SOLVED PAPER-4

Time: 3 Hours **MATHS-2B** Max. Marks : 75

SECTION -A

I. Answer ALL the following Very Short Answer Questions: $10 \times 2 = 20$

1. If the length of the tangent from (5, 4) to the circle $x^2 + y^2 + 2ky = 0$ is 1, then find k.
2. Find the equation of the polar of (1, – 2) with respect to circle $x^2 + y^2 - 10x - 10y + 25 = 0$.
3. Find the radical centre of the circles $x^2+y^2+4x-7=0$, $2x^2 + 2y^2+3x+5y-9=0$ and $x^2 + y^2 + y = 0$
4. Define latus rectum of a parabola. What is the length of the latus rectum of $y^2=4ax$?
5. Find the value of k if $3x-4y+k=0$ is a tangent to the hyperbola $x^2 - 4y^2 = 5$.
6. Evaluate $\int \sin mx \sin nx \, dx$
7. Find $\int \frac{(\log x)^2}{x} dx$
8. Evaluate $\int_{-\pi/2}^{\pi/2} \sin|x|\, dx$
9. Find the value of $\int_0^{2\pi} \sin^2 x.\cos^4 x dx$.
10. Find the I.F of the D.E $(\cos x)\frac{dy}{dx} + y\sin x = \tan x$ by transforming it into linear form.

SECTION-B

II. Answer any FIVE of the following Short Answer Questions: $5 \times 4 = 20$

11. Find the condition that the tangents drawn from the exterior point (0,0) to $S=x^2+y^2+2gx+2fy+c=0$ are perpendicular to each other.
12. Show that the circles $x^2+y^2-8x-2y+8=0$, $x^2+y^2-2x+6y+6=0$ touch each other and find the point of contact.
13. Find the condition for the line $lx+my+n=0$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
14. Prove that the equation of the chord joining the points α and β on the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$
15. Prove that the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ lies on the circle $x^2+y^2=a^2-b^2$.
16. Find the area of the region enclosed by the curves $y = 4x-x^2$, $y=5-2x$.
17. Solve $y^2dx+(x^2-xy)dy=0$

SECTION-C

III. Answer any FIVE of the following Long Answer Questions: $5 \times 7 = 35$

18. Find the values of c if the points (1, 2), (3,–4), (5, –6), (c, 8) are concyclic.

19. Find the equations of circles which touch $2x - 3y + 1 = 0$ at (1, 1) and having radius $\sqrt{13}$

20. Find the coordinates of vertex, focus, equation of the directrix and axis for the parabola $y^2 + 4x + 4y - 3 = 0$

21. Evaluate $\int \frac{1}{1+\sin x+\cos x} dx$.

22. Obtain the reduction formula for $I_n = \int \operatorname{cosec}^n x\,dx$, n being a positive integer, $n \geq 2$ and deduce that the value of $\int \operatorname{cosec}^5 x\,dx$

23. Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$

24. Solve the differential equation $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$.

SOLUTIONS

SECTION -A

1. If the length of the tangent from (5, 4) to the circle $x^2 + y^2 + 2ky = 0$ is 1, then find *k*.

Sol: The length of the tangent from (5, 4) to $S = x^2 + y^2 + 2ky = 0$ is $\sqrt{S_{11}} = 1$;

On squaring both sides, we get $S_{11} = 1$

$\Rightarrow 5^2 + 4^2 + 2k(4) = 1 \Rightarrow 25+16+8k=1 \Rightarrow 41+8k=1 \Rightarrow 8k = -40 \Rightarrow k = -40/8 = -5$

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2. Find the equation of the polar of (1, – 2) with respect to circle $x^2 + y^2 - 10x - 10y + 25 = 0$.

Sol: The equation of the polar of P(1,–2) w.r.t the circle $S = x^2 + y^2 - 10x - 10y + 25 = 0$. is $S_1 = 0$

$\Rightarrow x_1x + y_1y + g(x_1+x) + f(y_1+y) + c = 0$

$\Rightarrow 1(x) - 2(y) - 5(1+x) - 5(-2+y) + 25 = 0$

$\Rightarrow x - 2y - 5 - 5x + 10 - 5y + 25 = 0 \Rightarrow -4x - 7y + 30 = 0 \Rightarrow 4x + 7y - 30 = 0$

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3. Find the radical centre of the circles $x^2+y^2+4x-7=0$, $2x^2+2y^2+3x+5y-9=0$ and $x^2+y^2+y=0$

Sol: The given circles are $S' = x^2+y^2+4x-7=0 \Rightarrow S' = 2x^2 + 2y^2 + 8x - 14 = 0$ ....(1)

$S'' = 2x^2 + 2y^2 + 3x + 5y - 9 = 0$ ....(2)

$S''' = x^2 + y^2 + y = 0 \Rightarrow S''' = 2x^2 + 2y^2 + 2y = 0$ ....(3)

The radical axis of the first and second circle is $S' - S'' = 0$

$\Rightarrow (8x - 3x) - 5y - 14 + 9 = 0 \Rightarrow 5x - 5y - 5 = 0 \Rightarrow x - y - 1 = 0$ ....(4)

The radical axis of the first and third circle is $S' - S''' = 0$

$\Rightarrow 8x - 2y - 14 = 0 \Rightarrow 4x - y - 7 = 0$ ....(5)

(5) – (4) $\Rightarrow 3x - 6 = 0 \Rightarrow x = 2$

From (4) $\Rightarrow 2 - y - 1 = 0 \Rightarrow y = 1$

$\therefore$ the radical centre of the given 3 circles is (x, y) = (2, 1)

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4. Define latus rectum of a parabola. What is the length of the latus rectum of $y^2 = 4ax$?

Sol: The focal chord which is perpendicular to the axis of the parabola is called latus rectum of the parabola.

The length of the latus rectum of $y^2 = 4ax$ is 4a.

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5. Find the value of k if $3x - 4y + k = 0$ is a tangent to the hyperbola $x^2 - 4y^2 = 5$.

Sol: $x^2 - 4y^2 = 5 \Rightarrow \frac{x^2}{5} - \frac{4y^2}{5} = 1 \Rightarrow \frac{x^2}{5} - \frac{y^2}{5/4} = 1 \Rightarrow a^2 = 5$ and $b^2 = 5/4$

Comparing $3x - 4y + k = 0$ with $lx + my + n = 0$, we get $l = 3$, $m = -4$, $n = k$

Applying the tangential condition, $n^2 = a^2l^2 - b^2m^2$, we get

$(k)^2 = 5(3^2) - \frac{5}{4}(-4)^2 = 45 - 20 = 25$

$\therefore k^2 = 25 \Rightarrow k = \pm 5$

6. Evaluate $\int \sin mx \sin nx\, dx$

Sol: $\int \sin mx.\sin nx\, dx = \frac{1}{2}\int 2\sin mx.\sin nx\, dx$ $(\because 2\sin A \sin B = \cos(A-B) - \cos(A+B))$

$= \frac{1}{2}\int(\cos(m-n)x - \cos(m+n)x)dx$

$= \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{(m+n)}\right] + c$

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7. Find $\int \frac{(\log x)^2}{x} dx$

Sol: Put, $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore \int \frac{(\log x)^2}{x}dx = \int t^2 dt = \frac{t^3}{3} + c = \frac{1}{3}(\log x)^3 + c$

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8. Evaluate $\int_{-\pi/2}^{\pi/2} \sin|x|\,dx$

Sol: $f(x) = \sin|x| \Rightarrow f(-x) = \sin|-x| = \sin|x| = f(x)$

$\therefore$ f(x) is an even function

$\int_{-\pi/2}^{\pi/2} \sin|x|\,dx = 2\int_0^{\pi/2} \sin x\,dx = 2[-\cos x]_0^{\pi/2} = -2[\cos\frac{\pi}{2} - \cos 0] = -2(0-1) = 2$

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9. Find $\int_0^{2\pi} \sin^2 x \cos^4 x\,dx$

Sol: Let, $f(x) = \sin^2 x \cos^4 x$ Here, $f(2\pi - x) = f(\pi - x) = f(x)$

$\therefore \int_0^{2\pi} \sin^2 x \cos^4 x\,dx = 2\int_0^{\pi} \sin^2 x \cos^4 x\,dx = 2\times 2\int_0^{\pi/2} \sin^2 x \cos^4 x\,dx = 4\times\frac{[1][(3)(1)]}{(6)(4)(2)}\times\frac{\pi}{2} = \frac{\pi}{8}$

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10. Find the I.F of the D.E $(\cos x)\frac{dy}{dx} + y\sin x = \tan x$ by transforming it into linear form.

Sol: Given D.E is written as $\frac{dy}{dx} + y(\tan x) = \sec x.\tan x$.

This is a linear D.E in y.

Here P= tanx.

$I.F = e^{\int P dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$

SECTION-B

11. Find the condition that the tangents drawn from the exterior point (0,0) to $S = x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular to each other.

Sol: If θ is the angle between the pair of tangents from P(0, 0) to S = 0 then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$

$$\Rightarrow \tan\frac{90^0}{2} = \frac{\sqrt{g^2+f^2-c}}{\sqrt{0^2+0^2+2g(0)+2f(0)+c}} \Rightarrow \tan 45^0 = 1 = \frac{\sqrt{g^2+f^2-c}}{\sqrt{c}}$$

$$\Rightarrow c = g^2+f^2-c \Rightarrow g^2+f^2 = 2c$$

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12. Show that the circles $x^2 + y^2 - 8x - 2y + 8 = 0$, $x^2 + y^2 - 2x + 6y + 6 = 0$ touch each other and find the point of contact.

Sol: For the circle $S=x^2+y^2-8x-2y+8=0$; Centre $C_1=(4,1)$; Radius $r_1 = \sqrt{16+1-8} = \sqrt{9} = 3$

For the circle $S'=x^2+y^2-2x+6y+6=0$; Centre $C_2=(1,-3)$; Radius $r_2 = \sqrt{1+9-6} = \sqrt{4} = 2$

$$C_1C_2 = \sqrt{(4-1)^2+(1+3)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Also, $r_1 + r_2 = 3 + 2 = 5$. Here, $C_1C_2 = r_1 + r_2$

∴ The two circles touch each other externally.

Also the point of contact I divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 3:2$ internally.

$$\therefore \text{Point of contact } I = \left(\frac{3(1)+2(4)}{3+2}, \frac{3(-3)+2(1)}{3+2}\right) = \left(\frac{11}{5}, \frac{-7}{5}\right)$$

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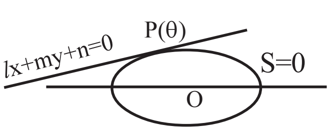
13. Find the condition for the line $lx+my+n=0$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol: Let, $lx + my + n = 0$ be the tangent at P(θ) =(a cosθ, b sinθ) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

∴ Equation of the tangent at P(θ) is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

Comparing the above equation with $lx + my = -n$, we get



$$\frac{\cos\theta}{al} = \frac{\sin\theta}{bm} = \frac{-1}{n} \Rightarrow \cos\theta = -\frac{al}{n}, \sin\theta = \frac{-bm}{n}$$

Now $\cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{a^2l^2}{n^2}+\frac{b^2m^2}{n^2} = 1 \Rightarrow a^2l^2 + b^2m^2 = n^2$

14. Prove that the equation of the chord joining the points α and β on the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

Proof: Given points on Ellipse are $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$

Slope of PQ = $\frac{b\sin\alpha-b\sin\beta}{a\cos\alpha-a\cos\beta}=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$

Equation of PQ is $y-b\sin\alpha=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}(x-a\cos\alpha)$

$$\Rightarrow \frac{x-a\cos\alpha}{a}(\sin\alpha-\sin\beta)=\frac{y-b\sin\alpha}{b}(\cos\alpha-\cos\beta)$$

$$\Rightarrow \left(\frac{x}{a}-\cos\alpha\right)2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=\left(\frac{y}{b}-\sin\alpha\right)(-2)\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \left(\frac{x}{a}-\cos\alpha\right)\cos\frac{\alpha+\beta}{2}+\left(\frac{y}{b}-\sin\alpha\right)\sin\frac{\alpha+\beta}{2}=0$$

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\alpha\cos\frac{\alpha+\beta}{2}+\sin\alpha\sin\frac{\alpha+\beta}{2}$$

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\left(\alpha-\frac{\alpha+\beta}{2}\right)\Rightarrow \frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\left(\frac{\alpha-\beta}{2}\right)$$

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15. Prove that the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ lies on the circle $x^2+y^2=a^2-b^2$.

Sol: Let, the equation of the hyperbola be $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$

Let, $P(x_1,y_1)$ be a point on the locus.

The equation of the tangent with slope m to the hyperbola S=0 is $y=mx\pm\sqrt{a^2m^2-b^2}$

If it passes through $P(x_1,y_1)$ then $y_1-mx_1=\pm\sqrt{a^2m^2-b^2}$

$\Rightarrow (y_1-mx_1)^2=a^2m^2-b^2 \Rightarrow y_1^2-2mx_1y_1+m^2x_1^2=a^2m^2-b^2$

$\Rightarrow (x_1^2-a^2)m^2-2mx_1y_1+(y_1^2+b^2)=0$ ....(1)

(1) is a quadratic equation in 'm' and its roots be taken as m_1, m_2.

Here, m_1, m_2 represent the slopes of the tangents.

If the tangents intersect at right angle then $m_1m_2=-1$

From (1) product of roots $\frac{y_1^2+b^2}{x_1^2-a^2}=-1\Rightarrow x_1^2+y_1^2=a^2-b^2$

| Product of roots of $ax^2+bx+c=0$ is c/a

The locus of $P(x_1, y_1)$ is $x^2+y^2=a^2-b^2$, which is called the director circle.

16. Find the area of the region enclosed by the curves $y = 4x - x^2$, $y = 5 - 2x$.

Sol: Given that, $y = 4x - x^2$, $y = 5 - 2x$

Solving $4x - x^2 = 5 - 2x \Rightarrow x^2 - 6x + 5 = 0 \Rightarrow (x-1)(x-5) = 0 \Rightarrow x = 1, 5$

Required area $A = \int_1^5 [(4x - x^2) - (5 - 2x)]dx = \int_1^5 (6x - x^2 - 5)dx = \left[3x^2 - \frac{x^3}{3} - 5x\right]_1^5$

$$= \left(75 - \frac{125}{3} - 25\right) - \left(3 - \frac{1}{3} - 5\right) = \frac{25}{3} + \frac{7}{3} = \frac{32}{3} \text{sq.units}$$

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17. Solve $y^2dx + (x^2 - xy)dy = 0$

Sol: Given D.E is $y^2dx + (x^2 - xy)dy = 0 \Rightarrow (x^2 - xy)dy = -y^2dx$

$\Rightarrow \frac{dy}{dx} = -\frac{y^2}{x^2 - xy} = \frac{y^2}{xy - x^2}$ ....(1). This is a homogeneous D.E

Put, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore (1) \Rightarrow v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2v - x^2} = \frac{v^2}{v-1} \Rightarrow x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1} = \frac{v}{v-1} \Rightarrow \frac{v-1}{v}dv = \frac{dx}{x}$

$\Rightarrow \int\left(1 - \frac{1}{v}\right)dv = \int\frac{dx}{x} \Rightarrow v - \log v = \log x + \log c \Rightarrow v = \log v + \log x + \log c = \log(cvx)$

$\Rightarrow e^v = cvx \Rightarrow e^{\frac{y}{x}} = c\frac{y}{x}x = cy \quad \therefore e^{\frac{y}{x}} = cy$

SECTION-C

18. Find the values of c if the points (1, 2), (3, –4), (5, –6), (c, 8) are concyclic.

Sol: Let the equation of the required circle be $S = x^2 + y^2 + 2gx + 2fy + k = 0$

(1, 2) lies on S=0 $\Rightarrow 1 + 4 + 2g(1) + 2f(2) + k = 0 \Rightarrow 2g + 4f + k + 5 = 0$ ....(1)

(3, –4) lies on S=0 $\Rightarrow 9 + 16 + 2g(3) + 2f(-4) + k = 0 \Rightarrow 6g - 8f + k + 25 = 0$ ....(2)

(5, –6) lies on S = 0 $\Rightarrow 25 + 36 + 2g(5) + 2f(-6) + k = 0 \Rightarrow 10g - 12f + k + 61 = 0$ .....(3)

(2) – (1) $\Rightarrow 4g - 12f + 20 = 0$ ....(4)

(3) – (2) $\Rightarrow 4g - 4f + 36 = 0$ ....(5)

(4) – (5) $\Rightarrow -8f - 16 = 0 \Rightarrow -8f = 16 = 0 \Rightarrow f = -2$

From (4) we get, $4g - 12(-2) + 20 = 0 \Rightarrow 4g + 24 = -20 \Rightarrow 4g = -44 \Rightarrow g = -11$

(1) $\Rightarrow 2(-11) + 4(-2) + k + 5 = 0 \Rightarrow -22 - 8 + k + 5 = 0 \Rightarrow k = 25$

Substituting the values of $g = -11$, $f = -2$ and $k = 25$ in $x^2 + y^2 + 2gx + 2fy + k = 0$, we get

$$x^2 + y^2 + 2(-11)x + 2(-2)y + 25 = 0 \Rightarrow x^2 + y^2 - 22x - 4y + 25 = 0$$

$\because$ the given 4 points are concyclic, (c, 8) lies on the above circle.

$\Rightarrow c^2 + 8^2 - 22(c) - 4(8) + 25 = 0 \Rightarrow c^2 - 22c + 57 = 0$

$\Rightarrow c^2 - 3c - 19c + 57 = 0 \Rightarrow c(c-3) - 19(c-3) = 0 \Rightarrow (c-19)(c-3) = 0$

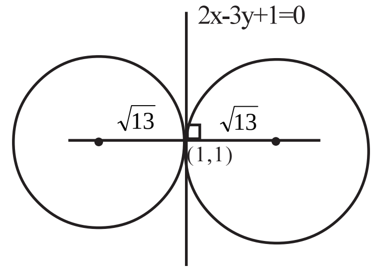
$\Rightarrow c = 19$ (or) $c = 3$

19. Find the equations of circles which touch $2x - 3y + 1 = 0$ at (1, 1) and having radius $\sqrt{13}$

Sol: The slope of the line $2x - 3y + 1 = 0$ is 2/3

$\Rightarrow$ the slope of its perpendicular is $-3/2$ $\Rightarrow \tan\theta = -3/2$

$$\Rightarrow \sin\theta = \frac{3}{\sqrt{13}}, \ \cos\theta = \frac{-2}{\sqrt{13}} \quad (\because \theta \in Q_2)$$



The centres of the required circles are at a distance $\sqrt{13}$ from (1, 1) on the line perpendicular to the given line.

Thus, the centres are given by $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

$$\Rightarrow \left(1+\sqrt{13}\left(\frac{-2}{\sqrt{13}}\right), 1+\sqrt{13}.\frac{3}{\sqrt{13}}\right) = (-1,4)$$

$$\left(1-\sqrt{13}\left(\frac{-2}{\sqrt{13}}\right), 1-\sqrt{13}\left(\frac{3}{\sqrt{13}}\right)\right) = (3,-2)$$

$\therefore$ The equation of the circle with centre (–1,4), radius $\sqrt{13}$ is

$$(x+1)^2 + (y-4)^2 = 13 \Rightarrow x^2 + y^2 + 2x - 8y + 4 = 0$$

$\therefore$ The equation of the circle with centre (3,–2), radius $\sqrt{13}$ is

$$(x-3)^2 + (y+2)^2 = 13 \Rightarrow x^2 + y^2 - 6x + 4y = 0$$

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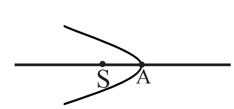
20. Find the coordinates of vertex, focus, equation of the directrix and axis for the parabola $y^2 + 4x + 4y - 3 = 0$

Sol: The given equation is $y^2 + 4x + 4y - 3 = 0$ $\Rightarrow y^2 + 4y = -4x + 3$

$\Rightarrow y^2 + 4y + 4 = -4x + 7 \Rightarrow (y+2)^2 = -4\left(x - \frac{7}{4}\right)$ This is a horizontal left side parabola.

Comparing it with $(y - k)^2 = -4a(x - h)$ we get $4a = 4$ $\Rightarrow a = 1$

Also, $h = 7/4$, $k = -2$



(i) Vertex $= (h,k) = \left(\frac{7}{4}, -2\right)$

(ii) Focus $= (h-a, k) = \left(\frac{7}{4} - 1, -2\right) = \left(\frac{3}{4}, -2\right)$

(iii) Equation of the directrix is $x = h + a \Rightarrow x = \frac{7}{4} + 1 = \frac{7+4}{4} = \frac{11}{4} \Rightarrow 4x = 11 \Rightarrow 4x - 11 = 0$

(iv) Equation of the axis is $y = k$ $\Rightarrow y = -2$ $\Rightarrow y + 2 = 0$

21. Evaluate $\int \frac{1}{1+\sin x+\cos x}dx.$

Sol: Put, $\tan\frac{x}{2}=t$ then $\sin x=\frac{2t}{1+t^2}$; $\cos x=\frac{1-t^2}{1+t^2}$ and $dx=\frac{2dt}{1+t^2}$

$$\therefore \int\frac{1}{1+\sin x+\cos x}dx=\int\frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}.\left(\frac{2dt}{1+t^2}\right)=\int\frac{1+t^2}{1+t^2+2t+1-t^2}.\frac{2dt}{1+t^2}$$

$$=2\int\frac{1}{2+2t}dt=\int\frac{1}{1+t}dt=\log|1+t|+c=\log\left|1+\tan\left(\frac{x}{2}\right)\right|+c$$

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22. Obtain the reduction formula for $I_n=\int \text{cosec}^n x\,dx$, n being a positive integer, $n\ge 2$ and deduce that the value of $\int \text{cosec}^5 x\,dx$

Sol: Let, $I_n=\int \text{cosec}^{n-2}x\,\text{cosec}^2 x\,dx=-\text{cosec}^{n-2}x\cot x-\int -\cot x(n-2)\text{cosec}^{n-3}x(-\text{cosec}x\cot x)dx$

$$=-\text{cosec}^{n-2}x\cot x-(n-2)\int \text{cosec}^{n-2}x\,\cot^2 x\,dx$$

$$=-\text{cosec}^{n-2}x\cot x-(n-2)\int \text{cosec}^{n-2}x\,(\text{cosec}^2 x-1)dx$$

$$=-\text{cosec}^{n-2}x\cot x-(n-2)\int \text{cosec}^{n}x\,dx+(n-2)\int \text{cosec}^{n-2}x\,dx$$

$$=-\text{cosec}^{n-2}x\cot x-(n-2)I_n+(n-2)I_{n-2}$$

$$I_n(1+n-2)=-\text{cosec}^{n-2}x\cot x+(n-2)I_{n-2}\Rightarrow I_n=-\frac{\text{cosec}^{n-2}x\cot x}{n-1}+\frac{n-2}{n-1}I_{n-2}$$

By putting n = 5 in the above recurrence relation, we have

$$I_5=\int \text{cosec}^5 x\,dx=\frac{-\text{cosec}^3 x\cot x}{4}+\frac{3}{4}I_3=-\frac{\text{cosec}^3 x\cot x}{4}+\frac{3}{4}\left[\frac{-\text{cosec}x+\cot x}{2}+\frac{1}{2}I_1\right]$$

$$=-\frac{\text{cosec}^3 x\cot x}{4}-\frac{3\,\text{cosec}\,x\cot x}{8}+\frac{3}{8}\int \text{cosec}\,x\,dx$$

$$=-\frac{\text{cosec}^3 x\cot x}{4}-\frac{3\,\text{cosec}\,x\cot x}{8}+\frac{3}{8}\log|\text{cosec}\,x-\cot x|+c$$

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23. Evaluate $\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$

Sol: We know $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$$\therefore I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx - \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx - I$$

$$\Rightarrow I + I = 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx = \pi \int_0^{\pi} \frac{1+\sin x - 1}{1+\sin x} dx = \pi \left[\int_0^{\pi} \left(1 - \frac{1}{1+\sin x}\right) dx \right]$$

$$= \pi \int_0^{\pi} 1\, dx - \pi \int_0^{\pi} \frac{1}{1+\sin x} dx = \pi[x]_0^{\pi} - \pi \int_0^{\pi} \frac{1}{1+\sin x} dx = \pi^2 - \pi \int_0^{\pi} \frac{1}{1+\sin x} dx \quad ....(1)$$

Now, $\int_0^{\pi} \frac{1}{1+\sin x} dx = \int_0^{\pi} \frac{1-\sin x}{(1+\sin x)(1-\sin x)} dx = \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx = \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$

$$= \int_0^{\pi} (\sec^2 x - \sec x \tan x)\, dx = [\tan x - \sec x]_0^{\pi} = \tan \pi - \sec \pi - \tan 0 + \sec 0$$

$= 0-(-1)-0+1=2$ .....(2)

Hence from (1) & (2), $2I = \pi^2 - 2\pi$ $\quad \therefore I = \frac{\pi^2}{2} - \pi$

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24. Solve the differential equation (2x + y + 1) dx + (4x + 2y – 1) dy = 0.

Sol: Given that, $(4x + 2y - 1)\, dy = -(2x + y + 1)\, dx \Rightarrow \frac{dy}{dx} = -\left(\frac{2x+y+1}{4x+2y-1}\right)$ ....(1)

Comparing, $\frac{2x+y+1}{4x+2y-1}$ with $\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$

We get, $\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}; \frac{b_1}{b_2} = \frac{1}{2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$

Here, we take the substitution $2x + y = v \Rightarrow 2 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 2$

$$\therefore \left(\frac{dv}{dx} - 2\right) = -\left(\frac{v+1}{2v-1}\right) \Rightarrow \frac{dv}{dx} = 2 - \left(\frac{v+1}{2v-1}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{2(2v-1)-(v+1)}{2v-1} = \frac{4v-2-v-1}{2v-1} = \frac{3v-3}{2v-1} = 3\left(\frac{v-1}{2v-1}\right) \Rightarrow \frac{dx}{dv} = \frac{1}{3}\left(\frac{2v-1}{v-1}\right)$$

$$\Rightarrow dx = \frac{1}{3}\left(\frac{2v-1}{v-1}\right) dv = \frac{1}{3}\left(\frac{2v-2+1}{v-1}\right) dv = \frac{1}{3}\left(\frac{2(v-1)+1}{v-1}\right) dv = \frac{1}{3} \int \left(2 + \frac{1}{v-1}\right) dv$$

Integrating on both sides, we have

$$\Rightarrow x = \frac{1}{3}(2v + \log(v-1)) + c \Rightarrow 3x = (2(2x+y) + \log(2x+y-1)) + c$$

$\Rightarrow 3x = 4x + 2y + \log(2x+y-1) + c \Rightarrow x + 2y + \log(2x+y-1) + c = 0$