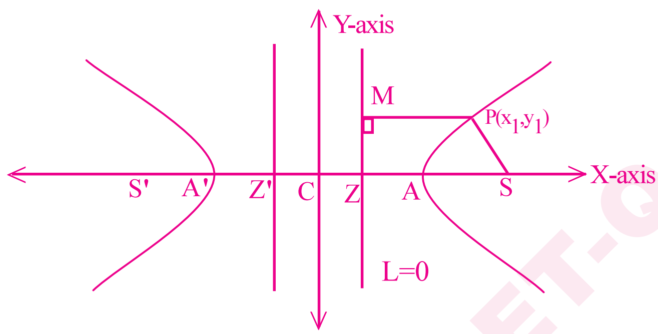
5. HYPERBOLA

(1 x 2) + (1 x 4) = 6 Marks

IMP FORMULAS, KEY CONCEPTS

I. Hyperbola: The locus of a point in a plane, which moves such that its distance from a fixed point (focus) bears a constant ratio e, e>1, to its distance from a fixed line (directrix) is called a hyperbola



1) The **eccentricity** of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\mathbf{e}=\frac{\sqrt{a^2+b^2}}{a}$ $\because b^2=a^2(e^2-1)$, Also $a^2e^2=a^2+b^2$

2) The **foci** of the hyperbola are **S=(ae, 0), S'(−ae, 0) i.e.,** $(\pm\sqrt{a^2+b^2},0)$

3) The **feet of the directrices** are $Z=\left(\frac{a}{e},0\right)$, $Z'=\left(-\frac{a}{e},0\right)$ & the equation of the directrices is $x=\pm\frac{a}{e}$

4.1) **AA'** is called the **transverse axis**, **BB'** is called the **conjugate axis** of the hyperbola

4.2) The **length** of the **transverse axis** is **2a** and the **equation** of the **transverse axis** is **y=0**
The **length** of **conjugate axis** is **2b** and the **equation** of the **conjugate axis** is **x=0**

5.1) **A(a,0), A'(-a,0)** are called the vertices of the hyperbola on the **transverse axis**
B(0,b), B'(0, -b) are called the vertices on the **conjugate axis**.

5.2) The **equation** of the **tangents** at the **vertices** is **x=±a**

6.1) The line segments passing through the foci and perpendicular to the transverse axis are called the **latusrecta** of the hyperbola.

6.2) The **equation** of the **latus recta** is **x=±ae**

6.3) The ends of the **latus recta** are $\left(ae,\pm\frac{b^2}{a}\right)$ and $\left(-ae,\pm\frac{b^2}{a}\right)$

6.4) The length of the **latus rectum** is $\frac{2b^2}{a}$

7.1) If P is any point on the hyperbola then **SP** is known as the **focal distance** of the point **P** w.r.t the focus **S**.

7.2) The **focal distance** of the point $P(x_1,y_1)$ on the hyperbola w.r.to the focus **S** is $SP=ex_1-a$ and the **focal distance** of the point **P** on the hyperbola w.r.to the focus **S'** is $S'P=ex_1+a$

8) The equation of the **auxiliary circle** of the hyperbola is $x^2+y^2=a^2$

9) The equation of the **director circle** of the hyperbola is $x^2+y^2=a^2-b^2$

II) The following results hold true w.r.to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

1) Notation: $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$; $S_1=\frac{x_1x}{a^2}-\frac{y_1y}{b^2}-1$; $S_{11}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$; $S_{12}=\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}-1$

2) **Relative positions** of a point $P(x_1,y_1)$ and the hyperbola S=0

(i) The point $P(x_1,y_1)$ lies on the hyperbola $S=0 \Leftrightarrow S_{11}=0$

(ii) The point $P(x_1,y_1)$ lies inside the hyperbola $S=0 \Leftrightarrow S_{11}>0$

(iii) The point $P(x_1,y_1)$ lies outside the hyperbola $S=0 \Leftrightarrow S_{11}<0$

3) The equation of the chord joining the points $A(x_1,y_1)$ & $B(x_2,y_2)$ on the hyperbola S=0 is $S_1+S_2=S_{12}$

4) The equation of the tangent at $P(x_1,y_1)$ on the hyperbola S=0 is $S_1=0$

5) The equation of the normal at $P(x_1,y_1)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

6) The condition for the line y=mx+c to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $c^2=a^2m^2-b^2$.

7) The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ having slope m is $y=mx\pm\sqrt{a^2m^2-b^2}$

8) Two tangents can be drawn from an external point $P(x_1,y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ &

if m_1,m_2 are the slopes of the two tangents then $m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}$, $m_1m_2=\frac{y_1^2+b^2}{x_1^2-a^2}$

9) The combined equation of the pair of tangents drawn from an external point $P(x_1,y_1)$ to the hyperbola S=0 is $S_1^2=S_{11}S$.

10) The equation of the **asymptotes** of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ (or) $y=\pm\frac{b}{a}x$

11) The angle between the **asymptotes** of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\text{Tan}^{-1}\frac{b}{a}$ (or) $2\text{Sec}^{-1}e$

III) PARAMETRIC TREATMENT:

12) The parametric point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $P(a\sec\theta, b\tan\theta)$ and is simply denoted by θ.

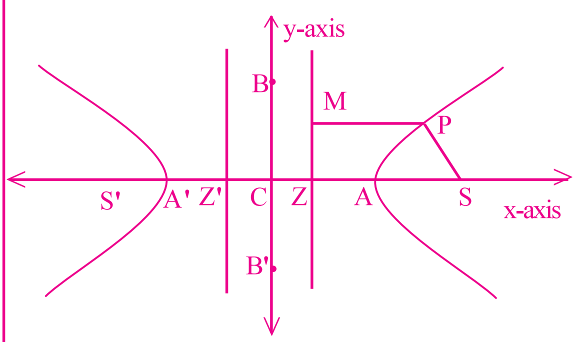
13) The equation of the tangent at P(θ) on the hyperbola S=0 is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

14) The equation of the normal at P(θ) on the hyperbola S=0 is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

IV) HYPERBOLA IN TWO STANDARD FORMS

$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Hyperbola in the standard form-I



1) Eccentricity $e = \frac{\sqrt{a^2+b^2}}{a}$

   ($\because$ $b^2=a^2(e^2-1)$ or $a^2e^2=a^2+b^2$)

2) Centre C=(0,0)

3) Foci S=(ae, 0) i.e., $S(\sqrt{a^2+b^2},0)$

   S'=(-ae,0) i.e., $S'(-\sqrt{a^2+b^2},0)$

4) Vertices, A=(a,0) and A'=(-a,0)

5) Ends of L.R=$\left(\pm ae, \pm\frac{b^2}{a}\right)$

6) Feet of the directrices = (±a/e, 0)

7) Equation of the directrices is x=±a/e (or)

   $(\sqrt{a^2+b^2})x = \pm a^2$

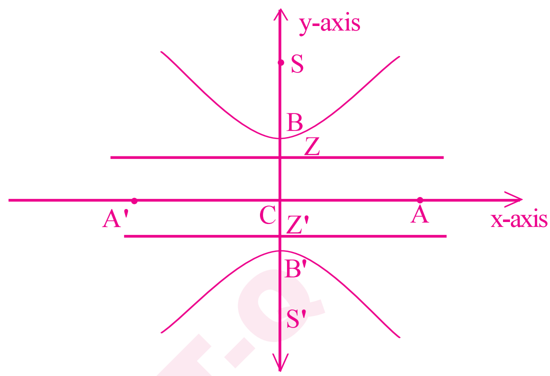
8) Equation of the L.R is x=±ae

9) Length of transverse axis is 2a and its equation is y=0
   Length of conjugate axis is 2b and its equation is x=0

10) Equation of the tangents at the vertices x=±a

$S' = \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

Conjugate Hyperbola (standard form-II)



1) Eccentricity $e' = \frac{\sqrt{b^2+a^2}}{b}$

   ($\because$ $a^2=b^2(e^2-1)$ or $b^2e^2=b^2+a^2$)

2) Centre C=(0,0)

3) Foci S=(0, be) i.e., $S(0,\sqrt{b^2+a^2})$

   S'=(0,-be) i.e., $S'(0,-\sqrt{b^2+a^2})$

4) Vertices, B=(0,b) and B'=(0,-b)

5) Ends of L.R= $\left(\pm\frac{a^2}{b}, \pm be\right)$

6) Feet of the directrices = (0, ±b/e)

7) Equation of the directrices is y=±b/e (or)

   $(\sqrt{b^2+a^2})y = \pm b^2$

8) Equation of the L.R is y=±be

9) Length of transverse axis is 2b and its equation is x=0
   Length of conjugate axis is 2a and its equation is y=0

10) Equation of the tangents at the vertices y=±b