BULLET MODEL PAPER

A 'MULTI QUESTION PAPER' WITH 'BULLET ANSWERS'

SAQ & LAQ SECTIONS

SAQ SECTION-B

Q11. MATRICES

• **If A is a non-singular matrix then P.T $A^{-1}=\dfrac{\text{Adj A}}{\det A}$**

A: We take $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

We take cofactors of a_1,b_1,c_1, ...as A_1,B_1,C_1 ......

$$\therefore \text{Adj } A=\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}^T \Rightarrow \text{AdjA}=\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A.(\text{AdjA})=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$=\begin{bmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{bmatrix}$$

$$=\begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}=(\det A)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=(\det A).I$$

$\therefore A(\text{Adj } A)=(\det A)\, I$;

Similarly, we can prove that $(\text{Adj } A)A=(\det A)\, I$

$\therefore A\left(\dfrac{\text{Adj } A}{\det A}\right)=I$ ($\because \det A\neq 0$, as A is non singular)

$\therefore A^{-1}=\dfrac{\text{Adj } A}{\det A}$ [$\because AB=I \Rightarrow A^{-1}=B$]

• **If $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then show that $A^2-4A-5I=O$.**

A: $A^2=A.A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A=4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}=\begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I=5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\Rightarrow A^2-4A-5I=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}-\begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}-\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}=O$$

• **S.T. $A=\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ is non-singular and find A^{-1}.**

A: $\det A=1(4-3)-2(6-3)+1(3-2)=1-6+1=-4\neq 0$

$\therefore$ A is non-singular

$$\text{Adj } A=\begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T=\begin{bmatrix} 1 & -3 & 1 \\ -3 & 1 & 1 \\ 4 & 0 & -4 \end{bmatrix}^T$$

$$\therefore \text{Adj } A=\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj } A)=\frac{1}{-4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}=\begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$$

• **If $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then S.T $(aI+bE)^3=a^3I+3a^2bE$**

A: $aI+bE=a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}+\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$$\text{L.H.S}=(aI+bE)^3=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^3=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$=\begin{bmatrix} a^2+0 & ab+ab \\ 0+0 & 0+a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$=\begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}=\begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix} \quad ....(1)$$

$$\text{R.H.S}=a^3I+3a^2bE=a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix}+\begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix} ......(2)$$

$\therefore$ from (1), (2) L.H.S= R.H.S. Hence, proved.

• **If $\theta-\phi=\pi/2$, then show that**

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}=O$$

A: Given that $\theta-\phi=\pi/2 \Rightarrow \theta=(\pi/2)+\phi$

$\cos\theta=\cos\left(\dfrac{\pi}{2}+\phi\right)=-\sin\phi$; $\sin\theta=\sin\left(\dfrac{\pi}{2}+\phi\right)=\cos\phi$

$$\therefore \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$=\begin{bmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$=\begin{bmatrix} \sin^2\phi\cos^2\phi-\sin^2\phi\cos^2\phi & \sin^3\phi\cos\phi-\sin^3\phi\cos\phi \\ -\sin\phi\cos^3\phi+\sin\phi\cos^3\phi & -\sin^2\phi\cos^2\phi+\sin^2\phi\cos^2\phi \end{bmatrix}=O$$

Q12. ADDITION OF VECTORS

- **Show that the four points $-\bar{a}+4\bar{b}-3\bar{c}, 3\bar{a}+2\bar{b}-5\bar{c}, -3\bar{a}+8\bar{b}-5\bar{c}, -3\bar{a}+2\bar{b}+\bar{c}$ are coplanar.**

A: We take $\overline{OP}=-\bar{a}+4\bar{b}-3\bar{c}, \overline{OQ}=3\bar{a}+2\bar{b}-5\bar{c}$,

$\overline{OR}=-3\bar{a}+8\bar{b}-5\bar{c}, \overline{OS}=-3\bar{a}+2\bar{b}+\bar{c}$

$\overline{PQ}=\overline{OQ}-\overline{OP}=(3\bar{a}+2\bar{b}-5\bar{c})-(-\bar{a}+4\bar{b}-3\bar{c})=4\bar{a}-2\bar{b}-2\bar{c}$

$\overline{PR}=\overline{OR}-\overline{OP}=(-3\bar{a}+8\bar{b}-5\bar{c})-(-\bar{a}+4\bar{b}-3\bar{c})=-2\bar{a}+4\bar{b}-2\bar{c}$

$\overline{PS}=\overline{OS}-\overline{OP}=(-3\bar{a}+2\bar{b}+\bar{c})-(-\bar{a}+4\bar{b}-3\bar{c})=-2\bar{a}-2\bar{b}+4\bar{c}$

$$[\overline{PQ}\ \overline{PR}\ \overline{PS}]=\begin{vmatrix}4&-2&-2\\-2&4&-2\\-2&-2&4\end{vmatrix}[\bar{a}\ \bar{b}\ \bar{c}]$$

$=[4(16-4)+2(-8-4)-2(4+8)][\bar{a}\ \bar{b}\ \bar{c}]$

$=[4(12)+2(-12)-2(12)][\bar{a}\ \bar{b}\ \bar{c}]$

$=[48-24-24][\bar{a}\ \bar{b}\ \bar{c}]=0[\bar{a}\ \bar{b}\ \bar{c}]=0$

So, $\overline{PQ}, \overline{PR}, \overline{PS}$ are coplanar. Hence P,Q,R,S are coplanar.

- **If the points whose position vectors are $3\bar{i}-2\bar{j}-\bar{k}, 2\bar{i}+3\bar{j}-4\bar{k}, -\bar{i}+\bar{j}+2\bar{k}, 4\bar{i}+5\bar{j}+\lambda\bar{k}$ are coplanar, then show that $\lambda=-146/17$**

A: We take $\overline{OP}=3\bar{i}-2\bar{j}-\bar{k}, \overline{OQ}=2\bar{i}+3\bar{j}-4\bar{k}$,

$\overline{OR}=-\bar{i}+\bar{j}+2\bar{k}, \overline{OS}=4\bar{i}+5\bar{j}+\lambda\bar{k}$

$\overline{PQ}=\overline{OQ}-\overline{OP}=(2\bar{i}+3\bar{j}-4\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=-\bar{i}+5\bar{j}-3\bar{k}$

$\overline{PR}=\overline{OR}-\overline{OP}=(-\bar{i}+\bar{j}+2\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=-4\bar{i}+3\bar{j}+3\bar{k}$

$\overline{PS}=\overline{OS}-\overline{OP}=(4\bar{i}+5\bar{j}+\lambda\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=\bar{i}+7\bar{j}+(\lambda+1)\bar{k}$

But $[\overline{PQ}\ \overline{PR}\ \overline{PS}]=0$ [Since P,Q,R,S are coplanar]

$$\Rightarrow\begin{vmatrix}-1&5&-3\\-4&3&3\\1&7&\lambda+1\end{vmatrix}=0$$

$\Rightarrow(-1)[3(\lambda+1)-21]-5[-4(\lambda+1)-3]-3[(-28)-3]=0$

$\Rightarrow-1(3\lambda-18)-5(-4\lambda-7)-3(-31)=0$

$\Rightarrow-3\lambda+18+20\lambda+35+93=0 \Rightarrow -3\lambda+20\lambda+35+93+18=0$

$\Rightarrow 17\lambda+146=0 \Rightarrow 17\lambda=-146 \Rightarrow \lambda=-146/17$

- **If ABCDEF is a regular hexagon with centre O, then, P.T $\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=3\overline{AD}=6\overline{AO}$**

A: Given ABCDEF is a regular hexagon with centre 'O'.

$\therefore(\overline{AB}+\overline{AC})+(\overline{AD})+(\overline{AE}+\overline{AF})$

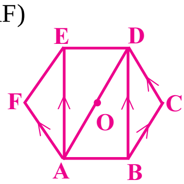
$=(\overline{AB}+\overline{AC})+(\overline{AD})+(\overline{BD}+\overline{CD})$

$[\because \overline{AE}=\overline{BD}, \overline{AF}=\overline{CD}]$

$=(\overline{AB}+\overline{BD})+\overline{AD}+(\overline{AC}+\overline{CD})$

$=(\overline{AD})+(\overline{AD})+\overline{AD}=3\overline{AD}$

$=3(2\overline{AO})\ [\because \overline{AD}=2\overline{AO}]=6\overline{AO}$



Q13. PRODUCT OF VECTORS

- **Find the area of the triangle formed with the points A(1,2,3), B(2,3,1), C(3,1,2).**

A: We take $\overline{OA}=\bar{i}+2\bar{j}+3\bar{k}, \overline{OB}=2\bar{i}+3\bar{j}+\bar{k}, \overline{OC}=3\bar{i}+\bar{j}+2\bar{k}$,

$\overline{AB}=\overline{OB}-\overline{OA}=(2\bar{i}+3\bar{j}+\bar{k})-(\bar{i}+2\bar{j}+3\bar{k})=\bar{i}+\bar{j}-2\bar{k}$

$\overline{AC}=\overline{OC}-\overline{OA}=(3\bar{i}+\bar{j}+2\bar{k})-(\bar{i}+2\bar{j}+3\bar{k})=2\bar{i}-\bar{j}-\bar{k}$

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\1&1&-2\\2&-1&-1\end{vmatrix}=\bar{i}(-1-2)-\bar{j}(-1+4)+\bar{k}(-1-2)$$

$=-3\bar{i}-3\bar{j}-3\bar{k}$

$|\overline{AB}\times\overline{AC}|=\sqrt{(-3)^2+(-3)^2+(-3)^2}$

$=\sqrt{9+9+9}=\sqrt{27}=\sqrt{9\times3}=3\sqrt{3}$

Area of $\Delta ABC=\frac{1}{2}|\overline{AB}\times\overline{AC}|=\frac{3\sqrt{3}}{2}$ sq.units

- **Find unit vector perpendicular to the plane passing through the points (1,2,3), (2,-1,1) and (1,2,-4)**

A: We take $\overline{OA}=\bar{i}+2\bar{j}+3\bar{k}, \overline{OB}=2\bar{i}-\bar{j}+\bar{k}, \overline{OC}=\bar{i}+2\bar{j}-4\bar{k}$ where 'O' is the origin.

Now $\overline{AB}=\overline{OB}-\overline{OA}=(2\bar{i}-\bar{j}+\bar{k})-(\bar{i}+2\bar{j}+3\bar{k})=\bar{i}-3\bar{j}-2\bar{k}$

$\overline{AC}=\overline{OC}-\overline{OA}=(\bar{i}+2\bar{j}-4\bar{k})-(\bar{i}+2\bar{j}+3\bar{k})=-7\bar{k}$

$$\Rightarrow\overline{AB}\times\overline{AC}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\1&-3&-2\\0&0&-7\end{vmatrix}$$

$=\bar{i}[(-3)(-7)-0(-2)]-\bar{j}[1(-7)-(0)(-2)]+\bar{k}[1(0)-(0)(-3)]$

$=\bar{i}(21)-\bar{j}(-7)=21\bar{i}+7\bar{j}=7(3\bar{i}+\bar{j})$

$|\overline{AB}\times\overline{AC}|=7\sqrt{3^2+1^2}=7\sqrt{9+1}=7\sqrt{10}$

Unit vector $=\pm\frac{\overline{AB}\times\overline{AC}}{|\overline{AB}\times\overline{AC}|}=\pm\frac{\not{7}(3\bar{i}+\bar{j})}{\not{7}\sqrt{10}}=\pm\frac{1}{\sqrt{10}}(3\bar{i}+\bar{j})$

Q14. TRIGONOMETRIC RATIOS

- **If $\tan\theta=\frac{b}{a}$ then prove that acos2θ+bsin2θ=a**

A: Given that $\tan\theta=\frac{b}{a}\Rightarrow\frac{\sin\theta}{\cos\theta}=\frac{b}{a}\Rightarrow b\cos\theta=a\sin\theta$

$\therefore$ L.H.S=acos2θ+bsin2θ= acos2θ+b(2sinθcosθ)

= acos2θ+2sinθ(bcosθ)= acos2θ+2sinθ (asinθ)

$=a(1-2\sin^2\theta)+2a.\sin^2\theta$.

$=a\ \not{-2a\sin^2\theta}\ \not{+2a\sin^2\theta}=a=$ R.H.S.

- **Show that $\frac{1}{\sin10^o}-\frac{\sqrt{3}}{\cos10^o}=4$**

A: $L.H.S=\frac{1}{\sin10^o}-\frac{\sqrt{3}}{\cos10^o}=\frac{\cos10^o-\sqrt{3}\sin10^o}{\sin10^o\cos10^o}$

$=\frac{2(\frac{1}{2}\cos10^o-\frac{\sqrt{3}}{2}\sin10^o)}{\sin10^o\cos10^o}=\frac{2(\sin30^o\cos10^o-\cos30^o\sin10^o)}{\sin10^o\cos10^o}$

$=\frac{2(\sin(30^o-10^o)}{\sin10^o\cos10^o}=\frac{2\sin20^o}{\sin10^o\cos10^o}$

$=\frac{2.2\sin10^o\cos10^o}{\sin10^o\cos10^o}=2.2=4=$ R.H.S

Q15. TRIGONOMETRIC EQUATIONS

- **Solve** $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

A: Given equation is $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

On dividing by $\sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$, we get

$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \sin\theta\left(\frac{\sqrt{3}}{2}\right) - \cos\theta\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\theta\cos 30^\circ - \cos\theta\sin 30^\circ = \sin 45^\circ$

$\Rightarrow \sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6} = \sin\frac{\pi}{4} \Rightarrow \sin\left(\theta - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}$.

Here P.V is $\alpha = \frac{\pi}{4}$

$\therefore$ General solution is $\theta = n\pi + (-1)^n\alpha, n \in Z$

$\Rightarrow \theta - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{4} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{6}, n \in Z$

- **Solve** $\sqrt{2}(\sin x + \cos x) = \sqrt{3}$

A: Given equation is $\sqrt{2}(\sin x + \cos x) = \sqrt{3}$

$\Rightarrow \sqrt{2}\sin x + \sqrt{2}\cos x = \sqrt{3}$

On dividing by $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{2+2} = \sqrt{4} = 2$, we get

$\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = \frac{\sqrt{3}}{2} \Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos x\left(\frac{1}{\sqrt{2}}\right) + \sin x\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos x\cos 45^\circ + \sin x\sin 45^\circ = \cos 30^\circ$

$\Rightarrow \cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4} = \cos\frac{\pi}{6}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{6}$. Here P.V is $\alpha = \frac{\pi}{6}$

$\therefore$ General solution is $\theta = 2n\pi \pm \alpha,\ n \in Z$

$\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{6} \Rightarrow x = 2n\pi \pm \frac{\pi}{6} + \frac{\pi}{4}, n \in Z$

- **Solve** $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$

A: Given equation is $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$

$\Rightarrow 2(1 - \sin^2\theta) - \sqrt{3}\sin\theta + 1 = 0$

$\Rightarrow 2 - 2\sin^2\theta - \sqrt{3}\sin\theta + 1 = 0$

$\Rightarrow 2\sin^2\theta + \sqrt{3}\sin\theta - 3 = 0$

$\Rightarrow 2\sin^2\theta + 2\sqrt{3}\sin\theta - \sqrt{3}\sin\theta - (\sqrt{3})^2 = 0$

$\Rightarrow 2\sin\theta(\sin\theta + \sqrt{3}) - \sqrt{3}(\sin\theta + \sqrt{3}) = 0$

$\Rightarrow (2\sin\theta - \sqrt{3})(\sin\theta + \sqrt{3}) = 0$

$\Rightarrow (2\sin\theta - \sqrt{3}) = 0$ (or) $(\sin\theta + \sqrt{3}) = 0$

$2\sin\theta = \sqrt{3}$ (or) $\sin\theta = -\sqrt{3}$ [This has no solution]

Now, $2\sin\theta = \sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

So, $\sin\theta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$, here P.V is $\alpha = \frac{\pi}{3}$

$\therefore$ General solution is $\theta = n\pi + (-1)^n\alpha, n \in Z$

$\Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{3}, n \in Z$

- **Solve** $1 + \sin^2\theta = 3\sin\theta\cos\theta$

A: Dividing the given equation by $\cos^2\theta$, we get

$\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{3\sin\theta\cos\theta}{\cos^2\theta} \Rightarrow \sec^2\theta + \tan^2\theta = 3\tan\theta$

$\Rightarrow (1 + \tan^2\theta) + \tan^2\theta = 3\tan\theta \Rightarrow 2\tan^2\theta - 3\tan\theta + 1 = 0$

$\Rightarrow (2\tan\theta - 1)(\tan\theta - 1) = 0 \Rightarrow \tan\theta = 1$ (or) $\tan\theta = 1/2$

Now, $\tan\theta = 1 = \tan\pi/4$. Here P.V is $\alpha = \pi/4$

$\therefore$ General solution is $\theta = n\pi + \frac{\pi}{4}, n \in Z$

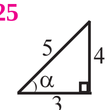
Also, $\tan\theta = \frac{1}{2} \Rightarrow \theta = \mathrm{Tan}^{-1}\frac{1}{2}$ $\therefore$ P.V is $\alpha = \mathrm{Tan}^{-1}\frac{1}{2}$

$\therefore$ General solution is $\theta = n\pi + \mathrm{Tan}^{-1}\frac{1}{2}, n \in Z$

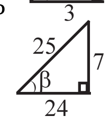
Q16. INVERSE TRIGONOMETRIC FUNCTIONS

- **Prove that** $\mathrm{Sin}^{-1}\frac{4}{5} + \mathrm{Sin}^{-1}\frac{7}{25} = \mathrm{Sin}^{-1}\frac{117}{125}$

Sol: Let $\mathrm{Sin}^{-1}\frac{4}{5} = \alpha \Rightarrow \sin\alpha = \frac{4}{5} \Rightarrow \cos\alpha = \frac{3}{5}$



$\mathrm{Sin}^{-1}\frac{7}{25} = \beta \Rightarrow \sin\beta = \frac{7}{25} \Rightarrow \cos\beta = \frac{24}{25}$

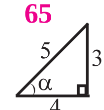


$\therefore \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

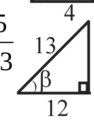
$= \frac{4}{5} \times \frac{24}{25} + \frac{3}{5} \times \frac{7}{25} = \frac{96 + 21}{125} = \frac{117}{125}$. Hence proved.

- **Prove that** $\mathrm{Sin}^{-1}\frac{3}{5} + \mathrm{Cos}^{-1}\frac{12}{13} = \mathrm{Cos}^{-1}\frac{33}{65}$

A: Let $\mathrm{Sin}^{-1}\frac{3}{5} = \alpha \Rightarrow \sin\alpha = \frac{3}{5} \Rightarrow \cos\alpha = \frac{4}{5}$



$\mathrm{Cos}^{-1}\frac{12}{13} = \beta \Rightarrow \sin\beta = \frac{12}{13} \Rightarrow \sin\beta = \frac{5}{13}$



$\therefore \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13} = \frac{48 - 15}{65} = \frac{33}{65}$. Hence proved.

- **Prove that** $\mathrm{Tan}^{-1}\frac{1}{2} + \mathrm{Tan}^{-1}\frac{1}{5} + \mathrm{Tan}^{-1}\frac{1}{8} = \frac{\pi}{4}$

A: **Formula:** $\mathrm{Tan}^{-1}x + \mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x + y}{1 - xy}\right)$

$\therefore \mathrm{Tan}^{-1}\frac{1}{2} + \mathrm{Tan}^{-1}\frac{1}{5} = \mathrm{Tan}^{-1}\left(\frac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{2}\cdot\frac{1}{5}}\right) = \mathrm{Tan}^{-1}\left(\frac{\frac{5+2}{10}}{\frac{10-1}{10}}\right) = \mathrm{Tan}^{-1}\frac{7}{9}$

$\therefore \mathrm{L.H.S} = \mathrm{Tan}^{-1}\frac{7}{9} + \mathrm{Tan}^{-1}\frac{1}{8} = \mathrm{Tan}^{-1}\left(\frac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{9}\cdot\frac{1}{8}}\right) = \mathrm{Tan}^{-1}\left(\frac{\frac{56+9}{72}}{\frac{72-7}{72}}\right)$

$= \mathrm{Tan}^{-1}\left(\frac{65}{65}\right) = \mathrm{Tan}^{-1}1 = \frac{\pi}{4} = \mathrm{R.H.S}$

Q17. PROPERTIES OF TRIANGLES

- **Prove that** $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\Delta}$

A: L.H.S= $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$

$$=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$=\frac{s(s-a)+s(s-b)+s(s-c)}{\Delta}$$

$$=\frac{s[(s-a)+(s-b)+(s-c)]}{\Delta}$$

$$=\frac{s[3s-(a+b+c)]}{\Delta}=\frac{s[3s-2s]}{\Delta}$$

$$=\frac{s[s]}{\Delta}=\frac{s^2}{\Delta}=\text{R.H.S}$$

- **Prove that cotA+cotB+cotC** $=\frac{a^2+b^2+c^2}{4\Delta}$

A: L.H.S=cotA+cotB+cotC$=\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}$

$$=\frac{b^2+c^2-a^2}{2bc(\sin A)}+\frac{c^2+a^2-b^2}{2ca(\sin B)}+\frac{a^2+b^2-c^2}{2ab(\sin C)}$$

$$=\frac{b^2+c^2-a^2}{4(\frac{1}{2}bc\sin A)}+\frac{c^2+a^2-b^2}{4(\frac{1}{2}ca\sin B)}+\frac{a^2+b^2-c^2}{4(\frac{1}{2}ab\sin C)}$$

$$=\frac{b^2+c^2-a^2}{4\Delta}+\frac{c^2+a^2-b^2}{4\Delta}+\frac{a^2+b^2-c^2}{4\Delta}$$

$$=\frac{b^2+c^2\cancel{-a^2}\cancel{+c^2}\cancel{+a^2}\cancel{-b^2}+a^2\cancel{+b^2}\cancel{-c^2}}{4\Delta}$$

$$=\frac{a^2+b^2+c^2}{4\Delta}=\text{RHS}$$

- **If** $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=3:5:7$**, S.T a:b:c=6:5:4**

Sol: Given that $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=3:5:7$

$$\Rightarrow\frac{s(s-a)}{\Delta}:\frac{s(s-b)}{\Delta}:\frac{s(s-c)}{\Delta}=3:5:7$$

$\Rightarrow$ (s–a) : (s–b) : (s–c) = 3 : 5 : 7

Let s–a=3k ....(1), s–b=5k ....(2), s–c=7k .....(3)

Now, (1)+(2)+(3) $\Rightarrow$ 3s–(a+b+c)=15k

$\Rightarrow$ 3s–2s=15k $\Rightarrow$ s=15k

From (1), s–a=3k $\Rightarrow$ 15k–a=3k $\Rightarrow$ a=15k–3k=12k

From (2), s–b=5k $\Rightarrow$ 15k–b=5k $\Rightarrow$ b=15k–5k=10k

From (3), s–c=7k $\Rightarrow$ 15k–c=7k $\Rightarrow$ c=15k–7k=8k

$\therefore$ a : b : c = 12k : 10k : 8k = 12 : 10 : 8 =6 : 5 : 4

- **If** $\sin\theta=\frac{a}{(b+c)}$ **then S.T** $\cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$

A: Given $\sin\theta=\frac{a}{b+c}\Rightarrow\sin^2\theta=\frac{a^2}{(b+c)^2}$

$\therefore$ $\cos^2\theta=1-\sin^2\theta$ $\quad[\because\sin^2\theta+\cos^2\theta=1]$

$$=1-\frac{a^2}{(b+c)^2}=\frac{(b+c)^2-a^2}{(b+c)^2}=\frac{(b^2+c^2+2bc)-a^2}{(b+c)^2}$$

$$=\frac{2bc+(b^2+c^2-a^2)}{(b+c)^2}=\frac{2bc+2bc\cos A}{(b+c)^2}$$

$$\left(\because\frac{b^2+c^2-a^2}{2bc}=\cos A\right)$$

$$=\frac{2bc(1+\cos A)}{(b+c)^2}=\frac{2bc.2\cos^2\frac{A}{2}}{(b+c)^2}=\frac{4bc\cos^2\frac{A}{2}}{(b+c)^2}$$

$$\therefore\cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$$

- **If a=(b–c)secθ, prove that** $\tan\theta=\frac{2\sqrt{bc}}{b-c}\sin\left(\frac{A}{2}\right)$

A: Given a=(b–c)secθ$\Rightarrow\sec\theta=\frac{a}{b-c}\Rightarrow\sec^2\theta=\frac{a^2}{(b-c)^2}$

$\therefore$ $\tan^2\theta=\sec^2\theta-1$ $\quad[\because\sec^2\theta-\tan^2\theta=1]$

$$=\frac{a^2}{(b-c)^2}-1=\frac{a^2-(b-c)^2}{(b-c)^2}$$

$$=\frac{a^2-(b^2+c^2-2bc)}{(b-c)^2}=\frac{a^2-b^2-c^2+2bc}{(b-c)^2}$$

$$=\frac{2bc-(b^2+c^2-a^2)}{(b-c)^2}=\frac{2bc-(2bc\cos A)}{(b-c)^2}$$

$$=\frac{2bc(1-\cos A)}{(b-c)^2}=\frac{2bc.\left(2\sin^2\frac{A}{2}\right)}{(b-c)^2}=\frac{4bc.\sin^2\frac{A}{2}}{(b-c)^2}$$

$$\therefore\tan\theta=\frac{2\sqrt{bc}}{b-c}\sin\left(\frac{A}{2}\right)$$

LAQ SECTION-C

Q18. FUNCTIONS

- **If $f:A\to B$, $g:B\to C$ are two bijective functions then prove that $gof:A\to C$ is also a bijective function.**

A: Given f,g are bijectives.So f,g are both one one and onto.

(i) To prove that $gof:A\to C$ is one one

Let $(gof)(a_1)=(gof)(a_2)$, [for $a_1, a_2\in A$]

$\Rightarrow g[f(a_1)]=g[f(a_2)]$

$\Rightarrow f(a_1)=f(a_2)$ ($\because$ g is one one)

$\Rightarrow a_1=a_2$ ($\because$ f is one one)

$\therefore$ gof: $A\to C$ is one one.

(ii) To prove that $gof:A\to C$ is onto.

Given $f:A\to B$ is onto, then f(a)=b........(1),

[$\because$ for all $b\in B$ there exist $a\in A$ such that f(a)=b]

Given $g:B\to C$ is onto, then g(b)=c......(2),

[$\because$ for all $c\in C$ there exist $b\in B$ such that g(b)=c]

Now (gof)(a) =g[f(a)]=g(b)=c, [From (2) and (1)]

$\therefore$ gof: $A\to C$ is onto.

Hence, we proved that $gof:A\to C$ is one one and onto, hence bijective.

- **If $f:A\to B$, $g:B\to C$ are two bijective functions then prove that $(gof)^{-1}=f^{-1}og^{-1}$**

A: **Part -1:** Given $f:A\to B$, $g:B\to C$ are two bijectives, so

(i) $gof:A\to C$ is bijection

$\Rightarrow (gof)^{-1}:C\to A$ is also a bijection

(ii) $f^{-1}:B\to A$, $g^{-1}:C\to B$ are both bijections

$\Rightarrow (f^{-1}og^{-1})$: $C\to A$ is also a bijection.

So, $(gof)^{-1}$ and $f^{-1}og^{-1}$, both have same domain 'C'

Part–2: $f:A\to B$ is bijection, then $f(a)=b \Rightarrow a=f^{-1}(b)$...(1)

$g:B\to C$ is bijection, then $g(b)=c \Rightarrow b=g^{-1}(c)$........(2)

$gof:A\to C$ is bijection, then $gof(a)=c \Rightarrow a=(gof)^{-1}(c)$....(3)

From (1)&(2), $(f^{-1}og^{-1})(c)=f^{-1}[g^{-1}(c)]=f^{-1}(b)=a$ ......(4)

From (3) & (4), $(gof)^{-1}(c)=(f^{-1}og^{-1})(c)$, $\forall c\in C$

Hence, we proved that $(gof)^{-1}=f^{-1}og^{-1}$

- **If $f:A\to B$ is a function and I_A, I_B are identities on A, B respectively then prove that $foI_A=f=I_Bof$.**

A: (i) **To prove that $foI_A= f$**

Part–1: Given $f:A\to B$ is a function.

We know $I_A:A\to A$

$\therefore foI_A:A\to B$

So, foI_A and f, both have same domain A

Part–2: For $a\in A$, $(foI_A)(a)=f[I_A(a)]$

=f(a) [$\because I_A(a)=a$ for all $a\in A$]

Hence, we proved that $foI_A= f$

(ii) **To prove that $I_Bof=f$**

Part–1: Given $f:A\to B$ is a function.

We know $I_B:B\to B$

$\therefore I_Bof:A\to B$

So, I_Bof and f, both have same domain A

Part–2: For $a\in A$, $(I_Bof)(a)=I_B[f(a)]$

=f(a) [$\because I_B(b)=b$ for all $b\in B$]

Hence, we proved that that $I_Bof=f$

- **If $f:A\to B$ is a bijective function then prove that (i) $fof^{-1}=I_B$ (ii) $f^{-1}of=I_A$**

A: (i) **To prove that $fof^{-1}=I_B$**

Part-1: Given $f:A\to B$ is a bijective function,

then f^{-1}: $B\to A$ is also a bijection

$\therefore fof^{-1}$: $B\to B$

We know, I_B: $B\to B$

So, fof^{-1} and I_B, both have same domain B

Part-2: For $b\in B$, $(fof^{-1})(b)=f[f^{-1}(b)]$

=f(a) [$\because f:A\to B$ is bijection

$\Rightarrow f(a)=b \Rightarrow f^{-1}(b)=a$, for $a\in A$]

$=b=I_B(b)$ [$\because I_B(b)=b$, for $b\in B$]

Hence we proved that $fof^{-1}=I_B$

(ii) **To prove that $f^{-1}of=I_A$**

Part-1: Given $f:A\to B$ is a bijective function,

then $f^{-1}:B\to A$ is also a bijection

$\therefore f^{-1}of$: $A\to A$

We know I_A: $A\to A$

So, $f^{-1}of$ and I_A, both have same domain A

Part-2: for $a\in A$, $(f^{-1}of)(a)=f^{-1}[f(a)]$

$= f^{-1}(b)= a$ [$\because f:A\to B$ is a bijection

$\Rightarrow f(a)=b \Rightarrow f^{-1}(b)=a$]

$= I_A(a)$[$\because I_A(a)=a$, for $a\in A$]

Hence we proved that $f^{-1}of=I_A$

Q19. MATHEMATICAL INDUCTION

- **Using P.M.I, P.T. 1.2.3 + 2.3.4 + 3.4.5 +...+n terms $=\frac{n(n+1)(n+2)(n+3)}{4}, \forall n \in N$**

A: n^{th} term is $T_n=n(n+1)(n+2)$.

S(n) : 1.2.3 + 2.3.4 + ...+n(n+1)(n+2)terms $=\frac{n(n+1)(n+2)(n+3)}{4}$

Step 1: L.H.S of S(1) = 1.2.3=6

R.H.S of S(1) $=\frac{1(1+1)(1+2)(1+3)}{4}=\frac{2.3.4}{4}=6$

$\therefore$ L.H.S = R.H.S. So, S(1) is true.

Step 2: Assume that S(k) is true, for k∈N

$S(k) : 1.2.3.+2.3.4+.....+k(k+1)(k+2)$

$=\frac{k(k+1)(k+2)(k+3)}{4}$ ......(1)

Step 3: We show that S(k+1) is true

On adding (k+1)(k+2)(k+3) to both sides of (1), we get

$L.H.S=[1.2.3.+2.3.4+.....+k(k+1)(k+2)]+(k+1)(k+2)(k+3)$

$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3)$

$=\frac{k(k+1)(k+2)(k+3)+4(k+1)(k+2)(k+3)}{4}$

$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$ =R.H.S

$\therefore$ L.H.S=R.H.S . So, S(k+1) is true

Hence, by P.M.I the given statement is true, ∀ n∈N

- **S.T. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+n$ terms $=\frac{n}{3n+1}$**

A: n^{th} term is $T_n=\frac{1}{(3n-2)(3n+1)}$

$S(n) : \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

Step 1: L.H.S of S(1) $=\frac{1}{1.4}=\frac{1}{4}$;

R.H.S of S(1) $=\frac{1}{3.1+1}=\frac{1}{4}$

$\therefore$ L.H.S = R.H.S.

So, S(1) is true

Step 2: Assume that S(k) is true ,for k∈N

$S(k) : \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ .....(1)

Step 3: We show that S(k+1) is true

On adding $(k+1)^{th}$ term to both sides of (1), we get

$L.H.S=\left[\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3k-2)(3k+1)}\right]+\frac{1}{(3k+1)(3k+4)}$

$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$

$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$=\frac{(k+1)\cancel{(3k+1)}}{\cancel{(3k+1)}(3k+4)}=\frac{k+1}{3k+4}=\frac{k+1}{3k+3+1}=\frac{k+1}{3(k+1)+1}=R.H.S$

$\therefore$ L.H.S =R.H.S. So, S(k+1) is true

Hence, by P.M.I the given statement is true, ∀n∈N

- **Using PMI, P.T. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+....+n$ terms $=\frac{n(n+1)^2(n+2)}{12}, \forall n\varepsilon N$**

A: n^{th} term $T_n=1^2+2^2+3^2+.....+n^2=\frac{n(n+1)(2n+1)}{6}$

Let $S(n) : 1^2+(1^2+2^2)+.....+\frac{n(n+1)(2n+1)}{6}$

$=\frac{n(n+1)^2(n+2)}{12}$

Step 1: L.H.S of S(1) = 1^2=1

R.H.S of S(1) $=\frac{1(1+1)^2(1+2)}{12}=\frac{1(2^2)3}{12}=\frac{1(4)3}{12}=\frac{12}{12}=1$

$\therefore$ L.H.S = R.H.S. So, S(1) is true

Step 2: Assume that S(k) is true for k ∈ N

$S(k) : 1^2+(1^2+2^2)+.....+\frac{k(k+1)(2k+1)}{6}$

$=\frac{k(k+1)^2(k+2)}{12}$ .....(1)

Step 3: We show that S(k+1) is true

$(k+1)^{th}$ term $=\frac{(k+1)(k+2)(2(k+1)+1)}{6}$

$=\frac{(k+1)(k+2)(2k+3)}{6}$

On adding $(k+1)^{th}$ term to both sides of (1), we have

$L.H.S=\left[1^2+(1^2+2^2)+.....+\frac{k(k+1)(2k+1)}{6}\right]+\frac{(k+1)(k+2)(2k+3)}{6}$

$=\frac{k(k+1)^2(k+2)}{12}+\frac{(k+1)(k+2)(2k+3)}{6}$

$=\frac{k(k+1)^2(k+2)+2(k+1)(k+2).(2k+3)}{12}$

$=\frac{(k+1)(k+2)[k(k+1)+2(2k+3)]}{12}$

$=\frac{(k+1)(k+2)(k^2+5k+6)}{12}$

$=\frac{(k+1)(k+2)(k+2)(k+3)}{12}=\frac{(k+1)(k+2)^2(k+3)}{12}=R.H.S$

$\therefore$L.H.S =R.H.S. So, S(k+1) is true

Hence, by P.M.I the given statement is true, ∀n∈N

- **By Mathematical Induction, show that 49^n+16^n-1 is divisible by 64 for all positive Integers n.**

A: Given S(n) : $49^n+16n-1=64q$, q∈Z

Step 1: L.H.S of S(1)=$49^{(1)}$+16 (1)–1

=49+16–1=64=64(1)

So, S(1) is true

Step 2: Assume that S(k) is true for k∈N

S(k):49^k+16k–1 =64q ....(1)

Step 3: We show that S(k+1) is true

Writing $(k+1)^{th}$ term from (1), we get

L.H.S = 49^{k+1}+16(k+1)–1=49^k.49+16k+16–1

= (64q–16k +1).49+16k+15 (from (1))

= 64q.49–16k.49+1.49+16k+15

=64q.49–16k.(49–1) + (49+15)

= 64q.49–16k.(48) + 64=64q.49–16k.(4.12)+64

= 64q.49–64k (12)+64

= 64(49q–12k+1)=64(an integer)

So, S(k+1) is true whenever S(k) is true

Hence, by P.M.I the given statement is true, ∀n∈N

Q20 & 21. MATRICES

12. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

A: $\text{L.H.S} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= abc\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad (\because C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1)$

$= (abc)\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$

$= (abc)(b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$

$= (abc)(b-a)(c-a)[(c+\not{a})1-(b+\not{a})1]$

$= (abc)(b-a)(c-a)[(c-b)]$

$= (abc)(a-b)(b-c)(c-a) = \text{R.H.S}$

• **S.T. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$**

A: $\text{L.H.S} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} \quad (\because R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$

$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & (b-a)(b+a) & (b-a)(b^2+ba+a^2) \\ 0 & (c-a)(c+a) & (c-a)(c^2+ca+a^2) \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & a+c & a^2+ac+c^2 \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & ac-ab+c^2-b^2 \end{vmatrix} \quad (\because R_3 \to R_3 - R_1)$

$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & a(c-b)+(c+b)(c-b) \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & (c-b)(a+b+c) \end{vmatrix}$

$= (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & 1 & (a+b+c) \end{vmatrix}$

$= (b-a)(c-a)(c-b)1[(a+b)(a+b+c)-(a^2+ab+b^2)]$

$= (b-a)(c-a)(c-b)[(a+b)^2+c(a+b)-(a^2+ab+b^2)]$

$= (b-a)(c-a)(c-b)[(\not{a^2}+\not{b^2}+2ab)+(ca+cb)-\not{a^2}-ab-\not{b^2}]$

$= (a-b)(b-c)(c-a)(ab+bc+ca) = \text{R.H.S}$

• **S.T $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$**

A: $\text{L.H.S} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix} \quad (\because C_1 \to C_1 + C_2 + C_3)$

$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$

$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix} \quad (\because R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$

$= 2(a+b+c)1[(a+b+c)^2-0] = 2(a+b+c)^3 = \text{R.H.S}$

• **S.T $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$**

A: Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(bc-a^2)-b(b^2-ac)+c(ab-c^2)$

$= abc-a^3-b^3+abc+abc-c^3 = -(a^3+b^3+c^3-3abc)$

$\Rightarrow \Delta^2 = (a^3+b^3+c^3-3abc)^2$ ....(1)

Now, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

On applying C_{23} on the first determinent, we have

$= -\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= \begin{vmatrix} -a^2+cb+bc & -ab+c^2+ab & -ac+ac+b^2 \\ -ab+ab+c^2 & -b^2+ac+a & -bc+a^2+cb \\ -ca+b^2+ac & -cb+bc+a^2 & -c^2+ba+ab \end{vmatrix}$

$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$ ....(2)

From (1) and (2) the given result follows.

- **By using Cramer's rule solve $2x - y + 3z = 9, x+y+z = 6, x-y+z = 2$**

A: Given equations in the matrix equation form: AX = D,

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$$\Delta = \det A = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+1)+1(1-1)+3(-1-1)$$

$$=2(2)+1(0)+3(-2)=4+0-6= -2$$

$$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 9(1+1)\ +1(6-2)+3(-6-2)$$

$$= 9(2)+1(4)+3(-8) = 18+4-24 = -2;$$

$$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2(6-2)-9(1-1)+3(2-6)$$

$$= 2(4)-9(0)+3(-4) = 8-0-12 = -4;$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2(2+6)+1(2-6)+9(-1-1)$$

$$= 2(8)+1(-4)+9(-2) = 16-4-18 = -6$$

By Cramer's rule,

$$x = \frac{\Delta_1}{\Delta} = \frac{\cancel{-2}}{\cancel{-2}} = 1;\ \ y = \frac{\Delta_2}{\Delta} = \frac{\cancel{-4}}{\cancel{-2}} = 2 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{\cancel{-6}}{\cancel{-2}} = 3$$

$\therefore$ x=1,y=2,z=3 are the solutions of the given equations.

- **Solve the equations $3x+4y+5z=18$, $2x-y+8z=13$, $5x-2y+7z=20$ by Matrix inversion method.**

A: Given equations in the matrix equation form: AX=D,

where $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$ x

In the Matrix Inverstion Method, solution is $X=A^{-1}D$

First we find A^{-1}

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix} = 3(-7+16)-4(14-40)+5(-4+5)$$

$$= 3(9)-4(-26)+5(1) = 27+104+5 = 136$$

$$\text{Adj}A = \begin{bmatrix} +\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} \\ -\begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} & +\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} \\ +\begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -7+16 & -(14-40) & -4+5 \\ -(28+10) & (21-25) & -(-6-20) \\ (32+5) & -(24-10) & (-3-8) \end{bmatrix}^T = \begin{vmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{vmatrix}^T$$

$$\Rightarrow \text{Adj}\,A = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix};\ A^{-1} = \frac{\text{Adj}\,A}{|A|} = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$\therefore X = A^{-1}D = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 9\times18-38\times13+37\times20 \\ 26\times18-4\times13-14\times20 \\ 1\times18+26\times13-11\times20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 162-494+740 \\ 468-52-280 \\ 18+338-220 \end{bmatrix} = \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} = \begin{bmatrix} \frac{\cancel{408}}{\cancel{136}} \\ \frac{\cancel{136}}{\cancel{136}} \\ \frac{\cancel{136}}{\cancel{136}} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ Hence, the solution is x=3, y=1, z=1

- **Solve the system of equations 2x-y+3z=9, x+y+z=6, x - y + z = 2 using Gauss Jordan method.**

A: Given equations in the matrix equation form: AX=D,

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}; D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Augmented Matrix $[AD] = \begin{bmatrix} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 9 \\ 1 & -1 & 1 & 2 \end{bmatrix} (R_1 \to R_{12})$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & -3 \\ 0 & -2 & 0 & -4 \end{bmatrix} \begin{matrix} (R_2 \to R_2 - 2R_1) \\ (R_3 \to R_3 - R_1) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 0 & -4 \end{bmatrix} (R_2 - R_3) \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -2 & -6 \end{bmatrix} (R_3 - 2R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \left(R_3 \to R_3\left(-\tfrac{1}{2}\right)\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} (R_1 \to R_1 - R_3) \\ (R_2 \to R_1 - R_2) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} (R_1 \to R_1 + R_2)$$

$\therefore$ x = 1, y = 2, z = 3

Q22. PRODUCT OF VECTORS

- **If $\bar{a}=\bar{i}-2\bar{j}+3\bar{k}, \bar{b}=2\bar{i}+\bar{j}+\bar{k}, \bar{c}=\bar{i}+\bar{j}+2\bar{k}$ then find $|(\bar{a}\times\bar{b})\times\bar{c}|$ and $|\bar{a}\times(\bar{b}\times\bar{c})|$**

A: Given $\bar{a}=\bar{i}-2\bar{j}+3\bar{k}, \bar{b}=2\bar{i}+\bar{j}+\bar{k}, \bar{c}=\bar{i}+\bar{j}+2\bar{k}$

1) To find $(\bar{a}\times\bar{b})\times\bar{c}$, first we have to find $\bar{a}\times\bar{b}$ (term in the bracket)

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 3\\ 2 & 1 & 1\end{vmatrix}$$

$=\bar{i}(-2-3)-\bar{j}(1-6)+\bar{k}(1+4)$

$=-5\bar{i}+5\bar{j}+5\bar{k}$

$$\therefore(\bar{a}\times\bar{b})\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ -5 & 5 & 5\\ 1 & 1 & 2\end{vmatrix}$$

$=\bar{i}(10-5)-\bar{j}(-10-5)+\bar{k}(-5-5)$

$=5\bar{i}+15\bar{j}-10\bar{k}=5(\bar{i}+3\bar{j}-2\bar{k})$

$\therefore|(\bar{a}\times\bar{b})\times\bar{c}|=5|\bar{i}+3\bar{j}-2\bar{k}|$

$=5\sqrt{1^2+3^2+(-2)^2}=5\sqrt{1+9+4}=5\sqrt{14}$

2) To find $\bar{a}\times(\bar{b}\times\bar{c})$ first we have to find $\bar{b}\times\bar{c}$ (term in the bracket)

$$\bar{b}\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & 1 & 1\\ 1 & 1 & 2\end{vmatrix}$$

$=\bar{i}(2-1)-\bar{j}(4-1)+\bar{k}(2-1)=\bar{i}-3\bar{j}+\bar{k}$

$$\therefore\bar{a}\times(\bar{b}\times\bar{c})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 3\\ 1 & -3 & 1\end{vmatrix}$$

$=\bar{i}(-2+9)-\bar{j}(1-3)+\bar{k}(-3+2)$

$=7\bar{i}+2\bar{j}-\bar{k}$

$\therefore|\bar{a}\times(\bar{b}\times\bar{c})|=|7\bar{i}+2\bar{j}-\bar{k}|$

$=\sqrt{7^2+2^2+(-1)^2}=\sqrt{49+4+1}=\sqrt{54}=\sqrt{9\times6}=3\sqrt{6}$

- **Find the shortest distance between the skew lines $\bar{r}=(6\bar{i}+2\bar{j}+2\bar{k})+t(\bar{i}-2\bar{j}+2\bar{k})$ and $\bar{r}=(-4\bar{i}-\bar{k})+s(3\bar{i}-2\bar{j}-2\bar{k})$**

A: Given skew lines $\bar{r}=(6\bar{i}+2\bar{j}+2\bar{k})+t(\bar{i}-2\bar{j}+2\bar{k})$;

$\bar{r}=(-4\bar{i}-\bar{k})+s(3\bar{i}-2\bar{j}-2\bar{k})$

Formula: For the skew lines $\bar{r}=\bar{a}+t\bar{b}, \ \bar{r}=\bar{c}+s\bar{d}$

Shortest distance(SD) $=\dfrac{|(\bar{a}-\bar{c}).(\bar{b}\times\bar{d})|}{|\bar{b}\times\bar{d}|}$

On comparing the given skew lines with $\bar{r}=\bar{a}+t\bar{b}, \ \bar{r}=\bar{c}+s\bar{d}$ we get

$\bar{a}=6\bar{i}+2\bar{j}+2\bar{k}, \quad \bar{b}=\bar{i}-2\bar{j}+2\bar{k}$ and

$\bar{c}=-4\bar{i}-\bar{k}, \ \bar{d}=3\bar{i}-2\bar{j}-2\bar{k}$

So, $\bar{a}-\bar{c}=(6\bar{i}+2\bar{j}+2\bar{k})-(-4\bar{i}-\bar{k})=10\bar{i}+2\bar{j}+3\bar{k}$

$$\bar{b}\times\bar{d}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}=\bar{i}(4+4)-\bar{j}(-2-6)+\bar{k}(-2+6)=8\bar{i}+8\bar{j}+4\bar{k}$$

$\therefore(\bar{a}-\bar{c}).(\bar{b}\times\bar{d})=(10\bar{i}+2\bar{j}+3\bar{k}).(8\bar{i}+8\bar{j}+4\bar{k})$

$=80+16+12=108$

Also, $|\bar{b}\times\bar{d}|=\sqrt{8^2+8^2+4^2}=\sqrt{64+64+16}=\sqrt{144}=12$

$\therefore$ Shortest distance(SD) $=\dfrac{|(\bar{a}-\bar{c}).(\bar{b}\times\bar{d})|}{|\bar{b}\times\bar{d}|}=\dfrac{108}{12}=9$

- **If A=(1,–2,–1),B=(4,0,–3), C=(1,2,–1), D=(2,–4,–5) then find the distance between $\overline{AB}$ and $\overline{CD}$.**

A: Given A=(1,–2,–1),B=(4,0,–3),C=(1,2,–1),D=(2,–4,–5)

$\overline{OA}=\bar{i}-2\bar{j}-\bar{k}, \overline{OB}=4\bar{i}-3\bar{k}, \overline{OC}=\bar{i}+2\bar{j}-\bar{k}, \overline{OD}=2\bar{i}-4\bar{j}-5\bar{k}$

(i) Vector equation of the line $\overline{AB}$ is $\bar{r}=\bar{a}+t\bar{b}, t\in R$,

where $\bar{a}=\overline{OA}=\bar{i}-2\bar{j}-\bar{k}$ and

$\bar{b}=\overline{AB}=\overline{OB}-\overline{OA}=(4\bar{i}-3\bar{k})-(\bar{i}-2\bar{j}-\bar{k})=3\bar{i}+2\bar{j}-2\bar{k}$

(ii) Vector equation of the line $\overline{CD}$ is $\bar{r}=\bar{c}+s\bar{d}, s\in R$,

where $\bar{c}=\overline{OC}=\bar{i}+2\bar{j}-\bar{k}$ and

$\bar{d}=\overline{CD}=\overline{OD}-\overline{OC}=(2\bar{i}-4\bar{j}-5\bar{k})-(\bar{i}+2\bar{j}-\bar{k})=\bar{i}-6\bar{j}-4\bar{k}$

So, $\bar{a}-\bar{c}=(\bar{i}-2\bar{j}-\bar{k})-(\bar{i}+2\bar{j}-\bar{k})=-4\bar{j}$

$$\bar{b}\times\bar{d}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & 2 & -2\\ 1 & -6 & -4\end{vmatrix}=\bar{i}(-8-12)-\bar{j}(-12+2)+\bar{k}(-18-2)$$

$=-20\bar{i}+10\bar{j}-20\bar{k}$

Now, $(\bar{a}-\bar{c}).(\bar{b}\times\bar{d})=(-4\bar{j}).(-20\bar{i}+10\bar{j}-20\bar{k})=-4(10)=-40$

Also, $|\bar{b}\times\bar{d}|=\sqrt{(-20)^2+10^2+(-20)^2}$

$=\sqrt{400+100+400}=\sqrt{900}=30$

$\therefore$ SD $=\dfrac{|(\bar{a}-\bar{c}).(\bar{b}\times\bar{d})|}{|\bar{b}\times\bar{d}|}=\dfrac{|-40|}{30}=\dfrac{40}{30}=\dfrac{4}{3}$ units

Q23. TRANSFORMATIONS

- **If A, B, C are angles in a triangle, then prove that**

$\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}$

A: L.H.S=(cosA+cosB)–cosC

$$=2\cos\frac{A+B}{2}.\cos\frac{A-B}{2}-\left(1-2\sin^2\frac{C}{2}\right)$$

$$\left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right);\cos\theta=1-2\sin^2\frac{\theta}{2}\right]$$

$$=-1+2\cos\left(90^o-\frac{C}{2}\right).\cos\frac{A-B}{2}+2\sin^2\frac{C}{2}$$

$$=-1+2\sin\frac{C}{2}.\cos\frac{A-B}{2}+2\sin^2\frac{C}{2}$$

$$=-1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\sin\frac{C}{2}\right)$$

$$=-1+2\sin\frac{C}{2}\left[\cos\frac{A-B}{2}+\sin\left(90^o-\frac{A+B}{2}\right)\right]$$

$$=-1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)$$

$$(\because \sin(90^o-\theta)=\cos\theta)$$

$$=-1+2\sin\frac{C}{2}\left(2\cos\frac{A}{2}.\cos\frac{B}{2}\right)$$

$$(\because \cos(A+B)+\cos(A-B)=2\cos A\cos B)$$

$$=-1+4\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}=\text{R.H.S}$$

- **If A,B,C are angles of a triangle, then prove that**

$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

A: L.H.S= $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}$

$$=\sin^2\frac{A}{2}+\left(1-\cos^2\frac{B}{2}\right)-\sin^2\frac{C}{2}$$

$$=1+(\sin^2\frac{A}{2}-\cos^2\frac{B}{2})-\sin^2\frac{C}{2}$$

$$=1+(-\cos\frac{A+B}{2}.\cos\frac{A-B}{2})-\sin^2\frac{C}{2}$$

$$[\because \sin^2A-\cos^2B=-\cos(A+B)\cos(A-B)]$$

$$=\left(1-\cos\left(\frac{180^o-C}{2}\right).\cos\frac{A-B}{2}\right)-\sin^2\frac{C}{2}$$

$$[\because (A+B)+C=180^o]$$

$$=\left(1-\cos(90^o-\frac{C}{2}).\cos\frac{A-B}{2}\right)-\sin^2\frac{C}{2}$$

$$=1-\sin\frac{C}{2}.\cos\frac{A-B}{2}-\sin^2\frac{C}{2}$$

$$=1-\sin\frac{C}{2}(\cos\frac{A-B}{2}+\sin\frac{C}{2})$$

$$=1-\sin\frac{C}{2}(\cos\frac{A-B}{2}+\sin\frac{180^o-(A+B)}{2})$$

$$=1-\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\sin\left(90^o-\frac{A+B}{2}\right)\right)$$

$$=1-\sin\frac{C}{2}(\cos\frac{A-B}{2}+\cos\frac{A+B}{2})$$

$$=1-\sin\frac{C}{2}(2\sin\frac{A}{2}\sin\frac{B}{2})$$

$$[\because \cos(A+B)+\cos(A-B)=2\cos A\cos B]$$

$$=1-(2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})=\text{R.H.S}$$

- **If A+B+C=π, then prove that**

$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})$

A: L.H.S $=\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$

$$=\cos^2\frac{A}{2}+\left(1-\sin^2\frac{B}{2}\right)+\cos^2\frac{C}{2}$$

$$=1+\left(\cos^2\frac{A}{2}-\sin^2\frac{B}{2}\right)+\cos^2\frac{C}{2}$$

$$=1+\left(\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right)+\left(1-\sin^2\frac{C}{2}\right)$$

$$\left[\because \cos^2A-\sin^2B=\cos(A+B)\cos(A-B)\right]$$

$$=2+\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-\sin^2\frac{C}{2}$$

$$=2+\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-\sin^2\frac{C}{2}$$

$$=2+\sin\frac{C}{2}\left(\cos\left(\frac{A-B}{2}\right)-\sin\frac{C}{2}\right)$$

$$=2+\sin\frac{C}{2}\left(\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right)$$

$$\left[\because \sin\frac{C}{2}=\cos\left(\frac{A+B}{2}\right)\right]$$

$$=2+\sin\frac{C}{2}\left(2\sin\frac{A}{2}.\sin\frac{B}{2}\right)$$

$$\left[\because \cos(A-B)-\cos(A+B)=2\sin A\sin B\right]$$

$$=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=\text{R.H.S}$$

Q24 .PROPERTIES OF TRIANGLES

- **If $r_1 = 2, r_2 = 3, r_3 = 6, r = 1$, P.T a=3, b=4, c=5.**

A: Given $r_1=2, r_2=3, r_3=6$ and r=1, then

$$\Delta = \sqrt{rr_1r_2r_3} = \sqrt{1\times 2\times 3\times 6} = \sqrt{36} = 6$$

We take, $r = \frac{\Delta}{s} \Rightarrow 1 = \frac{6}{s} \quad \therefore s = 6$

(i) $r_1 = \frac{\Delta}{s-a} \Rightarrow s-a = \frac{\Delta}{r_1} = \frac{6}{2} = 3$

$\therefore s-a = 3 \Rightarrow 6-a = 3 \Rightarrow a = 6-3 = 3$

(ii) $r_2 = \frac{\Delta}{s-b} \Rightarrow s-b = \frac{\Delta}{r_2} = \frac{6}{3} = 2$

$\therefore s-b = 2 \Rightarrow 6-b = 2 \Rightarrow b = 6-2 = 4$

(iii) $r_3 = \frac{\Delta}{s-c} \Rightarrow s-c = \frac{\Delta}{r_3} = \frac{6}{6} = 1$

$\therefore s-c = 1 \Rightarrow 6-c = 1 \Rightarrow c = 6-1 = 5$

- **In a ΔABC if $r_1=8, r_2=12, r_3=24$ find a,b,c.**

A: Given $r_1=8, r_2=12, r_3=24$, then $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$$\Rightarrow \frac{1}{r} = \frac{1}{8} + \frac{1}{12} + \frac{1}{24} = \frac{3+2+1}{24}$$

$$= \frac{6}{24} = \frac{1}{4} \quad \therefore \frac{1}{r} = \frac{1}{4} \Rightarrow r = 4$$

Now $\Delta = \sqrt{rr_1r_2r_3} = \sqrt{4\times 8\times 12\times 24}$

$$= \sqrt{4\times 8\times (3\times 4)\times (3\times 8)} = \sqrt{3^2\times 4^2\times 8^2} = 3\times 4\times 8 = 96$$

We take $r = \frac{\Delta}{s} \Rightarrow s = \frac{\Delta}{r} = \frac{96}{4} = 24 \quad \therefore s = 24$

(i) $r_1 = \frac{\Delta}{s-a} \Rightarrow s-a = \frac{\Delta}{r_1} = \frac{96}{8} = 12$

$\therefore s-a = 12 \Rightarrow 24-a = 12 \Rightarrow a = 24-12 = 12$

(ii) $r_2 = \frac{\Delta}{s-b} \Rightarrow s-b = \frac{\Delta}{r_2} = \frac{96}{12} = 8$

$\therefore s-b = 8 \Rightarrow 24-b = 8 \Rightarrow b = 24-8 = 16$

(iii) $r_3 = \frac{\Delta}{s-c} \Rightarrow s-c = \frac{\Delta}{r_3} = \frac{96}{24} = 4$

$\therefore s-c = 4 \Rightarrow 24-c = 4 \Rightarrow c = 24-4 = 20$

- **In a ΔABC if a=13, b=14, c=15 then show that $R = \frac{65}{8}, r = 4, r_1 = \frac{21}{2}, r_2 = 12, r_3 = 14$**

A: Given a=13, b=14, c=15 , then

$2s = a+b+c = 13+14+15 = 42 \Rightarrow 2s = 42 \Rightarrow s = 21$

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{21(21-13)(21-14)(21-15)}$$

$$= \sqrt{21\times(8)(7)(6)} = \sqrt{(3\times 7)(4\times 2)(7)(3\times 2)}$$

$$= \sqrt{3^2\times 4^2\times 7^2} = 3\times 4\times 7 = 84$$

(i) $R = \frac{abc}{4\Delta} = \frac{13\times 14\times 15}{4\times 84} = \frac{65}{8}$

(ii) $r = \frac{\Delta}{s} = \frac{84}{21} = 4$;

(iii) $r_1 = \frac{\Delta}{s-a} = \frac{84}{21-13} = \frac{84}{8} = \frac{21}{2}$

(iv) $r_2 = \frac{\Delta}{s-b} = \frac{84}{21-14} = \frac{84}{7} = 12$

(v) $r_3 = \frac{\Delta}{s-c} = \frac{84}{21-15} = \frac{84}{6} = 14$

- **Show that $r+r_3+r_1-r_2 = 4R\cos B$.**

A: L.H.S $= r+r_3+r_1-r_2 = (r_3+r_1)+(r-r_2)$

$$= \left(4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} + 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$+\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right)$$

$$= 4R\left(\cos\frac{B}{2}\left(\cos\frac{A}{2}\sin\frac{C}{2} + \sin\frac{A}{2}\cos\frac{C}{2}\right) - \sin\frac{B}{2}\left(\cos\frac{A}{2}\cos\frac{C}{2} - \sin\frac{A}{2}\sin\frac{C}{2}\right)\right)$$

$$= 4R\left(\cos\frac{B}{2}\sin\left(\frac{A+C}{2}\right) - \sin\frac{B}{2}\cos\left(\frac{A+C}{2}\right)\right)$$

$$= 4R\left(\cos\frac{B}{2}\cos\frac{B}{2} - \sin\frac{B}{2}\sin\frac{B}{2}\right)$$

$$= 4R\left[\cos^2\frac{B}{2} - \sin^2\frac{B}{2}\right] = 4R\cos B = R.H.S$$