BULLET

MODEL PAPER

A 'MULTI QUESTION PAPER' WITH 'BULLET ANSWERS'

SAQ & LAQ SECTIONS

SAQ SECTION-B

Q11. CIRCLES:

- **Find the length of the chord intercepted by the circle $x^2+y^2-8x-2y-8=0$ on the line $x+y+1=0$**

A: Given circle is $x^2+y^2+8x-2y-8=0$,

It's Centre C=(4,1), radius $r=\sqrt{16+1+8}=\sqrt{25}=5$

Perpendicular distance from the centre (4,1) to the line $x+y+1=0$ is $p=\frac{|4+1+1|}{\sqrt{1^2+1^2}}=\frac{6}{\sqrt{2}}=\frac{3\times 2}{\sqrt{2}}=3\sqrt{2}$

$\therefore$ Length of the chord $=2\sqrt{r^2-p^2}=2\sqrt{5^2-(3\sqrt{2})^2}$

$=2\sqrt{25-9(2)}=2\sqrt{25-18}=2\sqrt{7}$

- **Find the equation of tangent of $x^2+y^2-2x+4y=0$ at (3,–1). Also find the equation of other tangent parallel to it.**

A: **(i) To find the tangent at (3,–1)**

The equation of tangent at (3,–1) on $S\equiv x^2+y^2-2x+4y=0$ is $S_1=0$

$\Rightarrow x(3)+y(-1)-(x+3)+2(y-1)=0$

$\Rightarrow 3x-y-x-3+2y-2=0 \Rightarrow 2x+y-5=0$

(ii) To find the parallel tangent

Slope of the tangent $2x+y-5=0$ is $m=-2$.

Also $x^2+y^2-2x+4y=0 \Rightarrow g=-1;\ f=2$.

Radius $r=\sqrt{1^2+(-2)^2-0}=\sqrt{1+4-0}=\sqrt{5}$

Formula: Equation of tangents with slope m is

$y+f=m(x+g)\pm r\sqrt{1+m^2}$

$\Rightarrow (y+2)=-2(x-1)\pm\sqrt{5}\sqrt{1+4}$

$\Rightarrow (y+2)=-2(x-1)\pm 5 \Rightarrow 2x+y\pm 5=0$

$\therefore$ the equation of the parallel tangent is $2x+y+5=0$

- **Find the condition that the tangents drawn from the exterior point (0,0) to $S=x^2+y^2+2gx+2fy+c=0$ are perpendicular to each other.**

A: If θ is the angle between the pair of tangents from P(0,0) to S=0 then $\tan\frac{\theta}{2}=\frac{r}{\sqrt{S_{11}}}$

$\Rightarrow \tan\frac{90^\circ}{2}=\frac{\sqrt{g^2+f^2-c}}{\sqrt{0^2+0^2+2g(0)+2f(0)+c}}$

$\Rightarrow \tan 45^0=1=\frac{\sqrt{g^2+f^2-c}}{\sqrt{c}}$

$\Rightarrow 1^2=\frac{g^2+f^2-c}{c}$

$\Rightarrow c=g^2+f^2-c \Rightarrow g^2+f^2=2c$

- **Find the pole of the line $x+y+2=0$ w.r.t the circle $x^2+y^2-4x+6y-12=0$.**

A: Let $P(x_1,y_1)$ be the pole of $x+y+2=0$

Then the polar of $P(x_1,y_1)$ w.r.to $S=x^2+y^2-4x+6y-12=0$ is $S_1=0$

$\Rightarrow x_1x+y_1y-2(x_1+x)+3(y_1+y)-12=0$

$\Rightarrow (x_1-2)x+(y_1+3)y-2x_1+3y_1-12=0$

Comparing above with $x+y+2=0$, we have

$\frac{x_1-2}{1}=\frac{y_1+3}{1}=\frac{-2x_1+3y_1-12}{2}=k(\text{say})$

$\Rightarrow x_1-2=k;\ y_1+3=k$ and $-2x_1+3y_1-12=2k$

$\Rightarrow x_1=k+2,\ y_1=k-3$

$\therefore -2(k+2)+3(k-3)-12=2k$

$\Rightarrow -2k-4+3k-9-12=2k \Rightarrow k=-25$

$\therefore x_1=k+2=-25+2=-23$ and $y_1=k-3=-25-3=-28$

$\therefore$ Pole $P(x_1,y_1)=(-23,-28)$

- **Find the value of k if $x+y-5=0$ and $2x+ky-8=0$ are conjugate with respect to the circle $x^2+y^2-2x-2y-1=0$**

A: From the given lines, we get $l_1=1, m_1=1, n_1=-5$; $l_2=2, m_2=k, n_2=-8$;

The given cirlce is $x^2+y^2-2x-2y-1=0$

$\Rightarrow g=-1, f=-1, c=-1$

radius $r=\sqrt{(-1)^2+(-1)^2+1}=\sqrt{1+1+1}=\sqrt{3}$

Conjugate lines condition:

$r^2(l_1l_2+m_1m_2)=(l_1g+m_1f-n_1)(l_2g+m_2f-n_2)$

$\Rightarrow 3((1\times 2)+(1\times k))=(-1-1+5)(-2-k+8)$

$\Rightarrow 3(2+k)=(3)(6-k) \Rightarrow 2+k=6-k$

$\Rightarrow 2k=4 \Rightarrow k=2$

Q12: SYSTEM OF CIRCLES:

- **Find the radical centre of the circles $x^2+y^2-4x-6y+5=0, x^2+y^2-2x-4y-1=0, x^2+y^2-6x-2y=0$.**

A: The given circles are $S=x^2+y^2-4x-6y+5=0$, $S'=x^2+y^2-2x-4y-1=0$, $S''=x^2+y^2-6x-2y=0$

One radical axis is $S-S'=0$

$\Rightarrow (-4x+2x)+(-6y+4y)+(5+1)=0$

$\Rightarrow -2x-2y+6=0 \Rightarrow -2(x+y-3)=0 \Rightarrow x+y-3=0$ .... (1)

Another radical axis is $S-S''=0$

$\Rightarrow (-4x+6x)+(-6y+2y)+5=0 \Rightarrow 2x-4y+5=0$ .... (2)

Now, $(1)\times 2 \Rightarrow 2x+2y-6=0$ ....(3)

$(3)-(2) \Rightarrow 6y-11=0 \Rightarrow y=11/6$

From(1), $x=3-y=3-\frac{11}{6}=\frac{18-11}{6}=\frac{7}{6}$

$\therefore$ the radical centre is (7/6,11/6)

- **Find the equation and length of the common chord of the two circles $x^2+y^2+2x+2y+1=0$, $x^2+y^2+4x+3y+2=0$**

A: Given circles are $S\equiv x^2+y^2+2x+2y+1=0$ and $S'\equiv x^2+y^2+4x+3y+2=0$

Equation of the common chord is $S-S'=0$

$\Rightarrow -2x-y-1=0 \Rightarrow 2x+y+1=0$

For the circle $S\equiv x^2+y^2+2x+2y+1=0$

centre $C(-1,-1)$, radius $r=\sqrt{1^2+1^2-1}=\sqrt{1+1-1}=\sqrt{1}=1$

Length of the perpendicular from $C(-1,-1)$ to the line $2x+y+1=0$ is $p=\frac{|2(-1)-1+1|}{\sqrt{4+1}}=\frac{|-2|}{\sqrt{5}}=\frac{2}{\sqrt{5}}$

$\therefore$ Length of the common chord

$=2\sqrt{r^2-p^2}=2\sqrt{1-\frac{4}{5}}=2\sqrt{\frac{1}{5}}=\frac{2}{\sqrt{5}}$ units

- **If $x+y=3$ is the equation of the chord AB of the circle $x^2+y^2-2x+4y-8=0$, find the equation of the circle having $\overline{AB}$ as diameter.**

A: Given circle is $S\equiv x^2+y^2-2x+4y-8=0$ and the given line is $L\equiv x+y-3=0$

Equation of any circle passing through the points of intersection of $S=0$, $L=0$ is $S+\lambda L=0$

$\Rightarrow (x^2+y^2-2x+4y-8)+\lambda(x+y-3)=0$

$\Rightarrow x^2+y^2-(2-\lambda)x+(4+\lambda)y-(3\lambda+8)=0$....(1)

Centre of the above circle is $\left(\frac{2-\lambda}{2},-\frac{4+\lambda}{2}\right)$

Line $L=0$ becomes a diameter if the above centre lies on $L=x+y-3=0 \Rightarrow \left(\frac{2-\lambda}{2}\right)-\left(\frac{4+\lambda}{2}\right)-3=0$

$\Rightarrow 2-\lambda-4-\lambda-6=0 \Rightarrow 2\lambda=-8 \Rightarrow \lambda=-4$

From (1), required circle's equation is

$x^2+y^2-(2+4)x+(4-4)y-(-12+8)=0$

$\Rightarrow x^2+y^2-6x+4=0$

- **Find the equation of the circle passing through the points of intersection of the circles $x^2+y^2-8x-6y+21=0$, $x^2+y^2-2x-15=0$ and (1,2)**

A: Given circles are $S=x^2+y^2-8x-6y+21=0$, $S'=x^2+y^2-2x-15=0$

Radical axis of the circles is $L=S-S'=0$

$\Rightarrow -8x+2x-6y+21+15=0 \Rightarrow -6x-6y+36=0$

$\Rightarrow -6(x+y-6)=0 \Rightarrow x+y-6=0$

Equation of any circle passing through the points of intersection of $S'=0$, $L=0$ is $S'+\lambda L=0$

$\Rightarrow (x^2+y^2-2x-15)+\lambda(x+y-6)=0$ .....(1)

(1) passes through the point (1,2)

$\Rightarrow (1+4-2-15)+\lambda(1+2-6)=0 \Rightarrow -12+\lambda(-3)=0$

$\Rightarrow 3\lambda=-12 \Rightarrow \lambda=-4$

Put $\lambda=-4$ in (1) then $(x^2+y^2-2x-15)-4(x+y-6)=0$

$\Rightarrow x^2+y^2-2x-15-4x+4y+24=0$

$\Rightarrow x^2+y^2-6x-4y+9=0$

Q13 & 14: ELLIPSE:

- **Find the eccentricity, coordinates of foci, length of latus rectum and equations of directrices of the ellipse $9x^2+16y^2-36x+32y-92=0$.**

A: Given ellipse is $9x^2+16y^2-36x+32y-92=0$

$\Rightarrow (9x^2-36x)+(16y^2+32y)=92$

$\Rightarrow 9(x^2-4x+4)+16(y^2+2y+1)=92+36+16$

$\Rightarrow 9(x-2)^2+16(y+1)^2=144$

$\Rightarrow \frac{9(x-2)^2}{144}+\frac{16(y+1)^2}{144}=1 \Rightarrow \frac{(x-2)^2}{16}+\frac{(y-(-1))^2}{9}=1$,

Comparing with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we get

$a^2=16, b^2=9 \Rightarrow a=4, b=3 \Rightarrow a>b$.

Hence the ellipse is horizontal. Also $(h,k)=(2,-1)$

(i) $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\frac{\sqrt{7}}{4}$

(ii) Foci $=(h\pm ae,k)=(2\pm\frac{4\sqrt{7}}{4},-1)=(2\pm\sqrt{7},-1)$

(iii) Length of latus rectum $=\frac{2b^2}{a}=\frac{2(9)}{4}=\frac{9}{2}$

(iv) Equations of directrices

$x=h\pm\frac{a}{e}=2\pm\frac{4x4}{\sqrt{7}}=\frac{2\sqrt{7}\pm16}{\sqrt{7}} \Rightarrow \sqrt{7}x=(2\sqrt{7}\pm16)$

- **Find the equations of the tangents to $9x^2+16y^2=144$, which make equal intercepts on the coordinate axes.**

A: Given ellipse is $9x^2+16y^2=144 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$

$\Rightarrow a^2=16 \Rightarrow a=4$ and $b^2=9 \Rightarrow b=3$

The equation of any line which makes equal intercepts on the axes is taken as $x\pm y+k=0$

Its slope $m=\pm1$

Equation of the tangent with slope m is

$y=mx\pm\sqrt{a^2m^2+b^2}$

$\therefore$ Equations of required tangents are

$y=\pm x\pm\sqrt{16(1)+9}=\pm x\pm5 \Rightarrow y=\pm x\pm5$

- **Find the equation of the ellipse in the standard form whose distance between foci is 2 and the length of latus rectum is 15/2.**

A: Distance between foci $S(ae,0)$ and $S'(-ae,0)$ is 2

$\Rightarrow 2ae=2 \Rightarrow ae=1$ ....(1)

Length of latus rectum is 15/2

$\frac{2b^2}{a}=\frac{15}{2} \Rightarrow b^2=\frac{15}{4}a$ ......(2)

Now, $b^2=a^2(1-e^2)=a^2-a^2e^2=a^2-(ae)^2=a^2-1$ ......(3)

From (2) & (3), $\frac{15}{4}a=a^2-1 \Rightarrow 15a=4a^2-4$

$\Rightarrow 4a^2-15a-4=0 \Rightarrow (a-4)(4a+1)=0 \Rightarrow a=4$ (or) $-\frac{1}{4}$

If $a=4$ then $b^2=a^2-1=16-1=15$

$\therefore$ Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{15}=1$

- **Find the eccentricity, coordinates of foci, length of latus rectum and equations of directrices of the ellipse $9x^2+16y^2=144$**

A: Equation of ellipse is $9x^2+16y^2=144$

$\Rightarrow \frac{9x^2}{144}+\frac{16y^2}{144}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$.

Here, $a^2=16, b^2=9 \Rightarrow a>b$.

Hence the ellipse is horizontal

(i) Eccentricity $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}$

(ii) Foci $=(\pm ae,0)=\left(\pm 4\left(\frac{\sqrt{7}}{4}\right),0\right)=(\pm\sqrt{7},0)$

(iii) Length of latus rectum $=\frac{2b^2}{a}=\frac{2(9)}{4}=\frac{9}{2}$

(iv) Equation of directrices is $x=\pm\frac{a}{e}=\pm 4\left(\frac{4}{\sqrt{7}}\right)=\frac{\pm 16}{\sqrt{7}}$

$\Rightarrow \sqrt{7}x=\pm 16 \Rightarrow \sqrt{7}x\pm 16=0$

- **Find the equations of the tangents to the ellipse $2x^2+y^2=8$ which are (i) parallel to $x-2y-4=0$ (ii) perpendicular to $x+y+2=0$**

A: Given ellipse is $2x^2+y^2=8 \Rightarrow \frac{2x^2}{8}+\frac{y^2}{8}=1$

$\Rightarrow \frac{x^2}{4}+\frac{y^2}{8}=1 \Rightarrow a^2=4, b^2=8$

(i) Slope of the line $x-2y-4=0$ is $m=1/2$

$\therefore$ Tangent with slope m is $y=mx\pm\sqrt{a^2m^2+b^2}$

$\Rightarrow y=\frac{1}{2}x\pm\sqrt{4\times\frac{1}{4}+8}=\frac{1}{2}x\pm\sqrt{9}=\frac{1}{2}x\pm 3$

$\Rightarrow 2y=x\pm 6 \Rightarrow x-2y\pm 6=0$

(ii) Slope of the line $x+y+2=0$ is -1

$\Rightarrow$ Slope of its perpendicular is $m=1$

$\therefore$ Tangent with slope m=1 is $y=mx\pm\sqrt{a^2m^2+b^2}$

$\Rightarrow y=(1)x\pm\sqrt{4(1)^2+8}=x\pm\sqrt{12}=x\pm 2\sqrt{3}$

$\Rightarrow x-y\pm 2\sqrt{3}=0$

- **Show that the points of intersection of the perpendicular tangents to an ellipse lie on a circle.**

A: Let the equation of the ellipse be $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

Let $P(x_1,y_1)$ be a point on the locus

Tangent with slope m, is $y=mx\pm\sqrt{a^2m^2+b^2}$

If it pass through $P(x_1,y_1)$, then $y_1=mx_1\pm\sqrt{a^2m^2+b^2}$

$\Rightarrow y_1-mx_1=\pm\sqrt{a^2m^2+b^2} \Rightarrow (y_1-mx_1)^2=a^2m^2+b^2$

$\Rightarrow (y_1^2+m^2x_1^2-2x_1y_1m)-a^2m^2-b^2=0$

$\Rightarrow m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-b^2)=0$ ....(1)

(1) is a quadratic equation in m and its roots be taken as m_1,m_2 (Slopes of tangents)

If the tangents are perpendicular then $m_1m_2=-1$

From (1), product of roots $=\frac{y_1^2-b^2}{x_1^2-a^2}=-1$

$\Rightarrow y_1^2-b^2=-(x_1^2-a^2)=-x_1^2+a^2 \Rightarrow x_1^2+y_1^2=a^2+b^2$

$\therefore$ Locus of $P(x_1,y_1)$ is $x^2+y^2=a^2+b^2$.

Q15: HYPERBOLA:

- **Find the centre, eccentricity, foci, length of latus rectum and equations of the directrices of the Hyperbola $x^2-4y^2=4$**

A: Given hyperbola is $x^2-4y^2=4 \Rightarrow \frac{x^2}{4}-\frac{y^2}{1}=1$

Here $a^2=4, b^2=1$

(i) Centre C=(0,0)

(ii) Eccentricity $e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{4+1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$

(iii) Foci $=(\pm ae,0)=\left(\pm 2\left(\frac{\sqrt{5}}{2}\right),0\right)=(\pm\sqrt{5},0)$

(iv) Length of latusrectum $=\frac{2b^2}{a}=\frac{2(1)}{2}=1$

(v) Directrices: $x=\pm\frac{a}{e}\Rightarrow x=\pm\frac{2}{\sqrt{5}/2}\Rightarrow x=\pm\frac{4}{\sqrt{5}}$

- **Find the equations of the tangents to the hyperbola $3x^2-4y^2=12$ which are (a) Parallel (b) Perpendicular to the line $y=x-7$**

A: Given hyperbola is $3x^2-4y^2=12$

$\Rightarrow \frac{x^2}{4}-\frac{y^2}{3}=1 \Rightarrow a^2=4, b^2=3$

Slope of the given line $y=x-7$ is $m=1$

$\Rightarrow$ Slope of its perpendicular is -1

Tangent with slope m is $y=mx\pm\sqrt{a^2m^2-b^2}$

(i) Parallel tangent with slope m=1 is

$y=1.x\pm\sqrt{4(1)^2-3}=x\pm 1 \Rightarrow x-y\pm 1=0$

(ii) Perpendicular tangent with slope m=−1 is

$y=(-1)x\pm\sqrt{4(1)^2-3}=-x\pm 1 \Rightarrow x+y\pm 1=0$

- **P.T the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ lies on the circle $x^2+y^2=a^2-b^2$.**

A: Given hyperbola is $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$

Let $P(x_1,y_1)$ be a point on the locus.

Tangent with slope m, is $y=mx\pm\sqrt{a^2m^2-b^2}$

If it passes through $P(x_1,y_1)$ then

$y_1-mx_1=\pm\sqrt{a^2m^2-b^2} \Rightarrow (y_1-mx_1)^2=a^2m^2-b^2$

$\Rightarrow (y_1^2-2mx_1y_1+m^2x_1^2)-a^2m^2+b^2=0$

$\Rightarrow m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2+b^2)=0$ ....(1)

(1) is a quadratic equation in 'm' and its roots be taken as m_1, m_2. (Slopes of tangents)

If the tangents are perpendicular then $m_1m_2=-1$

From (1) product of roots $\frac{y_1^2+b^2}{x_1^2-a^2}=-1$

$\Rightarrow x_1^2+y_1^2=a^2-b^2$

The locus of $P(x_1,y_1)$ is $x^2+y^2=a^2-b^2$.

Q16 :DEFINITE INTEGRALS:

- **Evaluate** $\displaystyle\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$

A: We know $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx \quad .....(1)$$

$$= \int_0^{\pi/2} \frac{\sin^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} dx \quad ....(2)$$

Now adding (1) & (2) we get

$$I + I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx + \int_0^{\pi/2} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\cancel{\sin^5 x + \cos^5 x}}{\cancel{\sin^5 x + \cos^5 x}} dx$$

$$= \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

- **Evaluate** $\displaystyle\int_0^{\pi/2} \frac{dx}{4+5\cos x}$

A: Put $\tan\frac{x}{2} = t$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$.

Also, $x = 0 \Rightarrow t = 0,\ x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore I = \int_0^1 \frac{(2dt)/(1+t^2)}{4+5\left(\frac{1-t^2}{1+t^2}\right)} = \int_0^1 \frac{(2dt)/\cancel{(1+t^2)}}{\frac{4(1+t^2)+5(1-t^2)}{\cancel{1+t^2}}}$$

$$= 2\int_0^2 \frac{1}{9-t^2} dt = 2.\frac{1}{2.3} \log\left[\frac{3+t}{3-t}\right]_0^1 = \frac{1}{3} \log\frac{4}{2} = \frac{1}{3} \log 2$$

Q17:DIFFERENTIAL EQUATIONS:

- **Solve $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$**

A: The given D.E is in the variables and separable form

$\therefore (xy^2+x)dx+(yx^2+y)dy=0$

$\Rightarrow (xy^2+x)dx=-(yx^2+y)dy$

$\Rightarrow x(y^2+1)dx=-y(x^2+1)dy \Rightarrow \frac{x}{x^2+1}dx = -\frac{y}{y^2+1}dy$

$\Rightarrow \int \frac{2x}{x^2+1}dx = -\int \frac{2y}{y^2+1}dy$

$\Rightarrow \log(x^2+1) = -\log(y^2+1) + \log c$

$\Rightarrow \log(x^2+1) + \log(y^2+1) = \log c$

$\Rightarrow \log(x^2+1)(y^2+1) = \log c \Rightarrow (x^2+1)(y^2+1) = c$

$\therefore$ The solution is $(x^2+1)(y^2+1) = c$

- **Solve $\frac{dy}{dx} - x\tan(y-x) = 1$**

A: Put $y - x = t \Rightarrow \frac{dy}{dx} - 1 = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} + 1$

Hence, the given D.E becomes

$\left(\frac{dt}{dx}+1\right) - x\tan t = 1 \Rightarrow \frac{dt}{dx} = x\tan t \Rightarrow \frac{1}{\tan t}dt = xdx$

$\Rightarrow \int \cot t\,dt = \int x dx \Rightarrow \log(\sin t) = \frac{x^2}{2} + c$

$\Rightarrow 2\log\sin t = x^2 + c \Rightarrow 2\log\sin(y-x) = x^2 + c$

- **Solve $(1+x^2)\frac{dy}{dx} + y = e^{Tan^{-1}x}$**

A: Given D.E is $(1+x^2)\frac{dy}{dx} + y = e^{Tan^{-1}x}$

$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{1+x^2}\right) = \frac{e^{Tan^{-1}x}}{1+x^2}$

The above equation is in the form

$\frac{dy}{dx} + yP(x) = Q(x)$. This is a linear D.E in y.

Here $P = \frac{1}{1+x^2} \Rightarrow \int Pdx = \int \frac{1}{1+x^2}dx = Tan^{-1}x$

$\therefore IF = e^{\int Pdx} = e^{Tan^{-1}x}$.

Hence the solution is $y(I.F) = \int (I.F)Q\,dx$

$\Rightarrow ye^{Tan^{-1}x} = \int e^{Tan^{-1}x}\left(\frac{e^{Tan^{-1}x}}{1+x^2}\right)dx.$

Put $Tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$\therefore ye^t = \int e^t.e^t dt = \int e^{2t}dt = \frac{1}{2}e^{2t} + c$

$\Rightarrow ye^{Tan^{-1}x} = \frac{1}{2}e^{2Tan^{-1}x} + c$

LAQ SECTION-C

Q18 & 19: CIRCLES:

- **Find the equation of the circle passing through A(1,2), B(3,–4), C(5,–6)**

A: Given A=(1,2), B=(3,–4), C=(5,–6)

We take $S(x_1,y_1)$ as the centre of the circle

$\Rightarrow SA=SB=SC$

Now, $SA=SB \Rightarrow SA^2=SB^2$

$\Rightarrow (x_1-1)^2+(y_1-2)^2=(x_1-3)^2+(y_1+4)^2$

$\Rightarrow (\cancel{x_1^2}-2x_1+1)+(\cancel{y_1^2}-4y_1+4)$

$=(\cancel{x_1^2}-6x_1+9)+(\cancel{y_1^2}+8y_1+16)$

$\Rightarrow 4x_1-12y_1-20=0 \Rightarrow 4(x_1-3y_1-5)=0$

$\Rightarrow x_1-3y_1-5=0$ ....(1)

Also, $SA=SC \Rightarrow SA^2=SC^2$

$\Rightarrow (x_1-1)^2+(y_1-2)^2=(x_1-5)^2+(y_1+6)^2$

$\Rightarrow (\cancel{x_1^2}-2x_1+1)+(\cancel{y_1^2}-4y_1+4)$

$=(\cancel{x_1^2}-10x_1+25)+(\cancel{y_1^2}+12y_1+36)$

$\Rightarrow 8x_1-16y_1-56=0 \Rightarrow 8(x_1-2y_1-7)=0$

$\Rightarrow x_1-2y_1-7=0$ ....(2)

$(1)-(2) \Rightarrow -y_1+2=0 \Rightarrow y_1=2$

$(1) \Rightarrow x_1-6-5=0 \Rightarrow x_1=11$

$\therefore$ Centre of the circle $S(x_1,y_1)=(11,2)$.

Also, we have A=(1,2)

So, radius $r=SA \Rightarrow r^2=SA^2$

$\therefore r^2=(11-1)^2+(2-2)^2=10^2=100$

$\therefore$ Circle with centre (11, 2) and $r^2=100$ is

$(x-11)^2+(y-2)^2=100$

$\Rightarrow (x^2-22x+121)+(y^2-4y+4)=100$

$\Rightarrow x^2+y^2-22x-4y+25=0$

- **Show that the points (1, 1), (–6, 0), (–2, 2) and (–2, –8) are concyclic.**

A: Let A=(1,1), B=(–6,0), C=(–2,2), D=(–2,–8)

Take $S(x_1,y_1)$ as the centre of the circle

$\Rightarrow SA=SB=SC$

Now, $SA=SB \Rightarrow SA^2=SB^2$

$\Rightarrow (x_1-1)^2+(y_1-1)^2=(x_1+6)^2+(y_1-0)^2$

$\Rightarrow (\cancel{x_1^2}-2x_1+1)+(\cancel{y_1^2}-2y_1+1)=(\cancel{x_1^2}+12x_1+36)+\cancel{y_1^2}$

$\Rightarrow 14x_1+2y_1+34=0 \Rightarrow 2(7x_1+y_1+17)=0$

$\Rightarrow 7x_1+y_1+17=0$ ....(1)

Also, $SB=SC \Rightarrow SB^2=SC^2$

$\Rightarrow (x_1+6)^2+(y_1-0)^2=(x_1+2)^2+(y_1-2)^2$

$\Rightarrow (\cancel{x_1^2}+12x_1+36)+\cancel{y_1^2}=(\cancel{x_1^2}+4x_1+4)+(\cancel{y_1^2}-4y_1+4)$

$\Rightarrow 8x_1+4y_1+28=0 \Rightarrow 4(2x_1+y_1+7)=0$

$\Rightarrow 2x_1+y_1+7=0$ ......(2)

Solving (1) & (2) we get the centre $S(x_1,y_1)$

$(1)-(2) \Rightarrow 5x_1+10=0 \Rightarrow 5x_1=-10 \Rightarrow x_1=-2$

From (1), $7(-2)+y_1+17=0 \Rightarrow y_1+3=0 \Rightarrow y_1=-3$

$\therefore$ Centre of the circle is $S(x_1,y_1)=(-2,-3)$

Also, we have $A=(1,1) \Rightarrow r=SA \Rightarrow r^2=SA^2$

$\therefore r^2=(1+2)^2+(1+3)^2=9+16=25$

$\therefore$ Equation of the circle with centre (–2,–3) and $r^2=25$ is $(x+2)^2+(y+3)^2=25$

$\Rightarrow (x^2+4x+4)+(y^2+6y+9)=25$

$\Rightarrow x^2+y^2+4x+6y-12=0$

Now, put D(–2,–8) in the above equation, then we get $(-2)^2+(-8)^2+4(-2)+6(-8)-12$

$=4+64-8-48-12=68-68=0$

$\therefore$ D(–2,–8) lies on the circle

$\therefore$ the given 4 points are concyclic.

- **Find the equation of the circle passing through (4,1) (6,5) and having the centre on the line 4x+3y–24=0**

A: Let A=(4,1), B=(6,5).
Take $S(x_1,y_1)$ as the centre of the circle
$\Rightarrow SA=SB \Rightarrow SA^2=SB^2$.
$\Rightarrow (x_1-4)^2+(y_1-1)^2=(x_1-6)^2+(y_1-5)^2$
$\Rightarrow (x_1^2-8x_1+16)+(y_1^2-2y_1+1)$
$=(x_1^2-12x_1+36)+(y_1^2-10y_1+25)$
$\Rightarrow 17-8x_1-2y_1=61-12x_1-10y_1$
$\Rightarrow 17-8x_1-2y_1-61+12x_1+10y_1=0$
$\Rightarrow 4x_1+8y_1-44=0$..... (1)
But centre (x_1,y_1) lies on 4x+3y–24=0
$\Rightarrow 4x_1+3y_1-24=0$ .... (2)
$(2)-(1) \Rightarrow -5y_1+20=0 \Rightarrow 5y_1=20 \Rightarrow y_1=4$
From (2), $4x_1+3(4)-24=0 \Rightarrow 4x_1-12=0$
$\Rightarrow 4x_1=12 \Rightarrow x_1=3$
$\therefore$ Centre of the circle $S(x_1,y_1)=(3,4)$.
Also, we have A=(4,1)
So, radius $r=SA \Rightarrow r^2=SA^2$.

$\therefore r^2=(3-4)^2+(4-1)^2=1+9=10$
$\therefore$ circle with centre (3,4) and $r^2=10$ is
$(x-3)^2+(y-4)^2=10$
$\Rightarrow (x^2+9-6x)+(y^2+16-8y)=10 \Rightarrow x^2+y^2-6x-8y+15=0$

- **Show that the circles $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+18y+26=0$ touch each other. Find the point of contact and the equation of the common tangent at their point of contact.**

A: First circle is $S\equiv x^2+y^2-4x-6y-12=0$;
Centre $C_1=(2,3)$, radius $r_1=\sqrt{4+9+12}=\sqrt{25}=5$
Second circle is $S'\equiv x^2+y^2+2x-8y+13=0$
Centre $C_2=(-3,-9)$, radius $r_2=\sqrt{1+16-13}=2$

$C_1C_2=\sqrt{(2+3)^2+(3+9)^2}=\sqrt{25+144}=\sqrt{169}=13$
Also, $r_1+r_2=5+8=13=C_1C_2$.
$\therefore$ The circles touch each other externally.
Now, $r_1:r_2=5:8$
So, the Point of contact P divides the join of $C_1(2,3)$, $C_2(-3,-9)$ internally in the ratio 5 : 8

$$\therefore P=\left(\frac{5(-3)+8(2)}{5+8},\frac{5(-9)+8(3)}{5+8}\right)=\left(\frac{1}{13},\frac{-21}{13}\right)$$

The equation of the common tangent to the circles S=0, S'=0 at the point of contact is given by S–S'=0
$\Rightarrow (x^2+y^2-4x-6y-12)-(x^2+y^2+6x+18y+26)=0$
$\Rightarrow (-4x-6x)-6y-18y-12-26=0$
$\Rightarrow -10x-24y-38=0 \Rightarrow -2(5x+12y+19)=0$
$\Rightarrow 5x+12y+19=0$

- **Find the equation of direct common tangents to the circles $x^2+y^2+22x-4y-100=0$, $x^2+y^2-22x+4y+100=0$.**

A: For the circle $x^2+y^2+22x-4y-100=0$,
Centre $C_1=(-11,2)$,
Radius $r_1=\sqrt{121+4+100}=\sqrt{225}=15$
For the circle $x^2+y^2-22x+4y+100=0$,
Centre $C_2=(11,-2)$,
Radius $r_2=\sqrt{121+4-100}=\sqrt{25}=5$

Now, $C_1C_2=\sqrt{(-11-11)^2+(2+2)^2}$
$=\sqrt{484+16}=\sqrt{500}=10\sqrt{5}$
$r_1+r_2=15+5=20$. Hence $C_1C_2>r_1+r_2$
Also, $r_1:r_2=15:5=3:1$
$\therefore$ External centre of similitude E divides $\overline{C_1C_2}$ in the ratio 3:1 externally.

$$\Rightarrow E=\left(\frac{3(11)-(1)(-11)}{3-1},\frac{3(-2)-1(2)}{3-1}\right)=(22,-4)$$

Tangent through (22,–4) with slope 'm' is
$(y+4)=m(x-22) \Rightarrow mx-y-22m-4=0$....(1)
Perpendicular distance from $C_2(11,-2)$ to (1) is $r_2=5$

$$\Rightarrow \frac{|11m+2-22m-4|}{\sqrt{m^2+1}}=5 \Rightarrow \frac{|-11m-2|}{\sqrt{m^2+1}}=5$$

$\Rightarrow |11m+2|=5\sqrt{m^2+1} \Rightarrow (11m+2)^2=25(m^2+1)$
$\Rightarrow 121m^2+4+44m=25m^2+25$
$\Rightarrow 96m^2+44m-21=0$
$\Rightarrow 96m^2+72m-28m-21=0$
$\Rightarrow 24m(4m+3)-7(4m+3)=0$

$\Rightarrow (24m-7)(4m+3)=0 \Rightarrow m=\frac{7}{24},\frac{-3}{4}$

$\therefore$ Tangent through (22,–4) with slope 7/24 is

$(y+4)=\frac{7}{24}(x-22)$

$\Rightarrow 24(y+4)=7(x-22) \Rightarrow 7x-24y-250=0$
$\therefore$ Tangent through (22,–4) with slope –3/4 is

$(y+4)=\frac{-3}{4}(x-22) \Rightarrow 4(y+4)=-3(x-22)$

$\Rightarrow 4y+16=-3x+66 \Rightarrow 3x+4y-50=0$

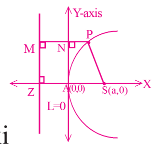
Q20:PARABOLA:

- **Derive the standard form of the parabola.**

A: Let S be the focus and L=0 be the directrix of the parabola.
Let Z be the projection of S on to the directrix
Let A be the mid point of SZ

$\Rightarrow SA=AZ \Rightarrow \frac{SA}{AZ}=1$

$\Rightarrow$ A is a point on the parabola.



Take AS, the principle axis asX-axi
Line perpendicular to AS through A as the Y-axis
$\Rightarrow A=(0,0)$
Let AS=a $\Rightarrow$ S=(a,0), Z=(–a,0)
$\Rightarrow$ the equation of the directrix is x=–a $\Rightarrow$ x+a=0
Let $P(x_1,y_1)$ be any point on the parabola.
N be the projection of P on to the Y-axis.
M be the projection of P on to the directrix.
Here $PM=PN+NM=x_1+a$
$(\because PN = x-\text{coordinate of } P \text{ and } NM = AZ = AS = a)$
Now, by the focus directrix property of the parabola,

we have $\frac{SP}{PM}=1 \Rightarrow SP=PM \Rightarrow SP^2=PM^2$

$\Rightarrow (x_1-a)^2+(y_1-0)^2=(x_1+a)^2$

$\Rightarrow y_1^2=(x_1+a)^2-(x_1-a)^2 \Rightarrow y_1^2=4ax_1$

$[\because (a+b)^2-(a-b)^2=4ab]$

$\therefore$ The equation of locus of $P(x_1,y_1)$ is $y^2=4ax$

- **Find the equation of the parabola passing through the points (–2,1), (1,2),(–1,3) and having its axis parallel to the x-axis.**

A: The equation of the parabola whose axis is parallel to the x-axis is $x=ly^2+my+n$
The parabola passes through (–2,1)
$\Rightarrow -2=l(1^2)+m(1)+n \Rightarrow l+m+n=-2$ ....(1)
The parabola passes through (1,2)
$\Rightarrow 1=l(2)^2+m(2)+n \Rightarrow 4l+2m+n=1$ ....(2)
The parabola passes through (–1,3)
$\Rightarrow -1=l(3)^2+m(3)+n \Rightarrow 9l+3m+n=-1$ ....(3)
(2)–(1) $\Rightarrow 3l+m=3$ .....(4)
(3)–(1) $\Rightarrow 8l+2m=1$ .....(5)
From 2×(4) $\Rightarrow 6l+2m=6$ ....(6)
(5)–(6) $\Rightarrow 2l=-5 \Rightarrow l=-5/2$

$(5) \Rightarrow 2m=1-8l=1+\cancel{8}\times\frac{5}{\cancel{2}}=1+20=21$

$\therefore 2m=21 \Rightarrow m=21/2$

$(1) \Rightarrow n=-2-l-m=-2+\frac{5}{2}-\frac{21}{2}=\frac{-4+5-21}{2}=\frac{-20}{2}=-10$

Put the values $l=-5/2$, m = 21/2 and n= –10 in $x=ly^2+my+n$ we get the required parabola as

$x=-\frac{5}{2}y^2+\frac{21}{2}y-10$

$\Rightarrow 2x=-5y^2+21y-20 \Rightarrow 5y^2+2x-21y+20=0$

- **Show that the equations of common tangents to the circle $x^2+y^2=2a^2$ and the parabola $y^2=8ax$ are $y=\pm(x+2a)$.**

A: Given parabola is $y^2=8ax \Rightarrow y^2=4(2a)x$ ....(1)

$\therefore$ Tangent to (1) with slope m is $y=mx+\frac{2a}{m}$ ....(2)

Comparing the above with y=mx+c we get $c=\frac{2a}{m}$

Given circle is $x^2+y^2=2a^2$.......(3)

Applying the tangential condition $c^2=r^2(1+m^2)$ between (2) and (3), we get

$\left(\frac{2a}{m}\right)^2=2a^2(1+m^2) \Rightarrow \frac{4\cancel{a^2}}{m^2}=2\cancel{a^2}(1+m^2) \Rightarrow \frac{2}{m^2}=(1+m^2)$

$\Rightarrow (1+m^2)m^2=2 \Rightarrow (1+m^2)m^2=(1+1^2)1^2 \Rightarrow m^2=1 \Rightarrow m=\pm1$

From (2), the equations of the required common tangents are $y=\pm x+\frac{2a}{\pm1} \Rightarrow y=\pm(x+2a)$

- **If y_1, y_2, y_3 are the y-coordinates of the vertices of the triangle inscribed in the parabola $y^2=4ax$ then show that the area of the triangle is $\frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$ sq. units.**

A: We take vertices as $P(x_1,y_1)=(at_1^2, 2at_1)$, $Q(x_2,y_2)=(at_2^2,2at_2)$, $R(x_3,y_3)=(at_3^2,2at_3)$

Area of ΔPQR

$=\frac{1}{2}\begin{vmatrix} x_1-x_2 & x_1-x_3 \\ y_1-y_2 & y_1-y_3 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} at_1^2-at_2^2 & at_1^2-at_3^2 \\ 2at_1-2at_2 & 2at_1-2at_3 \end{vmatrix}$

$=\frac{a.2a}{2}\begin{vmatrix} t_1^2-t_2^2 & t_1^2-t_3^2 \\ t_1-t_2 & t_1-t_3 \end{vmatrix}$

$=a^2\begin{vmatrix} (t_1-t_2)(t_1+t_2) & (t_1-t_3)(t_1+t_3) \\ t_1-t_2 & t_1-t_3 \end{vmatrix}$

$=a^2(t_1-t_2)(t_1-t_3)\begin{vmatrix} t_1+t_2 & t_1+t_3 \\ 1 & 1 \end{vmatrix}$

$=a^2(t_1-t_2)(t_1-t_3)|(\cancel{t_1}+t_2-\cancel{t_1}-t_3)|$

$=a^2|(t_1-t_2)(t_2-t_3)(t_3-t_1)|$

$=a^2\left|\left(\frac{y_1}{2a}-\frac{y_2}{2a}\right)\left(\frac{y_2}{2a}-\frac{y_3}{2a}\right)\left(\frac{y_3}{2a}-\frac{y_1}{2a}\right)\right|$

$\left(\because 2at_1=y_1 \Rightarrow t_1=\frac{y_1}{2a},......\right)$

$=\frac{a^2}{2a.2a.2a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$

$=\frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$ Sq.Units

Q21&22: INTEGRATION:

- **Evaluate the reduction formula for $I_n=\int \sin^n x dx$ and hence find $\int \sin^4 x dx$**

A: Given $I_n=\int \sin^n x dx=\int \sin^{n-1}x(\sin x)dx$.

We take First function $u=\sin^{n-1}x$ and

Second function $v=\sin x \Rightarrow \int v=-\cos x$

From By parts rule, we have

$$I_n=\sin^{n-1}x(-\cos x)-\int(n-1)\sin^{n-2}x\cos x(-\cos x)dx$$
$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\cos^2 x dx$$
$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x(1-\sin^2 x)dx$$
$$=-\sin^{n-1}x\cos x+(n-1)\left[\int \sin^{n-2}xdx-\int \sin^n xdx\right]$$
$$=-\sin^{n-1}x\cos x+(n-1)[I_{n-2}-I_n]$$
$$I_n=-\sin^{n-1}x\cos x+(n-1)I_{n-2}-(n-1)I_n$$
$$=-\sin^{n-1}x\cos x+(n-1)I_{n-2}-nI_n+I_n$$
$$\Rightarrow nI_n=-\sin^{n-1}x\cos x+(n-1)I_{n-2}+\cancel{I_n}-\cancel{I_n}$$
$$\Rightarrow I_n=\frac{-\sin^{n-1}x\cos x}{n}+\left(\frac{n-1}{n}\right)I_{n-2} \quad ....(1)$$

Put n=4, 2,0 successively in (1), we get

$$I_4=-\frac{\sin^3 x\cos x}{4}+\frac{3}{4}I_2$$
$$=-\frac{\sin^3 x\cos x}{4}+\frac{3}{4}\left[-\frac{\sin x\cos x}{2}+\frac{1}{2}I_0\right]$$
$$=\frac{-\sin^3 x\cos x}{4}-\frac{3}{8}\sin x\cos x+\frac{3}{8}I_0$$
$$=-\frac{\sin^3 x\cos x}{4}-\frac{3}{8}\sin x\cos x+\frac{3}{8}x \quad [\because I_0=x]$$

- **Evaluate the reduction formula for $I_n=\int \cos^n x dx$ and hence find $\int \cos^4 x dx$**

A: Given $I_n=\int \cos^n x dx=\int \cos^{n-1}x(\cos x)dx$.

We apply the "By parts rule"

We take First function $u=\cos^{n-1}x$ and

Second function $v=\cos x \Rightarrow \int v=\sin x$

From By parts rule, we have

$$I_n=\cos^{n-1}x(\sin x)-\int(n-1)\cos^{n-2}x(-\sin x)\sin x\,dx$$
$$=\cos^{n-1}x(\sin x)+(n-1)\int \cos^{n-2}x(\sin x)\sin x\,dx$$
$$=\cos^{n-1}x(\sin x)+(n-1)\int \cos^{n-2}x(\sin^2 x)\,dx$$
$$=\cos^{n-1}x(\sin x)+(n-1)\int \cos^{n-2}x(1-\cos^2 x)dx$$
$$=\cos^{n-1}x(\sin x)+(n-1)\int \cos^{n-2}xdx-(n-1)\int \cos^n x\,dx$$
$$=\cos^{n-1}x(\sin x)+(n-1)I_{n-2}-(n-1)I_n$$
$$\therefore I_n=\cos^{n-1}x(\sin x)+(n-1)I_{n-2}-nI_n+I_n$$
$$\Rightarrow nI_n=\cos^{n-1}x(\sin x)+(n-1)I_{n-2}+\cancel{I_n}-\cancel{I_n}$$
$$\therefore I_n=\frac{\cos^{n-1}x(\sin x)}{n}+\frac{(n-1)}{n}I_{n-2} ...(1)$$

Put n=4, 2,0 successively in (1), we get

$$I_4=\frac{\cos^3 x\sin x}{4}+\frac{3}{4}I_2$$
$$\Rightarrow \frac{\cos^3 x\sin x}{4}+\frac{3}{4}\left[\frac{\cos x\sin x}{2}+\frac{1}{2}I_0\right]$$
$$=\frac{\cos^3 x\sin x}{4}+\frac{3}{8}\cos x\sin x+\frac{3}{8}I_0$$
$$=\frac{\cos^3 x\sin x}{4}+\frac{3}{8}\cos x\sin x+\frac{3}{8}x+c$$

- **Evaluate $\int \tan^n x dx$, hence evaluate $\int \tan^5 x dx$**

A: Let $I_n=\int \tan^n xdx=\int(\tan^{n-2}x)\tan^2 xdx$

$$=\int(\tan^{n-2}x)(\sec^2 x-1)dx$$
$$=\int \tan^{n-2}x\sec^2 xdx-\int \tan^{n-2}xdx$$
$$=\frac{\tan^{n-1}}{n-1}-I_{n-2} .....(1) \quad [\because f(x)=\tan x, f'(x)=\sec^2 x]$$

Put n=5, 3,1 successively in (1), we get

$$I_5=\int \tan^5 xdx=\frac{\tan^4 x}{4}-I_3=\frac{\tan^4 x}{4}-\left(\frac{\tan^2 x}{2}-I_1\right)$$
$$=\frac{\tan^4 x}{4}-\frac{\tan^2 x}{2}+\int \tan xdx$$
$$=\frac{\tan^4 x}{4}-\frac{\tan^2 x}{2}+\log|\sec x|+c$$

- **Obtain the reduction formula for $I_n=\int \sec^n xdx$. Hence find $\int \sec^5 xdx$**

A: Given $I_n=\int \sec^n xdx=\int(\sec^{n-2}x)\sec^2 xdx$

We take first function $u=\sec^{n-2}x$

Second function $v=\sec^2 x \Rightarrow \int v=\tan x$

From Byparts rule, we have

$$I_n=(\sec^{n-2}x)\tan x-\int(n-2)(\sec^{n-3}x)\sec x\tan x(\tan x)dx$$
$$=(\sec^{n-2}x)(\tan x)-(n-2)\int(\sec^{n-2}x)\tan^2 xdx$$
$$=(\sec^{n-2}x)(\tan x)-(n-2)\int(\sec^{n-2}x)(\sec^2 x-1)dx$$
$$=(\tan x)(\sec^{n-2}x)-(n-2)\int(\sec^n x-\sec^{n-2}x)dx$$
$$=(\sec^{n-2}x)(\tan x)-(n-2)\int \sec^n xdx+(n-2)\int \sec^{n-2}xdx$$
$$\therefore I_n=(\sec^{n-2}x)(\tan x)-(n-2)I_n+(n-2)I_{n-2}$$
$$\Rightarrow I_n+(n-2)I_n=(\sec^{n-2}x)(\tan x)+(n-2)I_{n-2}$$
$$\Rightarrow I_n[1+(n-2)]=\sec^{n-2}x(\tan x)+(n-2)I_{n-2}$$
$$\Rightarrow I_n[(n-1)]=\sec^{n-2}x(\tan x)+(n-2)I_{n-2}$$
$$I_n=\frac{(\sec^{n-2}x)\tan x}{n-1}+\frac{(n-2)}{n-1}I_{n-2} ....(1)$$

Put n=5,3,1 successively in (1), we get

$$I_5=\int \sec^5 xdx=\frac{\sec^3 x\tan x}{4}+\frac{3}{4}I_3$$
$$=\frac{\sec^3 x\tan x}{4}+\frac{3}{4}\left(\frac{\sec x.\tan x}{2}\right)+\frac{3}{8}I_1$$
$$=\frac{\sec^3 x\tan x}{4}+\frac{3}{4}\left(\frac{\sec x.\tan x}{2}\right)+\frac{3}{8}\int \sec x\,dx$$
$$=\frac{\sec^3 x\tan x}{4}+\frac{3}{8}\sec x\tan x+\frac{3}{8}\log|\sec x+\tan x|+c$$

- **Evaluate $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$**

A: Let $(2\cos x + 3\sin x)$

$$= A\frac{d}{dx}(4\cos x + 5\sin x) + B(4\cos x + 5\sin x)$$

$$\Rightarrow (2\cos x + 3\sin x)$$

$$= A(-4\sin x + 5\cos x) + B(4\cos x + 5\sin x)$$

Equating the coefficients of sinx, we get

$-4A + 5B = 3$ ....(1)

Equating the coefficients of cosx, we get

$5A + 4B = 2$ ....(2)

$(1)\times 5 \Rightarrow -20A + 25B = 15$ .....(3)

$(2)\times 4 \Rightarrow 20A + 16B = 8$ ....(4)

$$(3)+(4) \Rightarrow 41B = 23 \Rightarrow B = \frac{23}{41}$$

$$\text{From (1), } 4A = 5B - 3 = 5\left(\frac{23}{41}\right) - 3 = \frac{115-123}{41} = \frac{-8}{41}$$

$$\therefore \cancel{4}A = \frac{-\cancel{8}}{41} \Rightarrow A = \frac{-2}{41}$$

$$\therefore\ I = \int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$$

$$= \int \frac{\frac{-2}{41}\frac{d}{dx}(4\cos x + 5\sin x) + \frac{23}{41}(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$= \frac{-2}{41}\int \frac{\frac{d}{dx}(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx + \frac{23}{41}\int 1\ dx$$

$$= \frac{-2}{41}\log|4\cos x + 5\sin x| + \frac{23}{41}x + c$$

$$\left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c\right]$$

- **Evaluate $\int \frac{2x+5}{\sqrt{x^2 - 2x + 10}} dx.$**

A: Let $2x + 5 = A\frac{d}{dx}(x^2 - 2x + 10) + B$

$$\Rightarrow 2x + 5 = A(2x - 2) + B = 2Ax - 2A + B$$

Equating the coefficient of x, we get $2A = 2 \Rightarrow A = 1$

Equating the constant terms, we get

$$-2A + B = 5 \Rightarrow B = 5 + 2A = 5 + 2(1) = 7 \Rightarrow B = 7$$

$$\therefore \int \frac{2x+5}{\sqrt{x^2 - 2x + 10}} dx$$

$$= \int \frac{\frac{d}{dx}(x^2 - 2x + 10)}{\sqrt{x^2 - 2x + 10}} dx + 7\int \frac{dx}{\sqrt{x^2 - 2x + 10}}$$

$$= 2\sqrt{x^2 - 2x + 10} + 7\int \frac{dx}{\sqrt{(x-1)^2 + 3^2}}$$

$$= 2\sqrt{x^2 - 2x + 10} + 7\text{Sinh}^{-1}\left(\frac{x-1}{3}\right) + c$$

Q23: DEFINITE INTEGRALS:

- **Evaluate $\int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x} dx$**

A: We know $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x} dx - I$$

$$\Rightarrow I + I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x} dx \Rightarrow 2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $\cos x = t \Rightarrow \sin x dx = -dt$

Also, $x=0 \Rightarrow t=\cos 0=1$ and $x=p \Rightarrow t=\cos\pi=-1$

$$\therefore I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}[\text{Tan}^{-1}t]_{-1}^{1}$$

$$= \frac{\pi}{2}[\text{Tan}^{-1}(1) - \text{Tan}^{-1}(-1)] = \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{2}.\frac{\pi}{2} = \frac{\pi^2}{4}$$

- **Evaluate $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x} dx$**

A: We know $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore I = \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin^3(\pi - x)}{1+\cos^2(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin^3 x}{1+\cos^2 x} dx = \pi\int_0^{\pi} \frac{\sin^3 x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x} dx$$

$$= \pi\int_0^{\pi} \frac{\sin^3 x}{1+\cos^2 x} dx - I \Rightarrow I + I = 2I = \pi\int_0^{\pi} \frac{\sin^3 x}{1+\cos^2 x} dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin^3 x dx}{1+\cos^2 x} = \frac{\pi}{2}\int_0^{\pi} \frac{\sin^2 x \sin x dx}{1+\cos^2 x} = \frac{\pi}{2}\int_0^{\pi} \frac{(1-\cos^2 x)\sin x dx}{1+\cos^2 x}$$

Put $\cos x = t \Rightarrow -\sin x dx = dt$;

$x=0 \Rightarrow t=\cos 0=1$ and $x=\pi \Rightarrow t=\cos\pi=-1$

$$\therefore I = \frac{\pi}{2}\int_1^{-1} -\frac{(1-t^2)}{1+t^2} dt = \frac{\pi}{2}\int_1^{-1} \frac{t^2 - 1}{1+t^2} dt = \frac{\pi}{2}\int_1^{-1} \frac{t^2 + 1 - 2}{t^2 + 1} dt$$

$$= \frac{\pi}{2}\int_1^{-1}\left(1 - \frac{2}{1+t^2}\right)dt = \frac{\pi}{2}[t - 2\text{Tan}^{-1}t]_1^{-1}$$

$$= \frac{\pi}{2}[-1 - 2\text{Tan}^{-1}(-1)] - [1 - 2\text{Tan}^{-1}1]$$

$$= \frac{\pi}{2}[-1 - 2\left(-\frac{\pi}{4}\right) - 1 + 2\left(\frac{\pi}{4}\right)] = \frac{\pi}{2}\left[\frac{\pi}{2} + \frac{\pi}{2} - 2\right] = \frac{\pi}{2}[\pi - 2]$$

- **Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$ (or) $\int_0^{\pi/4}\log(1+\tan x)dx$**

A: Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

Also, $x=0\Rightarrow\theta=0; x=1\Rightarrow\theta=\frac{\pi}{4}$

And $1+x^2=1+\tan^2\theta=\sec^2\theta$

$$\therefore \int_0^1\frac{\log(1+x)}{1+x^2}dx=\int_0^{\pi/4}\frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$=\int_0^{\pi/4}\log(1+\tan\theta)d\theta$$

$$\therefore I=\int_0^{\pi/4}\log[1+\tan\theta]d\theta=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta=\int_0^{\pi/4}\log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[\frac{(1+\tan\theta)+(1-\tan\theta)}{1+\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta=\int_0^{\pi/4}[\log 2-\log(1+\tan\theta)]\,d\theta$$

$$=\log 2\int_0^{\pi/4}1\,d\theta-\int_0^{\pi/4}\log(1+\tan\theta)d\theta=\log 2\int_0^{\pi/4}1\,d\theta-I$$

$$=\log 2[\theta]_0^{\pi/4}-I\Rightarrow I+I=(\log 2)\left(\frac{\pi}{4}\right)\Rightarrow 2I=\left(\frac{\pi}{4}\right)(\log 2)$$

$$\Rightarrow I=\left(\frac{\pi}{8}\right)(\log 2)$$

- **Evaluate $\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$**

A: Here, we take the substitution sinx–cosx=t.
Then (cosx+sinx)dx=dt.
Now x=0 ⇒ t=sin0–cos0 = 0–1 = –1 and

$$x=\frac{\pi}{4}\Rightarrow t=\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$$

Also, $(\sin x-\cos x)^2=t^2\Rightarrow \sin^2x+\cos^2x-2\sin x\cos x=t^2\Rightarrow 1-\sin 2x=t^2\Rightarrow \sin 2x=1-t^2$

$\therefore 9+16\sin 2x=9+16(1-t^2)=9+16-16t^2=25-16t^2$

$$\therefore I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx=\int_{-1}^0\frac{dt}{25-16t^2}$$

$$=\int_{-1}^0\frac{dt}{5^2-(4t)^2}=\frac{1}{4}\cdot\frac{1}{2(5)}\cdot\log\left[\frac{5+4t}{5-4t}\right]_{-1}^0$$

$$=\frac{1}{40}\left[\log\left[\frac{5+0}{5-0}\right]-\log\left[\frac{5-4}{5+4}\right]\right]=\frac{1}{40}\left[\log 1-\log\frac{1}{9}\right]$$

$$=\frac{1}{40}\left[0-\log 9^{-1}\right]=\frac{1}{40}[\log 9]=\frac{\log 3^2}{40}=\frac{2\log 3}{40}=\frac{\log 3}{20}$$

Q24: DIFFERENTIAL EQUATIONS:

- **Solve $(x^2+y^2)dx=2xydy$**

A: Given D.E is $(x^2+y^2)dx=2xydy$

$$\Rightarrow\frac{dy}{dx}=\frac{x^2+y^2}{2xy}\ .....(1)$$

(1) is a **homogeneous D.E**

Put $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore (1)\Rightarrow v+x\frac{dv}{dx}=\frac{x^2+(vx)^2}{2x(vx)}$$

$$=\frac{x^2+v^2x^2}{2x^2v}=\frac{x^2(1+v^2)}{2x^2v}=\frac{1+v^2}{2v}$$

$$v+x\frac{dv}{dx}=\frac{1+v^2}{2v}\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{2v}-v$$

$$=\frac{1+v^2-2v^2}{2v}=\frac{1-v^2}{2v}\Rightarrow x\frac{dv}{dx}=\frac{1-v^2}{2v}$$

$$\therefore\frac{2v}{1-v^2}dv=\frac{dx}{x}\Rightarrow\int\frac{2v}{1-v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow-\int\frac{-2v}{1-v^2}dv=\int\frac{dx}{x}\Rightarrow-\log(1-v^2)=\log x+\log c$$

$$\Rightarrow\log x+\log(1-v^2)=\log c$$

$$\Rightarrow\log(x(1-v^2))=\log c\Rightarrow x(1-v^2)=c$$

$$\Rightarrow x\left(1-\frac{y^2}{x^2}\right)=c;\quad\left(\because v=\frac{y}{x}\right)$$

$$\Rightarrow x\left(\frac{x^2-y^2}{x^2}\right)=c\Rightarrow x^2-y^2=cx.$$

∴ The solution is $x^2-y^2=cx$

- **Find the equation of a curve whose gradient is $\frac{dy}{dx}=\frac{y}{x}-\cos^2\left(\frac{y}{x}\right)$, which passes through $\left(1,\frac{\pi}{4}\right)$.**

A: Given D.E is $\frac{dy}{dx}=\frac{y}{x}-\cos^2\left(\frac{y}{x}\right)$ ....(1)

(1) is a homogeneous D.E.

Put y=vx then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$(1)\Rightarrow v+x\frac{dv}{dx}=\frac{vx-x\cos^2v}{x}=v-\cos^2v$$

$$\Rightarrow x\frac{dv}{dx}=-\cos^2v\Rightarrow\sec^2v\,dv=-\frac{dx}{x}$$

$$\Rightarrow\int\sec^2v\,dv=-\int\frac{dx}{x}\Rightarrow\tan v=-\log|x|+c$$

$$\Rightarrow\tan\frac{y}{x}+\log|x|=c\ ....(2)$$

If this curve passes through the point $(x,y)=\left(1,\frac{\pi}{4}\right)$

then $\tan\frac{\pi}{4}+\log 1=c\Rightarrow 1+0=c\Rightarrow c=1$

∴ From (2), required solution is $\tan\frac{y}{x}+\log|x|=1.$

- **Solve $(x^2y-2xy^2)dx=(x^3-3x^2y)dy$**

A: Given D.E is $(x^2y-2xy^2)dx=(x^3-3x^2y)dy$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2y-2xy^2}{x^3-3x^2y} \text{ ....(1)}$$

(1) is a homogeneous D.E

Put $y=vx \Rightarrow \frac{dy}{dx}=v+\frac{xdv}{dx}$

$$\therefore (1) \Rightarrow v+\frac{xdv}{dx}=\frac{x^2(v.x)-2.x.(v^2x^2)}{x^3-3x^2vx}$$

$$=\frac{vx^3-2v^2x^3}{x^3-3vx^3}=\frac{x^3(v-2v^2)}{x^3(1-3v)}=\frac{v-2v^2}{1-3v}$$

$$\therefore v+\frac{xdv}{dx}=\frac{v-2v^2}{1-3v} \Rightarrow x\frac{dv}{dx}=\frac{v-2v^2}{1-3v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-2v^2-v+3v^2}{1-3v}=\frac{v^2}{1-3v}$$

$$\Rightarrow \frac{dx}{x}=\frac{1-3v}{v^2}dv \Rightarrow \int\frac{dx}{x}=\int\left(\frac{1-3v}{v^2}\right)dv$$

$$\Rightarrow \int\frac{dx}{x}=\int\frac{1}{v^2}dv-\int\frac{3v}{v^2}dv \Rightarrow \log x+\log c=\frac{-1}{v}-3\log v$$

$$\Rightarrow \log x+\log c+3\log v=\frac{-1}{v} \Rightarrow \log xv^3c=\frac{-1}{v}$$

$$\Rightarrow \log x\left(\frac{y^3}{x^3}\right)c=\frac{-1}{y/x} \Rightarrow \log\left(\frac{y^3}{x^2}\right)c=\frac{-x}{y}$$

$$\Rightarrow \frac{y^3}{x^2}c=e^{-x/y} \Rightarrow c=\frac{x^2}{y^3}e^{-x/y}$$

- **Solve $\frac{dy}{dx}=\frac{x-y+3}{2x-2y+5}$**

A: Given that

$$\frac{dy}{dx}=\frac{x-y+3}{2x-2y+5}=\frac{(x-y)+3}{2(x-y)+5} \text{ .....(1)}$$

Comparing $\frac{x-y+3}{2x-2y+5}$ with

$\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$,we get

$$\frac{a_1}{a_2}=\frac{1}{2};\frac{b_1}{b_2}=\frac{-1}{-2}=\frac{1}{2} \Rightarrow \boxed{\frac{a_1}{a_2}=\frac{b_1}{b_2}}$$

So, we take the substitution

$$x-y=v \Rightarrow 1-\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=1-\frac{dv}{dx}$$

$$\therefore (1) \Rightarrow 1-\frac{dv}{dx}=\frac{v+3}{2v+5}$$

$$\Rightarrow \frac{dv}{dx}=1-\frac{v+3}{2v+5}=\frac{2v+5-v-3}{2v+5}=\frac{v+2}{2v+5}$$

$$\Rightarrow \frac{dv}{dx}=\frac{v+2}{2v+5}$$

$$\Rightarrow \frac{dx}{dv}=\frac{2v+5}{v+2}$$

$$\Rightarrow dx=\left(\frac{2v+5}{v+2}\right)dv=\left(\frac{2v+4+1}{v+2}\right)dv$$

$$=\left(\frac{2(v+2)+1}{v+2}\right)dv=\left(2+\frac{1}{v+2}\right)dv$$

Integrating on both sides, we get

$$\int dx=\int 2dv+\int\frac{dv}{v+2}$$

$$\Rightarrow x=2v+\log(v+2)+c$$

$$\Rightarrow x=2(x-y)+\log[(x-y)+2]+c$$

$$\Rightarrow x-2y+\log(x-y+2)=c$$