SOLVED PAPER - 4

MATHS- IB

Time: 3 Hours **Max. Marks : 75**

SECTION - A

I. Answer ALL the following Very Short Answer Questions: **10 × 2 = 20**

1. Find the value of k, if the straight lines $y - 3kx + 4 = 0$ and $(2k - 1)x - (8k - 1)y - 6 = 0$ are perpendicular.
2. Find the equation of the straight line passing through the point (–2, 4) and making intercepts, whose sum is zero
3. Find the ratio in which the point C(6, –17, –4) divides the line segment joining the points A(2, 3,4) and B(3,–2,2).
4. Reduce the equation $x+2y-3z-6=0$ of the plane to the normal form.
5. Evaluate $\lim_{x\to 0} \frac{\cos ax - \cos bx}{x^2}$
6. Find $\lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right)$
7. If $y = \log(\cosh 2x)$, then find $\frac{dy}{dx}$
8. Find the derivative of $\text{Tan}^{-1}\left(\frac{a-x}{1+ax}\right)$
9. If $y=x^2+x$, $x=10$, $\Delta x=0.1$, then find Δy and dy
10. State Rolle's Theorem.

SECTION - B

II. Answer any FIVE of the following Short Answer Questions: **5 × 4 = 20**

11. Find the equation of locus of P, if A=(4,0), B=(–4,0) and |PA–PB|=4
12. When the axes are rotated through an angle $\pi/4$, find the transformed equation of $3x^2+10xy+3y^2=9$
13. If the straight lines $ax+by+c=0$, $bx+cy+a=0$ and $cx+ay+b=0$ are concurrent, then prove that $a^3+b^3+c^3=3abc$
14. If f is given by $f(x) = \begin{cases} k^2x - k & \text{if } x \geq 1 \\ 2 & \text{if } x < 1 \end{cases}$ is a continuous function on R, then find k.
15. Find the derivative of sin2x from the first principles.
16. Show that at any point (x,y) on the curve $y=be^{x/a}$, the length of subtangent is a constant and the length of the subnormal is y^2/a.
17. A container in the shape of an inverted cone has height 8 m and radius 6 m at the top. If it is filled with water at the rate of $2m^3$/min, what is the rate of change in the height of water level when the tank is filled 4 m?

SECTION - C

III. Answer any FIVE of the following Long Answer Questions: **5 × 7 = 35**

18. Find the circumcentre of the triangle whose sides are given by $x+y+2=0$, $5x-y-2=0$ and $x-2y+5=0$
19. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines then P.T (i) $h^2=ab$ (ii) $af^2=bg^2$

    (iii) the distance between the parallel lines is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$ (or) $2\sqrt{\frac{f^2-bc}{b(a+b)}}$
20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line $x+2y=k$ are mutually perpendicular.
21. If a ray makes angle $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube then S.T $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3$.
22. Find the derivative $\frac{dy}{dx}$ of the function $y = \frac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1+7x)^{-6/7}}$.
23. Find the angle between the curves $x + y + 2 = 0$ and $x^2 + y^2 - 10y = 0$
24. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be 20 feet then find the maximum area.

SOLUTIONS

SECTION -A

1. Find the value of k, if the straight lines y–3kx+4=0 and (2k–1)x–(8k–1)y–6=0 are perpendicular.

Sol: The equation of the line is y–3kx+4=0 ⇒ 3kx–y–4=0 ⇒ Slope $m_1 = \frac{3k}{1} = 3k$

The equation of the other line is (2k–1)x–(8k–1)y–6=0 ⇒ slope $m_2 = \frac{2k-1}{8k-1}$

The two lines are perpendicular $\Rightarrow m_1 m_2 = -1 \Rightarrow (3k)\left(\frac{2k-1}{8k-1}\right) = -1$

$\Rightarrow (3k)(2k-1) = -(8k-1) \Rightarrow 6k^2-3k = -8k=1 \Rightarrow 6k^2+5k-1=0$

$\Rightarrow 6k^2+6k-k-1=0 \Rightarrow 6k(k+1)-(k+1)=0 \Rightarrow (6k-1)(k+1)=0 \Rightarrow k=1/6 \text{ or } -1$

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2. Find the equation of the straight line passing through the point (–2, 4) and making intercepts, whose sum is zero

Sol : Let the intercepts made by the required line be a, –a

Then the equation of the line is $\frac{x}{a} + \frac{y}{(-a)} = 1 \Rightarrow x - y = a$

But this line should pass through (–2,4) ⇒ –2–4=a ⇒ a= –6

∴ the equation of the required line is x–y= –6 ⇒ x–y+6=0

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3. Find the ratio in which the point C(6, –17,–4) divides the line segment joining the points A(2,3,4) and B(3,–2,2).

Sol: The ratio in which the point (x,y,z) divides the line segment joining $A(x_1,y_1,z_1)$,$B((x_2,y_2,z_2)$ is $\frac{x_1-x}{x-x_2}$

Hence, the ratio in which the point C(6, –17,–4) divides the line segment joining the points

A(2,3,4) and B(3,–2,2) is $\frac{x_1-x}{x-x_2} = \frac{2-6}{6-3} = \frac{-4}{3}$

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4. Reduce the equation x+2y–3z–6=0 of the plane to the normal form.

Sol: Then given equation is x+2y–3z–6=0 ⇒ x+2y–3z=6

dividing by $\sqrt{1^2+2^2+(-3)^2} = \sqrt{1+4+9} = \sqrt{14}$, we have

$\frac{x}{\sqrt{14}} + \frac{2y}{\sqrt{14}} + \frac{3z}{-\sqrt{14}} = \frac{6}{\sqrt{14}}$, which is in the normal form *l*x+my+nz=p

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5. Evaluate $\lim_{x\to 0} \frac{\cos ax - \cos bx}{x^2}$

Sol: $\lim_{x\to 0} \frac{\cos ax - \cos bx}{x^2} = \lim_{x\to 0} \frac{2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{bx-ax}{2}\right)}{x^2}$ $\left(\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right)$

$$= 2 \underset{x\to 0}{Lt}\left(\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\frac{\sin\left(\frac{b-a}{2}\right)x}{x}\right) = 2\left(\underset{x\to 0}{Lt}\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\underset{x\to 0}{Lt}\frac{\sin\left(\frac{b-a}{2}\right)x}{x}\right)$$

$$= 2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) = \frac{b^2-a^2}{2} \qquad \left(\because \underset{x\to 0}{Lt}\frac{\sin kx}{x} = k\right)$$

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6. Find $\underset{x\to\infty}{Lt}\left(\sqrt{x+1}-\sqrt{x}\right)$

Sol: $$\underset{x\to\infty}{Lt}\left(\sqrt{x+1}-\sqrt{x}\right) = \underset{x\to\infty}{Lt}\frac{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\sqrt{x+1}+\sqrt{x}} = \underset{x\to\infty}{Lt}\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$

$$= \underset{x\to\infty}{Lt}\frac{1}{\sqrt{x+1}+\sqrt{x}} = \underset{x\to\infty}{Lt}\frac{\frac{1}{\sqrt{x}}}{\sqrt{\frac{x}{x}+\frac{1}{x}}+\sqrt{\frac{x}{x}}} = \underset{x\to\infty}{Lt}\frac{\frac{1}{\sqrt{x}}}{\sqrt{1+\frac{1}{x}}+1} = \frac{0}{\sqrt{1+0}+1} = \frac{0}{2} = 0$$

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7. If y=log(cosh2x), then find $\frac{dy}{dx}$

Sol : $$\frac{dy}{dx} = \frac{d}{dx}(\log\cosh 2x) = \frac{1}{\cosh 2x}\frac{d}{dx}(\cosh(2x)) = \frac{1}{\cosh 2x}(\sinh 2x)(2) = 2\tanh 2x$$

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8. Find the derivative of $Tan^{-1}\left(\frac{a-x}{1+ax}\right)$

Sol: $$\frac{d}{dx}Tan^{-1}\left(\frac{a-x}{1+ax}\right) = \frac{d}{dx}\left(Tan^{-1}a - Tan^{-1}x\right) = \frac{d}{dx}Tan^{-1}a - \frac{d}{dx}Tan^{-1}x = 0 - \frac{1}{1+x^2} = \frac{-1}{1+x^2}$$

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9. If $y = x^2 + x$, $x = 10$, $\Delta x = 0.1$, then find Δy and dy

Sol: Let, $y=f(x)=x^2+x$ and $x=10$, $\Delta x=0.1$

(i) $\Delta y= f(x+\Delta x)-f(x)=(x+\Delta x)^2+(x+\Delta x)-x^2-x =x^2+2x\Delta x+(\Delta x)^2+x+\Delta x-x^2-2x$

$= \Delta x(\Delta x+2x+1)= 0.1\ (0.1+2(10)+1) = (0.1)(21.1) = 2.11$

(ii) $dy= f'(x)\Delta x=(2x+1)\Delta x=(2(10)+1)(0.1))= 21(0.1) = 2.1$

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10. State Rolle's Theorem.

Sol: **Rolle's Theorem:** Let $f:[a,b]\to R$ be a function satisfying the following conditions.

(i) f is continuous on [a,b]

(ii) f is differentiable on (a,b) and

(iii) f(a) = f(b)

then there exist at least one $c\in(a,b)$ such that $f'(c)=0$

SECTION-B

11. Find the equation of locus of P, if A=(4,0), B=(–4,0) and |PA–PB|=4

Sol: Let A=(4,0), B=(–4,0) and P(x,y) be any point on the locus

The given condition is $|PA-PB|=4 \Rightarrow PA-PB=\pm 4$

$\Rightarrow PA=4\pm PB \Rightarrow PA^2=(4\pm PB)^2 \Rightarrow PA^2=16+PB^2\pm 8PB \Rightarrow PA^2-PB^2=16\pm 8PB$

$\Rightarrow ((x-4)^2+(y-0)^2)-((x+4)^2+(y-0)^2)=16\pm 8PB$

$\Rightarrow (x-4)^2+y^2-(x+4)^2-y^2=16\pm 8PB \Rightarrow (x-4)^2-(x+4)^2=16\pm 8PB$

$\Rightarrow -4(x)(4)=16\pm 8PB \quad [\because (a-b)^2-(a+b)^2=-4ab]$

$\Rightarrow -16x=16\pm 8PB$

$\Rightarrow \pm 8PB=16+16x \Rightarrow \pm 8PB=16(x+1) \Rightarrow \pm PB=2(x+1) \Rightarrow PB^2=4(x+1)^2$

$\Rightarrow (x+4)^2+(y-0)^2=4(x^2+2x+1) \Rightarrow x^2+8x+16+y^2=4x^2+8x+4 \Rightarrow 3x^2-y^2-12=0$

$\therefore$ the equation of locus of P(x,y) is $3x^2-y^2-12=0$

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12. When the axes are rotated through an angle π/4, find the transformed equation of $3x^2+10xy+3y^2=9$ IPE'14

Sol: The angle of rotation is $\theta=\pi/4=45^o$

$$x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos 45^o - Y\sin 45^o = X\left(\frac{1}{\sqrt{2}}\right) - Y\left(\frac{1}{\sqrt{2}}\right) = \frac{X-Y}{\sqrt{2}}$$

$$y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos 45^o + X\sin 45^o = Y\left(\frac{1}{\sqrt{2}}\right) + X\left(\frac{1}{\sqrt{2}}\right) = \frac{X+Y}{\sqrt{2}}$$

$\therefore$ tranformed equation of $3x^2+10xy+3y^2=9$ is

$$3\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X-y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 3\left(\frac{X+Y}{\sqrt{2}}\right)^2 - 9 = 0$$

$$\Rightarrow 3\frac{X^2-2XY+Y^2}{2} + 10\frac{X^2-Y^2}{2} + 3\frac{X^2+2XY+Y^2}{2} - 9 = 0$$

$$\Rightarrow 3X^2 - 6XY + 3Y^2 + 10X^2 - 10Y^2 + 3X^2 + 6XY + 3Y^2 - 18 = 0$$

$$\Rightarrow 16X^2 - 4Y^2 - 18 = 0 \Rightarrow 2(8X^2 - 2Y^2 - 9) = 0 \Rightarrow 8X^2 - 2Y^2 - 9 = 0$$

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13. If the straight lines ax+by+c=0, bx+cy+a=0 and cx+ay+b=0 are concurrent, then prove that $a^3+b^3+c^3=3abc$

Sol: The given straight lines are $ax+by+c=0$ ....(1)
$bx+cy+a=0$ ....(2)

Solving (1) and (2), we have

$$\frac{x}{ab-c^2} = \frac{y}{bc-a^2} = \frac{1}{ca-b^2} \Rightarrow x = \frac{ab-c^2}{ca-b^2} \text{ and } y = \frac{bc-a^2}{ca-b^2}$$

$\therefore$ point of intersection of lines (1) and (2) is $P\left(\frac{ab-c^2}{ca-b^2}, \frac{bc-a^2}{ca-b^2}\right)$

Since the given lines are concurrent $\Rightarrow$ P lies on the line $cx+ay+b=0$

$$\Rightarrow c\left(\frac{ab-c^2}{ca-b^2}\right)+a\left(\frac{bc-a^2}{ca-b^2}\right)+b=0 \Rightarrow c(ab-c^2)+a(bc-a^2)+b(ca-b^2)=0$$

$$\Rightarrow cab-c^3+abc-a^3+bca-b^3=0 \Rightarrow a^3+b^3+c^3=3abc$$

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14. If f is given by $f(x)=\begin{cases} k^2x-k & \text{if } x\ge 1 \\ 2 & \text{if } x<1 \end{cases}$ is a continuous function on R, then find k.

Sol: Given that f(x) is continuous on R $\Rightarrow$ f(x) is continuous at x=1

At x=1, $f(1)=k^2(1)-k=k^2-k$

$\text{L.H.L} = \underset{x\to 1-}{Lt} f(x) = \underset{x\to 1-}{Lt} 2 = 2$ f(x) is continuous at x=1 $\Rightarrow$ f(1)=L.H.L=R.H.L

$\Rightarrow k^2-k=2 \Rightarrow k^2-k-2=0 \Rightarrow (k-2)(k+1)=0 \Rightarrow k=2$ or -1

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15. Find the derivative of sin2x from the first principles.

Sol: Let f(x)=sin2x $\Rightarrow f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

$$=\underset{h\to 0}{Lt}\frac{1}{h}(\sin 2(x+h)-\sin 2x)=\underset{h\to 0}{Lt}\frac{1}{h}(\sin(2x+2h)-\sin 2x)$$

$$=\underset{h\to 0}{Lt}\frac{1}{h}\left(2\cos\left(\frac{2x+2h+2x}{2}\right)\sin\left(\frac{2x+2h-2x}{2}\right)\right)=2\underset{h\to 0}{Lt}\frac{1}{h}\cos(2x+h)\sinh$$

$$=2\underset{h\to 0}{Lt}\cos(2x+h).\underset{h\to 0}{Lt}\frac{\sinh}{h}=2\cos(2x+0)(1)=2\cos 2x$$

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16. Show that at any point (x,y) on the curve $y=be^{x/a}$, the length of subtangent is a constant and the length of the subnormal is y^2/a.

TS 18

Sol: Given $y=be^{x/a}$, $\Rightarrow m=\frac{dy}{dx}=be^{\frac{x}{a}}\left(\frac{1}{a}\right)=\frac{b}{a}e^{\frac{x}{a}}$

(i)Length of subtangent $\left|\frac{y}{m}\right|=\left|\frac{be^{\frac{x}{a}}}{\frac{b}{a}e^{\frac{x}{a}}}\right|=a=\text{constant}$

(i) Length of subnormal $=|y.m|=\left|be^{\frac{x}{a}}.\frac{b}{a}e^{\frac{x}{a}}\right|=\left|\frac{\left(be^{\frac{x}{a}}\right)^2}{a}\right|=\left|\frac{y^2}{a}\right|$

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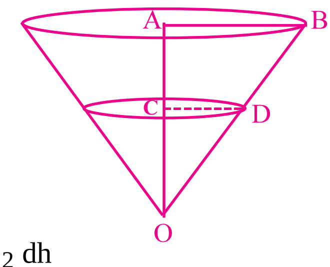
17. A container in the shape of an inverted cone has height 8 m and radius 6 m at the top. If it is filled with water at the rate of $2m^3$/min, what is the rate of change in the height of water level when the tank is filled 4 m?

Sol : Let OC be height of water level at t sec.

Let OC=h, CD=r and volume = V. Given that AB=6, OA= 8 , $\frac{dV}{dt}=2$

The triangles OAB and OCD are similar triangles.

$$\therefore \frac{CD}{AB}=\frac{OC}{OA}\Rightarrow\frac{r}{6}=\frac{h}{8}\Rightarrow r=\frac{3h}{4} \quad ....(1)$$



Volume of the cone V is given by $V=\frac{\pi r^2 h}{3}$ ....(2)

From (1), we have $V=\frac{\pi}{3}\left(\frac{3h}{4}\right)^2\times h=\frac{9\pi h^3}{48}$ ....(3)

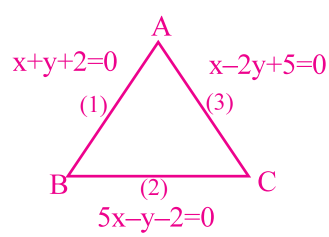
Differentiating (3) w.r.to t, we get $\frac{dV}{dt}=\frac{9\pi}{48}.3h^2\frac{dh}{dt}=\frac{9\pi}{16}h^2\frac{dh}{dt}$

$$\Rightarrow\frac{dh}{dt}=\frac{16}{9\pi h^2}\frac{dV}{dt}=\frac{16}{9\pi 4^2}(2)=\frac{2}{9\pi}=\left(\frac{1}{\pi}\right)\frac{4}{8^2}(12)=\frac{3}{4\pi}cm/sec$$

SECTION-C

18. Find the circumcentre of the triangle whose sides are given by x+y+2=0, 5x–y–2=0 and x–2y+5=0 **IPE'14**

***Sol*:** Let x+y+2=0 .... (1), 5x–y–2=0 .... (2), x–2y+5=0 ....(3)
represent the sides AB,BC,CA of ΔABC respectively



Solving (1) & (3),we get A; $x+y+2=0$
$x-2y+5=0$
$(1)-(3)\Rightarrow\ 3y-3=0\ \Rightarrow y=1$
$\therefore (1)\Rightarrow x+1+2=0\Rightarrow x=-3\quad \therefore A=(-3,1)$

Solving (1) & (2),we get B; $x+y+2=0$
$5x-y-2=0$
$(1)+(2)\Rightarrow 6x=0\ \Rightarrow x=0$
$\therefore (1)\Rightarrow 0+y+2=0\Rightarrow y=-2\quad \therefore B=(0,-2)$

Solving (2) & (3),we get C; $x-2y+5=0$
$(2)\times 2\Rightarrow\ 10x-2y-4=0$
$\Rightarrow\ -9x+9=0\ \Rightarrow 9x=9\Rightarrow x=1$
$\therefore (3)\Rightarrow 1-2y+5=0\Rightarrow 2y=6\Rightarrow y=3\quad \therefore C=(1,3)$

Let S(x,y) be the circumcentre of ΔABC with vertices A(–3,1), B(0,–2), C(1,3)

$$\Rightarrow SA=SB\Rightarrow SA^2=SB^2\Rightarrow(x+3)^2+(y-1)^2=(x-0)^2+(y+2)^2$$

$$\Rightarrow(x^2+6x+9)+(y^2-2y+1)=x^2+(y^2+4y+4)$$

$$\Rightarrow 6x-6y+6=0\Rightarrow 6(x-y+1)=0\Rightarrow x-y+1=0 \quad ....(4)$$

Also $SB=SC\Rightarrow SB^2=SC^2\Rightarrow(x-0)^2+(y+2)^2=(x-1)^2+(y-3)^2$

$$\Rightarrow x^2+(y^2+4y+4)=(x^2-2x+1)+(y^2-6y+9)$$

$$\Rightarrow 2x+10y-6=0\Rightarrow 2(x+5y-3)=0\Rightarrow x+5y-3=0 \quad ....(5)$$

Solving (4) & (5) we get the circumcentre S; (4)–(5)⇒ –6y +4=0⇒ 6y =4⇒ y =2/3

$$(4)\Rightarrow x-\frac{2}{3}+1=0\Rightarrow x=\frac{2}{3}-1=\frac{2-3}{3}=\frac{-1}{3}\quad \therefore \text{ the circumcentre } \Delta ABC \text{ is } S=\left(\frac{-1}{3},\frac{2}{3}\right)$$

19. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines then P.T (i) $h^2=ab$

(ii) $af^2=bg^2$ (iii) the distance between the parallel lines is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$ (or) $2\sqrt{\frac{f^2-bc}{b(a+b)}}$

Sol: Let the 2 parallel lines of the given pair of lines be $lx+my+n_1=0$, $lx+my+n_2=0$

$\Rightarrow ax^2+2hxy+by^2+2gx+2fy+c \equiv (lx+my+n_1)(lx+my+n_2)$

Now, comparing the coefficients of like terms in the above product, we get

$a=l^2$, $h=lm$, $b=m^2$, $2g=l(n_1+n_2)$, $2f=m(n_1+n_2)$, $c=n_1n_2$

Now, (i) $h^2=(lm)^2=l^2m^2=ab \Rightarrow h^2=ab$

$$\text{(ii) } af^2=l^2\left(\frac{m(n_1+n_2)}{2}\right)^2=\frac{l^2m^2(n_1+n_2)^2}{4}=\frac{m^2l^2(n_1+n_2)^2}{4}=m^2\left(\frac{l(n_1+n_2)}{2}\right)^2=bg^2$$

(iii) The distance between the parallel lines $lx+my+n_1=0$, $lx+my+n_2=0$ is $\frac{|n_1-n_2|}{\sqrt{l^2+m^2}}$

$$=\frac{\sqrt{(n_1+n_2)^2-4n_1n_2}}{\sqrt{a+b}}=\sqrt{\frac{\left(\frac{2g}{l}\right)^2-4c}{a+b}}=\sqrt{\frac{\frac{4g^2}{l^2}-4c}{a+b}}=\sqrt{\frac{\frac{4g^2}{a}-4c}{a+b}}=\sqrt{\frac{4g^2-4ac}{a(a+b)}}=2\sqrt{\frac{g^2-ac}{a(a+b)}}$$

Similarly, by considering $n_1+n_2=\frac{2f}{m}$ we get, the distance between the lines $2\sqrt{\frac{f^2-bc}{b(a+b)}}$

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20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line $x+2y=k$ are mutually perpendicular.

Sol: The given line is $x+2y=k \Rightarrow \frac{x+2y}{k}=1$ ...(1)

Now, we homogenise the equation $2x^2-2xy+3y^2+2x-y-1=0$ using (1)

$\Rightarrow 2x^2-2xy+3y^2+2x(1)-y(1)-(1^2)=0$

$\Rightarrow 2x^2-2xy+3y^2+2x\left(\frac{x+2y}{k}\right)-y\left(\frac{x+2y}{k}\right)-\frac{(x+2y)^2}{k^2}=0$, from(1)

$\Rightarrow k^2(2x^2-2xy+3y^2)+k(2x^2+4xy)-k(xy+2y^2)-(x^2+4xy+4y^2)=0$

If the above pair of lines are perpendicular, then sum of coefficients of x^2 and y^2 is zero

$\Rightarrow 2k^2+3k^2+2k-2k-1-4=0$

$\Rightarrow 5k^2-5=0 \Rightarrow 5(k^2-1)=0 \Rightarrow k^2-1=0 \Rightarrow k^2=1 \Rightarrow k=\pm 1$

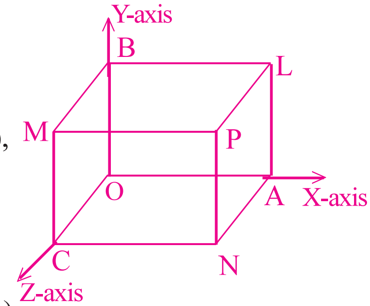
21. If a ray makes angle $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube then show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = 4/3$.

Sol: Let one of the vertices of the cube coincides with the origin O(0,0,0) and edges coincide with the coordinate axes.

Let A,B,C be the vertices of the cube on the x-axis, y-axis, z-axis so that OA=OB=OC=a

The vertices of the cube on xy-plane, yz-plane, zx-plane be L,M,N and P be the remaining vertex in the space.

The 4 diagonals of the cube are $\overleftrightarrow{OP}, \overleftrightarrow{CL}, \overleftrightarrow{NB}, \overleftrightarrow{AM}$



The coordinate of the vertices are A(a,0,0), B(0,a,0), C(0,0,a), L(a,a,0), M(0,a,a), N(a,0,a) and P(a,a,a)

The d.r's of $\overleftrightarrow{OP}$ =(a–0, a–0, a–0) = (a,a,a)=(1,1,1)

The d.r's of $\overleftrightarrow{CL}$ =(a–0, a–0, 0–a)=(a,a,–a)=(1,1,–1)

The d.r's of $\overleftrightarrow{NB}$ =(0–a,a–0,0–a)=(–a, a,–a)=(–1,1,–1)

The d.r's of $\overleftrightarrow{AM}$ =(0–a, a–0, a–0)=(–a,a,a) = (–1,1,1)

Let (l,m,n) be the d.c's of the given ray, then we have $l^2+m^2+n^2=1$

The ray makes angle $\alpha,\beta,\gamma,\delta$ with $\overleftrightarrow{OP}, \overleftrightarrow{CL}, \overleftrightarrow{NB}, \overleftrightarrow{AM}$

$$\Rightarrow \cos\alpha = \frac{1.l+1.m+1.n}{\sqrt{(1^2+1^2+1^2)(l^2+m^2+n^2)}} = \frac{(l+m+n)}{\sqrt{3(1)}} = \frac{(l+m+n)}{\sqrt{3}} \quad \Rightarrow \cos\alpha = \frac{l+m+n}{\sqrt{3}}$$

Similarly, $\cos\beta = \dfrac{l+m-n}{\sqrt{3}}$, $\cos\gamma = \dfrac{-l+m-n}{\sqrt{3}}$, $\cos\delta = \dfrac{-l+m+n}{\sqrt{3}}$

$$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$$

$$= \left(\frac{l+m+n}{\sqrt{3}}\right)^2 + \left(\frac{l+m-n}{\sqrt{3}}\right)^2 + \left(\frac{-l+m-n}{\sqrt{3}}\right)^2 + \left(\frac{-l+m+n}{\sqrt{3}}\right)^2$$

$$= \frac{1}{3}[(l+m+n)^2+(l+m-n)^2+(-l+m-n)^2+(-l+m+n)^2]$$

$$= \frac{1}{3}[(l^2+m^2+n^2+2lm+2mn+2ln)+(l^2+m^2+n^2+2lm-2mn-2ln)+$$
$$(l^2+m^2+n^2-2lm-2mn+2ln)+(l^2+m^2+n^2-2lm+2mn-2nl)]$$

$$= \frac{1}{3}(4l^2+4m^2+4n^2) = \frac{4}{3}(l^2+m^2+n^2) = \frac{4}{3}(1) = \frac{4}{3}$$

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22. Find the derivative $\dfrac{dy}{dx}$ of the function $y = \dfrac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1+7x)^{-6/7}}$.

Sol: Given $y = \dfrac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1+7x)^{-6/7}} \Rightarrow \log y = \log\left[\dfrac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1+7x)^{-6/7}}\right]$

We know that $\log ab = \log a + \log b, \log\dfrac{a}{b} = \log a - \log b$

$\therefore \log y = \log (1-2x)^{2/3} + \log (1+3x)^{-3/4} - [\log (1-6x)^{5/6} + \log (1+7x)^{-6/7}]$

$$= \frac{2}{3}\log(1-2x) - \frac{3}{4}.\log(1+3x) - \frac{5}{6}\log(1-6x) + \frac{6}{7}\log(1+7x)$$

Differentiating w.r.to x , $\dfrac{1}{y}.\dfrac{dy}{dx} = \dfrac{2}{3}.\dfrac{1(-2)}{1-2x} - \dfrac{3}{4}.\dfrac{1}{1+3x}.3 - \dfrac{5}{6}.\dfrac{1}{1-6x}.(-6) + \dfrac{6}{7}.\dfrac{1}{1+7x}.7$

$$= \frac{-4}{3(1-2x)} - \frac{9}{4(1+3x)} + \frac{5}{1-6x} + \frac{6}{1+7x} \Rightarrow \frac{dy}{dx} = y\left(\frac{5}{1-6x} + \frac{6}{1+7x} - \frac{4}{3(1-2x)} - \frac{9}{4(1+3x)}\right)$$

23. Find the angle between the curves x+y+2=0 and $x^2+y^2-10y=0$

Sol: Equation of the given curves x+y+2=0 ......(1); $x^2+y^2-10y=0$....(2)

$(y+2)^2+y^2-10y=0 \Rightarrow y^2+4y+4+y^2-10y=0 \Rightarrow 2y^2-6y+4=0 \Rightarrow y^2-3y+2=0 \Rightarrow (y-1)(y-2)=0$

$\Rightarrow y=1$ (or) $y=2$; $y=1 \Rightarrow x=-(1+2)=-3$; $y=2 \Rightarrow x=-(2+2)=-4$

The points of intersection are P(–3,1) and Q(–4,2)

Equation of the curves is $x^2+y^2-10y=0$

Differentiating w.r.to x $2x+2y\frac{dy}{dx}-10\frac{dy}{dx}=0 \Rightarrow \not{2}\frac{dy}{dx}(y-5)=-\not{2}x \Rightarrow \frac{dy}{dx}=\frac{-x}{y-5}$

Equation of the given line is x+y+2=0, differentiating w.r.to x $1+\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-1$

Case (i): At P(–3,1), $m_1=\left(\frac{dy}{dx}\right)_{(-3,1)}=-\frac{x}{y-5}=\frac{3}{1-5}=\frac{-3}{4}, m_2=\left(\frac{dy}{dx}\right)_{(-3,1)}=-1$

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{-\frac{3}{4}+1}{1+\frac{3}{4}}\right|=\frac{1}{7} \Rightarrow \theta=\text{Tan}^{-1}\left(\frac{1}{7}\right)$$

Case(ii): At Q(–4,2), $m_1=\left(\frac{dy}{dx}\right)_{(-4,2)}=\frac{4}{2-5}=-\frac{4}{3}, m_2=\left(\frac{dy}{dx}\right)_{(-4,2)}=-1$

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{-\frac{4}{3}+1}{1+\frac{4}{3}}\right|=\left|\frac{-4+3}{3+4}\right|=\frac{1}{7} \Rightarrow \theta=\text{Tan}^{-1}\left(\frac{1}{7}\right)$$

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24. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be 20 feet then find the maximum area.

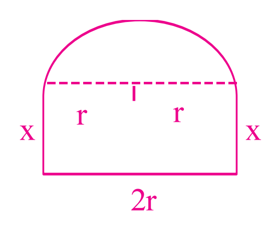
Sol: Let 'r' be the radius of the semi circle

$\Rightarrow$ a side of rectangle is 2r

Let the other side of rectangle be x.

Given that the perimeter of the window is 20

$\Rightarrow 2r+2x+\pi r=20 \Rightarrow 2x=20-2r-\pi r$ .....(1)



Area of the window $A=(2r)x+\frac{\pi r^2}{2}=r(2x)+\frac{\pi r^2}{2}$

$(1) \Rightarrow A(r)=r(20-2r-\pi r)+\frac{\pi r^2}{2}=20r-2r^2-\pi r^2+\frac{\pi r^2}{2}=20r-r^2\left(2+\pi-\frac{\pi}{2}\right)$

$\Rightarrow 20r-r^2\left(\frac{4+\pi}{2}\right) \Rightarrow A'(r)=20-2r\left(\frac{4+\pi}{2}\right)=20-r(4+\pi)$ ....(2)

The extreme values of A(r) are attained when A'(r)=0 $\Rightarrow 20-r(4+\pi)=0 \Rightarrow r=\frac{20}{4+\pi}$

From (2), $A''(r)=0-(4+\pi)<0$ $\therefore$ A(r) has a maximum value

$\therefore$ the maximum area is $A=20\left(\frac{20}{4+\pi}\right)-\frac{400}{(4+\pi)^2}\left(\frac{4+\pi}{2}\right)=\frac{400}{4+\pi}-\frac{200}{4+\pi}=\frac{200}{4+\pi}$ sq.ft