SOLVED PAPER-4

MATHS-1A

Time: 3 Hours Max. Marks : 75

SECTION-A

I. Answer ALL the following VSAQ: **10 × 2 = 20**

1. If $f = \{(1,2),(2,-3),(3,-1)\}$ then find (i) $2 + f$ (ii) $\sqrt{f}$ **2.** Find the domain of $\sqrt{9-x^2}$
3. For any square matrix A, show that AA' is symmetric.
4. Find the determinant of the matrix $\begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{bmatrix}$
5. Find the unit vector in the direction of vector $\bar{a}=2\bar{i}+3\bar{j}+\bar{k}$
6. If vectors $-3\bar{i}+4\bar{j}+\lambda\bar{k}$, $\mu\bar{i}+8\bar{j}+6\bar{k}$ are collinear vectors then find λ & μ.
7. Find the angle between the planes $\bar{r}.(2\bar{i}-\bar{j}+2\bar{k})=3$, $\bar{r}.(3\bar{i}+6\bar{j}+\bar{k})=4$
8. If $0<A<\pi/4$ and $\cos A=4/5$, then find the values of sin2A and cos2A
9. Find the maximum and minimum value of $f(x)=3\sin x-4\cos x$
10. Prove that $\cosh^2 x+\sinh^2 x=\cosh 2x$

SECTION-B

II. Answer any FIVE of the following SAQs: **5 × 4 = 20**

11. Examine whether the following system of equations are consistent or inconsistent and if consistent, find the complete solution. $x + y + z = 1$, $2x + y + z = 2$, $x + 2y + 2z = 1$.
12. $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors. Prove that the following four points are coplanar. $6\bar{a}+2\bar{b}-\bar{c}$, $2\bar{a}-\bar{b}+3\bar{c}$, $-\bar{a}+2\bar{b}-4\bar{c}$, $-12\bar{a}-\bar{b}-3\bar{c}$.
13. Find the volume of the tetrahedron, whose vertices are (1,2,1), (3,2,5), (2,–1,0) and (–1,0,1).
14. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$
15. Solve $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$ **16.** Prove that $\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$
17. If $\sin\theta=\frac{a}{(b+c)}$ then show that $\cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$

SECTION-C

III. Answer any FIVE of the following LAQs: **5 × 7 = 35**

18. If $f:A\to B$ is a bijective function then prove that (i) $fof^{-1}=I_B$ (ii) $f^{-1}of=I_A$
19. Using Mathematical Induction, prove that statement for all $n\in N$

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\cdots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

20. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=abc(a-b)(b-c)(c-a)$

21. Solve the following system of equations by using Cramer's rule. $x-y+3z=5$, $4x+2y-z=0$, $-x+3y+z=5$.
22. If $\bar{a}=2\bar{i}+\bar{j}-3\bar{k}$, $\bar{b}=\bar{i}-2\bar{j}+\bar{k}$, $\bar{c}=-\bar{i}+\bar{j}-4\bar{k}$, and $\bar{d}=\bar{i}+\bar{j}+\bar{k}$, then compute $\left|(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})\right|$.
23. If A,B,C are angles of a triangle, prove that $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$
24. Show that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\frac{r}{2R}$

SOLUTIONS

SECTION-A

1. If $f = \{(1,2),(2,-3),(3,-1)\}$ then find (i) 2+f (ii) $\sqrt{f}$

Sol: Given that $f = \{(1, 2), (2, -3), (3, -1)\}$ $\Rightarrow f(1) = 2, f(2) = -3, f(3) = -1$

(i) $(2 + f)(x) = f(x) + 2$ $\Rightarrow (2 + f)(1) = f(1) + 2 = 2 + 2 = 4; (2 + f)(2) = f(2) + 2 = -3+2 = -1$

$(2 + f)(3) = f(3) + 2 = -1+2 = 1$ Hence, $2 + f = \{(1, 4), (2, -1), (3, 1)\}$

(ii) $\sqrt{f}(x) = \sqrt{f(x)} \Rightarrow \sqrt{f}(1) = \sqrt{f(1)} = \sqrt{2}$; $\sqrt{f}(2) = \sqrt{f(2)} = \sqrt{-3}$ not defined

$\sqrt{f}(3) = \sqrt{f(3)} = \sqrt{-1}$ not defined. Hence, $\sqrt{f} = \{(1, \sqrt{2})\}$

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2. Find the domain of $\sqrt{9 - x^2}$

Sol: $\sqrt{9 - x^2}$ is defined when $9 - x^2 \geq 0$ $\Rightarrow x^2 - 9 \leq 0 \Rightarrow (x+3)(x-3) \leq 0$ $\Rightarrow x \in [-3,3]$

$\therefore$ Domain of f(x) is $[-3,3]$

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3. For any square matrix A, show that AA' is symmetric.

Sol: To show that AA' is Symmetric, we have to show that (AA')'=AA'

L.H.S=(AA')'=(A')'A' [since (AB)' = B'A']

=AA' [since (A')'=A]

= R.H.S

Thus (AA')'=AA'

Hence, AA' is a symmetric matrix.

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4. Find the determinant of the matrix $\begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{bmatrix}$

Sol : Given $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix} = 1(9\times25-16\times16)-4(4\times25-9\times16)+9(16\times4-9\times9)$

$=1(225-256)-4(100-144)+9(64-81)$

$=-31-4(-44)+9(-17)=-31+176-153=176-184=-8=$

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5. Find the unit vector in the direction of vector $\bar{a}=2\bar{i}+3\bar{j}+\bar{k}$

Sol: Given $\bar{a}=2\bar{i}+3\bar{j}+\bar{k}$, then $|\bar{a}| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14}$

$\therefore$ Required unit vector $= \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{2\bar{i}+3\bar{j}+\bar{k}}{\sqrt{14}}$

6. If vectors $-3\bar{i}+4\bar{j}+\lambda\bar{k}$, $\mu\bar{i}+8\bar{j}+6\bar{k}$ are collinear vectors then find λ & μ.

Sol: Given that the vectors $\bar{a}=-3\bar{i}+4\bar{j}+\lambda\bar{k}$, $\bar{b}=\mu\bar{i}+8\bar{j}+6\bar{k}$ are collinear.

$$\therefore \frac{-3}{\mu}=\frac{4}{8}=\frac{\lambda}{6} \Rightarrow \frac{-3}{\mu}=\frac{1}{2} \Rightarrow \mu = 2\times -3=-6 \text{ and } \frac{\lambda}{6}=\frac{1}{2} \Rightarrow \lambda=\frac{6}{2}=3 \qquad \therefore \lambda=3,\ \mu=-6$$

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7. Find the angle between the planes $\bar{r}.(2\bar{i}-\bar{j}+2\bar{k})=3$, $\bar{r}.(3\bar{i}+6\bar{j}+\bar{k})=4$

Sol: Given planes $\bar{r}.(2\bar{i}-\bar{j}+2\bar{k})=3$, $\bar{r}.(3\bar{i}+6\bar{j}+\bar{k})=4$ are in the form $\bar{r}.\bar{n}_1=p_1$, $\bar{r}.\bar{n}_2=p_2$, then

$\bar{n}_1=2\bar{i}-\bar{j}+2\bar{k}$, $\bar{n}_2=3\bar{i}+6\bar{j}+\bar{k}$

$$\therefore \cos\theta=\frac{\bar{n}_1.\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}=\frac{(2\bar{i}-\bar{j}+2\bar{k}).(3\bar{i}+6\bar{j}+\bar{k})}{\sqrt{4+1+4}.\sqrt{9+36+1}}=\frac{2(3)-1(6)+2(1)}{\sqrt{9}.\sqrt{46}}=\frac{6-6+2}{\sqrt{9}.\sqrt{46}}=\frac{2}{3\sqrt{46}}$$

$$\therefore \cos\theta=\frac{2}{3\sqrt{46}} \Rightarrow \theta=\cos^{-1}\frac{2}{3\sqrt{46}}$$

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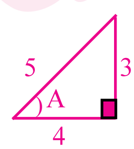
8. If $0<A<\pi/4$ and $\cos A = 4/5$, then find the values of $\sin 2A$ and $\cos 2A$

Sol: Given $0<A<\pi/4 \Rightarrow A$ is in Q_1

Given $\cos A = 4/5 \Rightarrow \sin A=3/5$



(i) $\sin 2A = 2\sin A\cos A = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)=\frac{24}{25}$

(ii) $\cos 2A=\cos^2 A-\sin^2 A=\frac{16}{25}-\frac{9}{25}=\frac{16-9}{25}=\frac{7}{25}$

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9. Find the maximum and minimum value of $f(x) = 3\sin x - 4\cos x$

Sol: Given function is $3\sin x-4\cos x$

On comparing with $a\sin x+b\cos x+c$, we get $a=3$, $b=-4$, $c=0$

Now $\sqrt{a^2+b^2}=\sqrt{3^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5$

$\therefore$ Maximum value $=c+\sqrt{a^2+b^2}=0+5=5$

Minimum value $=c-\sqrt{a^2+b^2}=0-5=-5$

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10. Prove that $\cosh^2 x + \sinh^2 x = \cosh 2x$

Sol: L.H.S $=\cosh^2 x+\sinh^2 x=\left(\frac{e^x+e^{-x}}{2}\right)^2+\left(\frac{e^x-e^{-x}}{2}\right)^2$

$$=\frac{1}{4}[(e^x+e^{-x})^2+(e^x-e^{-x})^2]=\frac{1}{4}\left[2\left[(e^x)^2+(e^{-x})^2\right]\right][\because (a+b)^2+(a-b)^2=2(a^2+b^2)]$$

$$=\frac{2}{4}(e^{2x}+e^{-2x})=\frac{e^{2x}+e^{-2x}}{2}=\cosh 2x = \text{R.H.S}$$

SECTION-B

11. Examine whether the following system of equations are consistent or inconsistent and if consistent, find the complete solution. $x + y + z = 1,\ 2x + y + z = 2,\ x + 2y + 2z = 1$.

Sol: The matrix equation corresponding to the given system of equations be AX=D, where

$$A=\begin{bmatrix}1&1&1\\2&1&1\\1&2&2\end{bmatrix};\ X=\begin{bmatrix}x\\y\\z\end{bmatrix};\ D=\begin{bmatrix}1\\2\\1\end{bmatrix}\Rightarrow \text{The augmented matrix is } [AD]=\begin{bmatrix}1&1&1&1\\2&1&1&2\\1&2&2&1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&1\\2-2&1-2&1-2&2-2\\1-1&2-1&2-1&1-1\end{bmatrix}\begin{matrix}(\because R_2\to R_2-2R_1\\ R_3\to R_3-R_1)\end{matrix}=\begin{bmatrix}1&1&1&1\\0&-1&-1&0\\0&1&1&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&1\\0&-1&-1&0\\0&0&0&0\end{bmatrix}(R_3\to R_3+R_2)\quad \text{.........(1)}$$

Hence, from (1), we observe that $\rho(A)=\rho(AB)=2$

Here, Rank of the coefficient matrix =2 and number of unknowns =3

$\therefore$ The given system of equations is consistent and has infinitely many solutions.

From (1), we have $x+y+z=1$, $-y-z=0 \Rightarrow y=-z$

$\therefore x-z+z=1\Rightarrow x=1$

If $z=k$, $k\in R$ then the complete solution of the given system of equations is $x=1$, $y=-k$, $z=k$

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12. $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors. Prove that the following four points are coplanar.

$6\bar{a}+2\bar{b}-\bar{c},\ 2\bar{a}-\bar{b}+3\bar{c},\ -\bar{a}+2\bar{b}-4\bar{c},\ -12\bar{a}-\bar{b}-3\bar{c}$.

Sol: We take $\overline{OP}=6\bar{a}+2\bar{b}-\bar{c},\ \overline{OQ}=2\bar{a}-\bar{b}+3\bar{c}$,

$\overline{OR}=-\bar{a}+2\bar{b}-4\bar{c},\ \overline{OS}=-12\bar{a}-\bar{b}-3\bar{c}$, where 'O' is the origin.

$\overline{PQ}=\overline{OQ}-\overline{OP}=(2\bar{a}-\bar{b}+3\bar{c})-(6\bar{a}+2\bar{b}-\bar{c})=-4\bar{a}-3\bar{b}+4\bar{c}$

$\overline{PR}=\overline{OR}-\overline{OP}=(-\bar{a}+2\bar{b}-4\bar{c})-(6\bar{a}+2\bar{b}-\bar{c})=-7\bar{a}-3\bar{c}$

$\overline{PS}=\overline{OS}-\overline{OP}=(-12\bar{a}-\bar{b}-3\bar{c})-(6\bar{a}+2\bar{b}-\bar{c})=-18\bar{a}-3\bar{b}-2\bar{c}$

$$[\overline{PQ}\ \overline{PR}\ \overline{PS}]=\begin{vmatrix}-4&-3&4\\-7&0&-3\\-18&-3&-2\end{vmatrix}[\bar{a}\ \bar{b}\ \bar{c}]$$

$=[-4(0-9)+3(14-54)+4(21-0)][\bar{a}\ \bar{b}\ \bar{c}]=[-4(-9)+3(-40)+4(21)=[36-120+84][\bar{a}\ \bar{b}\ \bar{c}]=0[\bar{a}\ \bar{b}\ \bar{c}]=0$

So, $\overline{PQ},\overline{PR},\overline{PS}$ are coplanar.

Hence proved that the four points P,Q,R,S are coplanar.

13. Find the volume of the tetrahedron, whose vertices are (1,2,1), (3,2,5), (2,–1,0) and (–1,0,1).

Sol: Let O be the origin of reference so that $\overline{OA}=\bar{i}+2\bar{j}+\bar{k}$, $\overline{OB}=3\bar{i}+2\bar{j}+5\bar{k}$,

$\overline{OC}=2\bar{i}-\bar{j}$, $\overline{OD}=-\bar{i}+\bar{k}$

Then $\overline{AB}=\overline{OB}-\overline{OA}=(3\bar{i}+2\bar{j}+5\bar{k})-(\bar{i}+2\bar{j}+\bar{k})=2\bar{i}+4\bar{k}$

$\overline{AC}=\overline{OC}-\overline{OA}=(2\bar{i}-\bar{j})-(\bar{i}+2\bar{j}+\bar{k})=\bar{i}-3\bar{j}-\bar{k}$

$\overline{AD}=\overline{OD}-\overline{OA}=(-\bar{i}+\bar{k})-(\bar{i}+2\bar{j}+\bar{k})=-2\bar{i}-2\bar{j}$

Now, $[\overline{AB}\ \overline{AC}\ \overline{AD}]=\begin{vmatrix}2 & 0 & 4\\ 1 & -3 & -1\\ -2 & -2 & 0\end{vmatrix}$

=[2(0–2)+4(–2–6)]=[– 4 –32]= –36

$\therefore$ Volume of the tetrahedron $=\frac{1}{6}|-36|=6$ cubic unit

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14. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$

Sol: $\cos\frac{\pi}{10}=\cos\frac{180^\circ}{10}=\cos18^\circ;\cos\frac{3\pi}{10}=\cos\frac{3(180^\circ)}{10}=\cos54^\circ$

$\cos\frac{7\pi}{10}=\cos\frac{7(180^\circ)}{10}=\cos136^\circ=\cos(180^\circ-54^\circ)=-\cos54^\circ$

$\cos\frac{9\pi}{10}=\cos\frac{9(180^\circ)}{10}=\cos172^\circ=\cos(180^\circ-18^\circ)=-\cos18^\circ$

$\therefore$ L.H.S= $(1+\cos18^\circ)(1+\cos54^\circ)(1-\cos54^\circ)(1-\cos18^\circ)=(1-\cos^2 18^\circ)(1-\cos^2 54^\circ)$

$=\sin^2 18^\circ\sin^2 54^\circ=\left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2=\left(\frac{5-1}{16}\right)^2=\left(\frac{4}{16}\right)^2=\left(\frac{1}{4}\right)^2=\frac{1}{16}=$ R.H.S

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15. Solve $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$

Sol: Given that $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$

Dividing the above equation by $\sqrt{(\sqrt{3})^2+(-1)^2}=\sqrt{3+1}=\sqrt{4}=2$, we have

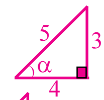
$\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta=\frac{\sqrt{2}}{2}\Rightarrow\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6}=\frac{1}{\sqrt{2}}\Rightarrow\sin\left(\theta-\frac{\pi}{6}\right)=\frac{1}{\sqrt{2}}$

Here, the principal solution is $\theta=\frac{\pi}{6}$

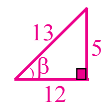
$\therefore$ The general solution is $\theta-\frac{\pi}{6}=n\pi+(-1)^n\frac{\pi}{4}, n\in Z\Rightarrow\theta=n\pi+(-1)^n\frac{\pi}{4}+\frac{\pi}{6}, n\in Z$

16. Prove that $\text{Sin}^{-1}\frac{3}{5}+\text{Cos}^{-1}\frac{12}{13}=\text{Cos}^{-1}\frac{33}{65}$

Sol: Let $\text{Sin}^{-1}\frac{3}{5}=\alpha \Rightarrow \sin\alpha=\frac{3}{5} \Rightarrow \cos\alpha=\frac{4}{5}$



$\text{Cos}^{-1}\frac{12}{13}=\beta \Rightarrow \cos\beta=\frac{12}{13} \Rightarrow \sin\beta=\frac{5}{13}$



Claim: $\alpha+\beta=\text{Cos}^{-1}\frac{33}{65} \Rightarrow \cos(\alpha+\beta)=\frac{33}{65}$

Now, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\frac{4}{5}\text{x}\frac{12}{13}-\frac{3}{5}\text{x}\frac{5}{13}=\frac{48-15}{65}=\frac{33}{65}$

$\Rightarrow \alpha+\beta=\text{Cos}^{-1}\frac{33}{65} \Rightarrow \text{Sin}^{-1}\frac{3}{5}+\text{Cos}^{-1}\frac{12}{13}=\text{Cos}^{-1}\frac{33}{65}$

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17. If $\sin\theta=\frac{a}{(b+c)}$ then show that $\cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$

Sol: Given $\sin\theta=\frac{a}{b+c} \Rightarrow \sin^2\theta=\frac{a^2}{(b+c)^2}$

$\therefore \cos^2\theta=1-\sin^2\theta$ $\quad [\because \sin^2\theta+\cos^2\theta=1]$

$=1-\frac{a^2}{(b+c)^2}=\frac{(b+c)^2-a^2}{(b+c)^2}=\frac{(b^2+c^2+2bc)-a^2}{(b+c)^2}$

$=\frac{2bc+(b^2+c^2-a^2)}{(b+c)^2}=\frac{2bc+2bc\cos A}{(b+c)^2}$ $\quad \left(\because \frac{b^2+c^2-a^2}{2bc}=\cos A\right)$

$=\frac{2bc(1+\cos A)}{(b+c)^2}=\frac{2bc.2\cos^2\frac{A}{2}}{(b+c)^2}=\frac{4bc\cos^2\frac{A}{2}}{(b+c)^2}$ $\quad \therefore \cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\left(\frac{A}{2}\right)$

SECTION-C

18. If f:A→B is a bijective function then prove that (i) $fof^{-1}=I_B$ (ii) $f^{-1}of=I_A$

Sol: (i) To prove that $f\,of^{-1}=I_B$

***Part-1*:** Given f:A→B is a bijective function, then $f^{-1}:B\to A$ is also a bijection

$\therefore fof^{-1}:B\to B$

We know, $I_B:B\to B$. So, fof^{-1} and I_B, both have same domain B

***Part-2*:** For $b\in B$, $(fof^{-1})(b)=f\,[f^{-1}(b)]$

$=f(a)$ $[\because f:A\to B$ is bijection $\Rightarrow f(a)=b \Rightarrow f^{-1}(b)=a$, for $a\in A$]

$=b=I_B(b)$ $[\because I_B(b)=b$, for $b\in B]$

Hence we proved that $fof^{-1}=I_B$

(ii) To prove that $f^{-1}of=I_A$

***Part-1*:** Given f:A→B is a bijective function, then $f^{-1}:B\to A$ is also a bijection

$\therefore f^{-1}of: A\to A$

We know $I_A: A\to A$ So, $f^{-1}of$ and I_A, both have same domain A

***Part-2*:** for $a\in A$, $(f^{-1}of)(a)=f^{-1}[f(a)]$

$=f^{-1}(b)=a$ $[\because f:A\to B$ is a bijection $\Rightarrow f(a)=b \Rightarrow f^{-1}(b)=a$]

$=I_A(a)[\because I_A(a)=a$, for $a\in A]$

Hence we proved that $f^{-1}of=I_A$

19. Using Mathematical Induction, prove that statement for all $n \in N$

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)..........\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

Sol: Let S(n): $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...........\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$

(a) L.H.S of $S(1)=1+\frac{3}{1}=4$, R.H.S of S(1)= $(1+1)^2=4$

$\therefore$ L.H.S = R.H.S $\therefore$ S(1) is true.

(b) Assume that S(k) is true. for $k \in N$.

S(k): $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...........\left(1+\frac{2k+1}{k^2}\right)=(k+1)^2$ ....(1)

(c) Now we show that S(k+1) is true

L.H.S of S(k+1)=

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...........\left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)=(k+1)^2\left(1+\frac{2k+3}{(k+1)^2}\right) \text{ [From (1)]}$$

$=(k+1)^2+2k+3=k^2+4k+4=(k+2)^2=$ R.H.S of S(k+1)

$\therefore$ S(k+1) is true, whenever S(k) is true.

Hence, by the principle of finite Mathematical Induction S(n) is true for all $n \in N$

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20. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

Sol: L.H.S $=\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = abc\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$ $(\because C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1)$

$$=(abc)\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} = (abc)(b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$=(abc)(b-a)(c-a)[(c+\not{a})1-(b+\not{a})1]$

$=(abc)(b-a)(c-a)[(c-b)]$

$=(abc)(a-b)(b-c)(c-a)=$ R.H.S

21. Solve the following system of equations by using Cramer's rule $x - y + 3z = 5,\ 4x + 2y - z = 0,\ x + 3y + z = 5$.

Sol: The matrix equation corresponding to the given system of equations be AX = D, where

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$$

$$\Delta = \det A = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix} = 1(2+3) + 1(4+1) + 3(12-2)$$

$$= 5 + 5 + 30 = 40 \neq 0 \Rightarrow A \text{ is non singular}$$

Hence, we can solve the given equations by using Cramer's rule.

$$\Delta_1 = \begin{vmatrix} 5 & -1 & 3 \\ 0 & 2 & -1 \\ 5 & 3 & 1 \end{vmatrix} = 5(2\times1-(-1)\times3) + 1(0\times1-(-1)\times5) + 3(0\times3+5\times2) = 25+5-30 = 0;$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 & 3 \\ 4 & 0 & -1 \\ 1 & 5 & 1 \end{vmatrix} = 1(0\times1-(-1)\times5) + 5(4\times1-(-1)\times-1) + 3(4\times5-0\times-1) = 5-25+60 = 40;$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 5 \\ 4 & 2 & 0 \\ 1 & 3 & 5 \end{vmatrix} = 1(2\times5+0\times3) + 1(4\times5-0\times1) + 5(4\times3-2\times1) = 10+20+50 = 80$$

Hence by Cramer's rule,

$$x = \frac{\Delta_1}{\Delta} = \frac{0}{40} = 0;\quad y = \frac{\Delta_2}{\Delta} = \frac{40}{40} = 1 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{80}{40} = 2$$

$\therefore$ the solution of the given system of equations is $x = 0,\ y = 1,\ z = 2$

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22. If $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k},\ \bar{b} = \bar{i} - 2\bar{j} + \bar{k},\ \bar{c} = -\bar{i} + \bar{j} - 4\bar{k}$ and $\bar{d} = \bar{i} + \bar{j} + \bar{k}$, then compute $\left|(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})\right|$.

Sol: Given $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k},\ \bar{b} = \bar{i} - 2\bar{j} + \bar{k},\ \bar{c} = -\bar{i} + \bar{j} - 4\bar{k},\ \bar{d} = \bar{i} + \bar{j} + \bar{k}$,

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = \bar{i}(1-6) - \bar{j}(2+3) + \bar{k}(-4-1) = -5\bar{i} - 5\bar{j} - 5\bar{k} \quad \text{............(1)}$$

$$\bar{c}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 1 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \bar{i}(1+4) - \bar{j}(-1+4) + \bar{k}(-1-1) = 5\bar{i} - 3\bar{j} - 2\bar{k} \quad \text{............(2)}$$

From (1) & (2),

$$\therefore (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -5 & -5 & -5 \\ 5 & -3 & -2 \end{vmatrix} = \bar{i}(10-15) - \bar{j}(10+25) + \bar{k}(15+25) = -5\bar{i} - 35\bar{j} + 40\bar{k}$$

$$= 5(-\bar{i} - 7\bar{j} + 8\bar{k})$$

$$\therefore |(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})| = 5\sqrt{(-1)^2 + (-7)^2 + 8^2} = 5\sqrt{1+49+64} = 5\sqrt{114}$$

23. If A,B,C are angles of a triangle, prove that cos2A+cos2B+cos2C=–4cosAcosBcosC–1

Sol: L.H.S = cos 2A + cos 2B + cos 2C

$= 2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\cos 2C \left[\because \cos C-\cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

$= 2\cos(A+B)\cos(A-B)+(2\cos^2C-1)$ $\quad\left[\because \cos 2\theta = 2\cos^2\theta-1\right]$

$= 2\cos(180^\circ-C)\cos(A-B)+2\cos^2C-1$ $\quad\left[\because A+B+C=180^\circ\right]$

$= -2\cos C\cos(A-B)+2\cos^2C-1$ $\quad\left[\because \cos(180^\circ-\theta)=-\cos\theta\right]$

$= -2\cos C\,[\cos(A-B)-\cos C]-1$ $\quad$[Taking –2cosC common]

$= -2\cos C\,[\cos(A-B)-\cos(180^\circ-(A+B))]-1$ $\quad\left[\because A+B+C=180^\circ\right]$

$= -2\cos C\,[\cos(A-B)+\cos(A+B)]-1$ $\quad\left[\because \cos(180^\circ-\theta)=-\cos\theta\right]$

$= -2\cos C\,[2\cos A\cos B]-1$ $\quad$[$\because$ cos(A+B)+cos(A–B)=2cosAcosB]

$= -4\cos A\cos B\cos C-1$ = R.H.S

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24. Show that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\frac{r}{2R}$

Sol: L.H.S= $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\left(\frac{1-\cos A}{2}\right)+\left(\frac{1-\cos B}{2}\right)+\left(\frac{1-\cos C}{2}\right)$

$=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)$ .....(1)

Now, consider $\cos A+\cos B+\cos C = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$

$=2\cos\left(90^\circ-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\left(1-2\sin^2\frac{C}{2}\right)=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}\right)+\left(1-2\sin^2\frac{C}{2}\right)$

$=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)=1+2\sin\frac{C}{2}\left(\cos\left(\frac{A-B}{2}\right)-\sin\left(90^\circ-\left(\frac{A+B}{2}\right)\right)\right)$

$=1+2\sin\frac{C}{2}\left(\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right)=1+2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ from (1), $\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)=\frac{3}{2}-\frac{1}{2}\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$=\frac{3}{2}-\frac{1}{2}-\frac{1}{2}\left(4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=\frac{2}{2}-\frac{1}{2}\left(\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\right)=1-\frac{1}{2}\left(\frac{r}{R}\right)=1-\frac{r}{2R}$ =R.H.S