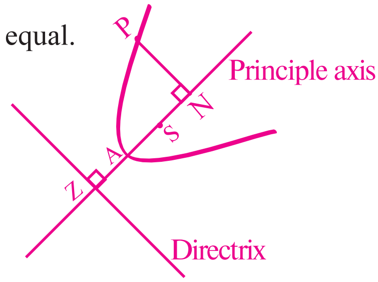
$$\Rightarrow(2x+y-7)^2=x^2+y^2-4x-10y+28$$

$$\Rightarrow 4x^2+y^2+49+4xy-28x-14y=x^2+y^2-4x-10y+28\Rightarrow 3x^2+4xy-24x-4y+21=0$$

20. Find the equation of the parabola whose focus is S(3,5) and vertex is A(1,3)

Sol: Given the focus S = (3, 5) and vertex A = (1,3)

Neither the x-coordinates nor the y-coordinates of A,S are equal.



Hence the parabola is an oblique one.

Let the foot of the directrix Z = (α, β)

Mid point of ZS = A

$$\Rightarrow \left(\frac{\alpha+3}{2}, \frac{\beta+5}{2}\right) = (1,3) \Rightarrow \alpha+3=2 \Rightarrow \alpha=-1\,;\ \beta+5=6 \Rightarrow \beta=1$$

∴ Z= (–1,1)

Slope of the axis joining A (1,3) and Z(–1,1) is $m = \frac{3-1}{1+1} = \frac{2}{2} = 1$

Since the directrix is perpendicular to the axis, slope of the directrix is –1.

∴ Equation of the directrix passing through Z(–1, 1) with slope –1 is

$\Rightarrow y-1 = -1(x+1) \Rightarrow y-1 = -x-1 \Rightarrow x+y = 0$

Let $P(x_1,y_1)$ be a point on the parabola with focus S(3,5) and directrix x+y=0

From the focus directrix property of the parabola we have SP= PM

$$\Rightarrow \sqrt{(x_1-3)^2+(y_1-5)^2} = \frac{|x_1+y_1|}{\sqrt{1^2+1^2}} \qquad \Rightarrow (x_1-3)^2+(y_1-5)^2 = \frac{(x_1+y_1)^2}{2}$$

$$\Rightarrow 2[(x_1-3)^2+(y_1-5)^2] = (x_1+y_1)^2$$

$$\Rightarrow 2(x_1^2-6x_1+9+y_1^2-10y_1+25) = x_1^2+2x_1y_1+y_1^2$$

$$\Rightarrow 2(x_1^2+y_1^2-6x_1-10y_1+34) = x_1^2+2x_1y_1+y_1^2$$

$$\Rightarrow 2x_1^2+2y_1^2-12x_1-20y_1+68 = x_1^2+2x_1y_1+y_1^2$$

$$\Rightarrow x_1^2-2x_1y_1+y_1^2-12x_1-20y_1+68=0$$

∴ The equation of the parabola is $x^2-2xy+y^2-12x-20y+68=0$

21. **Evaluate** $\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1} dx$

Sol: Let $\cos x + 3\sin x + 7 = A\frac{d}{dx}(\cos x + \sin x + 1) + B(\cos x + \sin x + 1) + C$

$\Rightarrow \cos x + 3\sin x + 7 = A(-\sin x + \cos x) + B(\cos x + \sin x + 1) + C$ ....(I)

$\Rightarrow \cos x + 3\sin x + 7 = \cos x(A + B) + \sin x(-A + B) + (B + C)$

Equating the coefficients of cosx, we have A+B=1 ....(1)

Equating the coefficients of sinx, we have –A+B=3 ....(2)

Equating the constant terms, we have B+C=7 ....(3)

Now (1) + (2) ⇒ 2B=4 ⇒ B=2

From (1) , A=1–B =1–2= –1

From (3) , C=7–B =7–2= 5

Putting A= –1, B=2, C= 5 in (I) we get Nr.

$\cos x + 3\sin x + 7 = -1(-\sin x + \cos x) + 2(\cos x + \sin x + 1) + 5$

$\therefore I = \int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1} dx = \int \frac{-1(-\sin x + \cos x) + 2(\cos x + \sin x + 1) + 5}{\cos x + \sin x + 1} dx$

$= -\int \frac{-\sin x + \cos x}{\cos x + \sin x + 1} dx + 2\int \frac{\cancel{\cos x + \sin x + 1}}{\cancel{\cos x + \sin x + 1}} dx + 5\int \frac{1}{\cos x + \sin x + 1} dx$

$= -\log|\cos x + \sin x + 1| + 2x + 5\int \frac{1}{\cos x + \sin x + 1} dx$ ......(II) $\left(\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c\right)$

Now, we find $I_1 = \int \frac{1}{\cos x + \sin x + 1} dx$

Put $\tan\frac{x}{2} = t \Rightarrow \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$

$\therefore I_1 = \int \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} + 1}\left(\frac{2dt}{1+t^2}\right) = \int \frac{1}{\frac{1-t^2+2t+1+t^2}{\cancel{1+t^2}}} \cdot \frac{2dt}{\cancel{1+t^2}}$

$= \int \frac{2dt}{2+2t} = \int \frac{\cancel{2}dt}{\cancel{2}(1+t)} = \int \frac{dt}{1+t} = \log|1+t| + c = \log\left|1+\tan\frac{x}{2}\right| + c$

From (II), $I = -\log|\cos x + \sin x + 1| + 2x + 5\log\left|1+\tan\frac{x}{2}\right| + c$

22. Obtain the reduction formula for $I_n = \int \csc^n x\,dx$, and hence find $\int \csc^5 x\,dx$

Sol: $I_n = \int \csc^n x\,dx = \int \csc^{n-2}x \csc^2 x\,dx$.

We take first function $u=\csc^{n-2}x$

Second function $v=\csc^2 x \Rightarrow \int v = -\cot x$

From Byparts Rule, we have $I_n = -\csc^{n-2}x \cot x$

$- \int (n-2)\csc^{n-3}x(-\csc x \cot x)(-\cot x)dx$

$= -\csc^{n-2}x \cot x \;\; -(n-2)\int \csc^{n-2}x \cot^2 x\,dx$

$= -\csc^{n-2}x \cot x \;\; -(n-2)\int \csc^{n-2}x (\csc^2 x - 1)dx$

$= -\csc^{n-2}x \cot x \;\; -(n-2)\int \csc^n x\,dx + (n-2)\int \csc^{n-2}x\,dx$

$I_n = -\csc^{n-2}x \cot x - (n-2)I_n + (n-2)I_{n-2}$

$\Rightarrow I_n + (n-2)I_n = \csc^{n-2}x \cot x + (n-2)I_{n-2}$

$\Rightarrow I_n(1+n-2) = -\csc^{n-2}x \cot x + (n-2)I_{n-2}$

$\Rightarrow I_n(n-1) = -\csc^{n-2}x \cot x + (n-2)I_{n-2}$

$$\Rightarrow I_n = -\frac{\csc^{n-2}x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Put n=5,3,1 successively in (1), we get

$$I_5 = \int \csc^5 x\,dx = \frac{-\csc^3 x \cot x}{4} + \frac{3}{4} I_3 = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4}\left[\frac{-\csc x + \cot x}{2} + \frac{1}{2} I_1\right]$$

$$= -\frac{\csc^3 x \cot x}{4} - \frac{3\csc x \cot x}{8} + \frac{3}{8}\int \csc x\,dx$$

$$= -\frac{\csc^3 x \cot x}{4} - \frac{3\csc x \cot x}{8} \quad + \frac{3}{8}\log|\csc x - \cot x| + c$$

$$\therefore I_5 = \frac{\csc^3 x . \cot x}{4} - \frac{3}{8}\csc x \cot x \,] \quad + \frac{3}{8}\log\left|\tan\frac{x}{2}\right| + c$$

23. Evaluate $\int_0^{\pi/4} \log(1+\tan x)dx$

Sol: We know $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore I = \int_0^{\pi/4} \log(1+\tan x)dx = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx = \int_0^{\pi/4} \log\left(\frac{1+\cancel{\tan x}+1-\cancel{\tan x}}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx \quad = \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]dx$$

$$= \log 2\int_0^{\pi/4} 1dx - \int_0^{\pi/4} \log(1+\tan x)dx \quad = \log 2[x]_0^{\pi/4} - I$$

$$\Rightarrow I + I = (\log 2)\left(\frac{\pi}{4}\right) \Rightarrow 2I = \left(\frac{\pi}{4}\right)(\log 2)$$

$$\Rightarrow I = \left(\frac{\pi}{8}\right)(\log 2)$$

24. Solve $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-xy}$

Sol: Given D.E is $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-xy}$ ....(1). This is a homogeneous D.E

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

From (1), $v+x\frac{dv}{dx}=\frac{v^2x^2-2x(vx)}{x^2-x(vx)}=\frac{x^2(v^2-2v)}{x^2(1-v)}=\frac{(v^2-2v)}{(1-v)}$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-2v}{1-v}-v=\frac{v^2-2v-v+v^2}{1-v}=\frac{2v^2-3v}{1-v}$$

$$\therefore x\frac{dv}{dx}=\frac{2v^2-3v}{1-v}\Rightarrow\frac{(1-v)dv}{2v^2-3v}=\frac{dx}{x}\Rightarrow\int\frac{(1-v)dv}{2v^2-3v}=\int\frac{dx}{x}\Rightarrow\int\frac{(1-v)dv}{2v^2-3v}=\log x-\log c........(2)$$

Now $\Rightarrow \int\frac{1-v}{v(2v-3)}dv=-\frac{1}{3}\int\frac{3v-3}{v(2v-3)}dv=-\frac{1}{3}\int\frac{(2v-3)+v}{v(2v-3)}dv$

$$=-\frac{1}{3}\int\left(\frac{1}{v}+\frac{1}{2v-3}\right)dv=-\frac{1}{3}\left[\log v+\frac{1}{2}\log(2v-3)\right]$$

From (2), $-\frac{1}{3}\left[\log v+\frac{1}{2}\log(2v-3)\right]=\log\frac{x}{c}\Rightarrow-\frac{1}{3}\log(v\sqrt{2v-3})=\log\frac{x}{c}$

$$\Rightarrow \log v\sqrt{2v-3}=-3\log\frac{x}{c}=\log\frac{c^3}{x^3}\Rightarrow v\sqrt{2v-3}=\frac{c^3}{x^3}\Rightarrow x^3v\sqrt{2v-3}=c^3\Rightarrow x^6v^2(2v-3)=c^6$$

$$\Rightarrow x^6\left(\frac{y^2}{x^2}\right)\left(2\frac{y}{x}-3\right)=c\Rightarrow x^6\left(\frac{y^2}{x^2}\right)\left(\frac{2y-3x}{x}\right)=c\Rightarrow x^3y^2(2y-3x)=c$$