SOLVED PAPER-3

Time: 3 Hours

MATHS-2B

Max. Marks : 75

SECTION -A

I. Answer ALL the following Very Short Answer Questions: $10 \times 2 = 20$

1. Find the equation of the circle passing through the point (–2,14) and concentric with $x^2+y^2-6x-4y-12=0$
2. Find a if $2x^2+ay^2-3x+2y-1=0$ represents a circle and also find its radius.
3. Find the angle between the circles given by the equations $x^2+y^2+6x-10y-135=0$, $x^2+y^2-4x+14y-116=0$
4. Find the equation of the tangent and normal at the positive end of L.R on the parabola $y^2=6x$.
5. If the eccentricity of a hyperbola is 5/4, then find the eccentricity of its conjugate hyperbola.
6. Evaluate $\int \frac{1}{1+\sin 2x}\, dx$
7. Evaluate $\int e^x(1+\tan^2 x+\tan x)\, dx$
8. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x+\cos^3 x} dx$
9. Find the area enclosed by the curves $y = x^2+1$, $y=2x-2$, $x=-1$, $x=2$
10. Find the general solution of $\frac{dy}{dx}=\frac{2y}{x}$

SECTION-B

II. Answer any FIVE of the following Short Answer Questions: $5 \times 4 = 20$

11. Find the length of the chord intercepted by the circle $x^2+y^2-8x-2y-8=0$ on the line $x+y+1=0$
12. If the straight line $2x + 3y = 1$ intersects the circle $x^2 + y^2 = 4$ at the points A and B, find the equation of the circle having AB as diameter
13. Find the equations of the tangents to $9x^2 + 16y^2 = 144$, which make equal intercepts on the coordinate axes.
14. If PN is the ordinate of a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the tangent at P meets the X-axis at T then show that $(CN)(CT)=a^2$ where C is the centre of the ellipse.
15. Find the equation of the tangents to the hyperbola $x^2-4y^2=4$ which are

    (i) parallel to and (ii) perpendicular to the line $x+2y=0$
16. Evaluate $\int_0^a x(a^2-x^2)^{7/2}\, dx$
17. Solve $\frac{dy}{dx} - y\tan x = e^x \sec x$

SECTION-C

III. Answer any FIVE of the following Long Answer Questions: $5 \times 7 = 35$

18. Show that the points (1, –6), (5, 2), (7, 0), (–1, –4) are concyclic and find the equation of the circle on which they lie.

19. If the chord of contact of a point P with respect to the circle $x^2 + y^2 = a^2$ with centre O is cutting the circle at A,B such that $\angle AOB = 90^o$, then show that P lies on the circle $x^2+y^2=2a^2$.

20. Find the equation of the parabola whose focus is (–2, 3) and directrix is the line 2x+3y–4=0. Also find the length of the latusrectum and the equation of the axis of the parabola.

21. Evaluate $\int \frac{1}{(1+\sqrt{x})\sqrt{(x-x^2)}} dx$

22. If $I_n = \int \cos^n x dx$, then show that $I_n = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n} I_{n-2}$ and hence deduce the value of $\int \cos^4 x dx$

23. Evaluate $\int_0^1 x \mathrm{Tan}^{-1} x \, dx$

24. Solve $\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3}$

SOLUTIONS

SECTION -A

1. Find the equation of the circle passing through the point (–2, 14) and concentric with $x^2 + y^2 - 6x - 4y - 12 = 0$

Sol: The equation of the required concentric circle is $x^2 + y^2 - 6x - 4y + k = 0$

It passes through P(–2, 14)

$\Rightarrow (-2)^2 + 14^2 - 6(-2) - 4(14) + k = 0 \Rightarrow 4 + 196 + 12 - 56 + k = 0 \Rightarrow k = -156$

$\therefore$ the equation of the required circle is $x^2 + y^2 - 6x - 4y - 156 = 0$

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2. Find a if $2x^2 + ay^2 - 3x + 2y - 1 = 0$ represents a circle and also find its radius.

Sol: In the equation of a circle, we have Coefficient of x^2 = Coefficient of $y^2 \Rightarrow 2=a$

Equation of the circle is $2x^2 + 2y^2 - 3x + 2y - 1 = 0 \Rightarrow x^2 + y^2 - \frac{3}{2}x + y - \frac{1}{2} = 0$

Radius $r = \sqrt{\left(\frac{-3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{1}{2}} = \sqrt{\frac{9}{16} + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{9+4+8}{16}} = \frac{\sqrt{21}}{4}$

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3. Find the angle between the circles given by the equations $x^2 + y^2 + 6x - 10y - 135 = 0$, $x^2 + y^2 - 4x + 14y - 116 = 0$

Sol: Equations of the given circles are $x^2 + y^2 + 6x - 10y - 135 = 0$, $x^2 + y^2 - 4x + 14y - 116 = 0$

Centre $C_1 = (-3, 5)$, $C_2 = (2, -7)$; $d = C_1C_2 = \sqrt{(2+3)^2 + (-7-5)^2} = \sqrt{25+144} = \sqrt{169} = 13$

$r_1 = \sqrt{9+25+135} = \sqrt{169} = 13$; $r_2 = \sqrt{4+49+116} = \sqrt{169} = 13$

If θ is the angle between the given circles, then $\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$

$\Rightarrow \cos\theta = \frac{13^2 - 13^2 - 13^2}{2(13)(13)} = -\frac{1}{2} = \cos 120^\circ$

Angle between the circles is $\theta = 120^\circ = \frac{2\pi}{3}$

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4. Find the equation of the tangent and normal at the positive end of L.R on the parabola $y^2 = 6x$.

Sol: Given that, $y^2 = 6x \Rightarrow 4a = 6 \Rightarrow 2a = 3 \Rightarrow a = 3/2$

$\therefore$ positive end of L.R is (a, 2a)= (3/2, 3)

The equation of the tangent at (x_1, y_1) on $S = y^2 - 4ax = 0$ is $S_1 = yy_1 - 2a(x + x_1) = 0$

$\therefore$ the equation of the tangent at (3/2, 3) on $y^2 = 6x$ is $y(3) - 3(x + 3/2) = 0 \Rightarrow 2x - 2y + 3 = 0$

The slope of the tangent $2x - 2y + 3 = 0$ is $1 \Rightarrow$ the slope of its normal is -1

$\therefore$ the equation of the normal at (3/2, 3) with slope -1 is $y - 3 = -1(x - 3/2) \Rightarrow 2x + 2y - 9 = 0$

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5. If the eccentricity of a hyperbola is 5/4, then find the eccentricity of its conjugate hyperbola.

Sol: Let, e= 5/4 and the eccentricity of the conjugate hyperbola be e_1

Then, $\frac{1}{e^2} + \frac{1}{e_1^2} = 1 \Rightarrow \frac{1}{(5/4)^2} + \frac{1}{e_1^2} = 1 \Rightarrow \frac{1}{e_1^2} = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow e_1^2 = \frac{25}{9} \Rightarrow e_1 = \frac{5}{3}$

6. Evaluate $\int \frac{1}{1+\sin 2x}dx$

Sol: $\int \frac{1}{1+\sin 2x} = \int \frac{dx}{1+\frac{2\tan x}{1+\tan^2 x}} = \int \frac{(1+\tan^2 x)dx}{1+\tan^2 x+2\tan x} = \int \frac{\sec^2 x dx}{(1+\tan x)^2}$

Put, 1+tanx = t $\Rightarrow \sec^2 x\, dx = dt$ $\quad \therefore I = \int \frac{dt}{t^2} = -\frac{1}{t}+c = \frac{-1}{1+\tan x}+c$

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7. Evaluate $\int e^x(1+\tan^2 x+\tan x)\,dx$

Sol: $1+\tan^2 x = \sec^2 x$ $\quad \therefore \int e^x\left[(1+\tan^2 x)+\tan x\right]dx = \int e^x(\sec^2 x+\tan x)\,dx$

If $f(x) = \tan x$, then $f'(x) = \sec^2 x$ $\quad \therefore \int e^x(\tan x+\sec^2 x)\,dx = e^x \tan x + c$

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8. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x+\cos^3 x}dx$

Sol: We know $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x+\cos^3 x}dx \quad .....(1) = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)-\cos^2\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right)+\cos^3\left(\frac{\pi}{2}-x\right)}dx = \int_0^{\pi/2}\frac{\cos^2 x-\sin^2 x}{\cos^3 x+\sin^3 x}dx \quad ....(2)$$

Adding (1) & (2) we get, $2I = \int_0^{\pi/2} \frac{0}{\cos^3 x+\sin^3 x}dx = 0 \Rightarrow I = 0$

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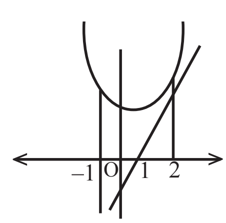
9. Find the area enclosed by the curves $y = x^2+1$, $y = 2x-2$, $x = -1$, $x =$

Sol: Required Area $= \int_{-1}^{2}[(x^2+1)-(2x-2)]\,dx = \left[\frac{x^3}{3}-x^2+3x\right]_{-1}^{2}$

$$= \left(\frac{8}{3}-4+6\right)-\left(-\frac{1}{3}-1-3\right) = \frac{9}{3}-4+6+1+3 = 9 \text{ Sq.units}$$



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10. Find the general solution of $\frac{dy}{dx} = \frac{2y}{x}$

Sol: Given that $\frac{dy}{dx} = \frac{2y}{x}$

$\Rightarrow \int \frac{dy}{y} = 2\int \frac{dx}{x}$

$\Rightarrow \log y = 2\log x + \log c$

$\Rightarrow \log y = \log x^2 + \log c$

$\Rightarrow y = cx^2.$

SECTION-B

11. Find the length of the chord intercepted by the circle $x^2 + y^2 - 8x - 2y - 8 = 0$ on the line $x + y + 1 = 0$

Sol: For the circle $x^2 + y^2 + 8x - 2y - 8 = 0$, Centre $C = (4,1)$, Radius $r = \sqrt{16+1+8} = \sqrt{25} = 5$

Let, the perpendicular distance from the centre (4, 1) to the line $x + y + 1 = 0$ be p.

$$\Rightarrow p = \frac{|4+1+1|}{\sqrt{1^2+1^2}} = \frac{6}{\sqrt{2}} = \frac{3\times 2}{\sqrt{2}} = 3\sqrt{2}$$

$\therefore$ Length of the chord $= 2\sqrt{r^2 - p^2} = 2\sqrt{5^2 - (3\sqrt{2})^2} = 2\sqrt{25 - 9(2)} = 2\sqrt{25-18} = 2\sqrt{7}$

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12. If the straight line $2x + 3y = 1$ intersects the circle $x^2 + y^2 = 4$ at the points A and B, find the equation of the circle having AB as diameter.

Sol: The equations of the given circle is $S \equiv x^2 + y^2 - 4 = 0$ and the line is $L \equiv 2x + 3y - 1 = 0$

The equation of any circle passing through the points of intersection of $S = 0$, $L = 0$ is $S + \lambda L = 0$

$\Rightarrow (x^2 + y^2 - 4) + \lambda(2x + 3y - 1) = 0 \quad \Rightarrow x^2 + y^2 + 2\lambda x + 3\lambda y - 4 - \lambda = 0 \ldots.(1)$

The centre of the above circle is $\left(-\lambda, -\frac{3\lambda}{2}\right)$

The line L =0 becomes a diameter if the above centre lies on L =0

$\Rightarrow 2(-\lambda) + 3(-3\lambda/2) - 1 = 0 \Rightarrow -4\lambda - 9\lambda - 2 = 0 \Rightarrow 13\lambda = -2 \Rightarrow \lambda = -2/13$

From (1), the equation of the required circle is

$$x^2 + y^2 + 2\left(-\frac{2}{13}\right)x + 3\left(-\frac{2}{13}\right)y - 4 + \frac{2}{13} = 0 \quad \Rightarrow 13(x^2+y^2) - 4x - 6y - 50 = 0$$

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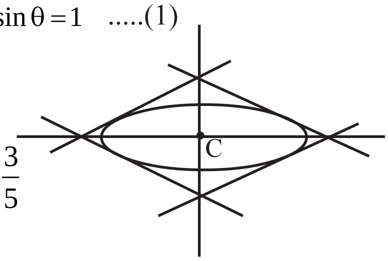
13. Find the equations of the tangents to $9x^2 + 16y^2 = 144$, which make equal intercepts on the coordinate axes.

Sol: Equation of the ellipse is $9x^2 + 16y^2 = 144 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow a^2 = 16 \Rightarrow a = 4;\ b^2 = 9 \Rightarrow b = 3$

The equation of any line which makes equal intercepts on the axes is taken as $x \pm y + k = 0$

Its slope $= \pm 1$

Equation of the tangent at $P(\theta)$ on the ellipse is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \ldots..(1)$



$\Rightarrow$ Slope of the tangent $= -\frac{b\cos\theta}{a\sin\theta}$

$\therefore -\frac{b\cos\theta}{a\sin\theta} = \pm 1 \Rightarrow \cot\theta = \pm\frac{a}{b} = \pm\frac{4}{3} \Rightarrow \cos\theta = \pm\frac{4}{5}, \sin\theta = \pm\frac{3}{5}$

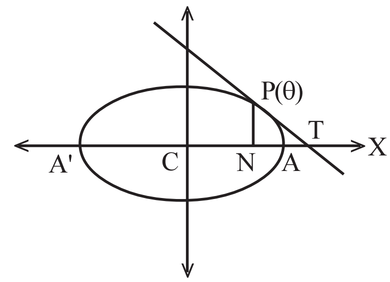
From (1), Equation of the tangent is $\frac{x}{4}\left(\pm\frac{4}{5}\right) + \frac{y}{3}\left(\pm\frac{3}{5}\right) = 1 \Rightarrow x \pm y = \pm 5 \Rightarrow x \pm y \pm 5 = 0$

14. If PN is the ordinate of a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the tangent at P meets the X-axis at T then show that (CN)(CT) = a^2 where C is the centre of the ellipse.

Sol: The parametric point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P(\theta) = (a\cos\theta, b\sin\theta)$

Given that PN = y-Coordinate of P then

x-coordinate of P = CN ⇒ CN = $a\cos\theta$



The equation of the tangent at P(θ) is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

$\Rightarrow \frac{x}{\left(\frac{a}{\cos\theta}\right)}+\frac{y}{\left(\frac{b}{\sin\theta}\right)}=1 \Rightarrow$ x-intercept = CT = $\frac{a}{\cos\theta}$

$\therefore (CN).(CT) = (a\cos\theta)\left(\frac{a}{\cos\theta}\right) = a^2$

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15. Find the equation of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are (i) parallel to and (ii) perpendicular to the line x + 2y = 0

Sol: The equation of the hyperbola is $x^2–4y^2=4 \Rightarrow \frac{x^2}{4}-\frac{y^2}{1}=1 \Rightarrow a^2=4, b^2=1$

The slope of the given line x+2y=0 is –1/2 ⇒ Slope of its perpendicular is m = 2

The equation of the tangent with slope m to the hyperbola S=0 is $y = mx \pm \sqrt{a^2m^2-b^2}$

(i) The equation of the parallel tangent with slope –1/2 is $y=-\frac{1}{2}x \pm \sqrt{4\left(\frac{1}{4}\right)-1} \Rightarrow x+2y=0$

(ii) The equation of the perpendicular tangent with slope 2 is $y = 2x \pm \sqrt{4(2^2)-1} \Rightarrow y = 2x \pm \sqrt{15}$

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16. Evaluate $\int_0^a x(a^2-x^2)^{7/2}\,dx$

Sol: Put, $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$; x = 0 , a ⇒ θ = 0,π/2

$$\int_0^a x(a^2-x^2)^{7/2}dx = \int_0^{\pi/2} a\sin\theta(a^2-a^2\sin^2\theta)^{7/2}a\cos\theta\, d\theta = a^9\int_0^{\pi/2}\sin\theta\cos^8\theta\, d\theta = a^9\times\frac{1}{9}=\frac{a^9}{9}$$

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17. Solve $\frac{dy}{dx} - y\tan x = e^x \sec x$

Sol : Given that, $\frac{dy}{dx} - y\tan x = e^x\sec x$

$\frac{dy}{dx}+(-\tan x)y = e^x\sec x$ which is a linear D.E. in y. Here, P = –tan x, Q = e^x sec x

I.F = $e^{\int p dx} = e^{\int -\tan x\, dx} = e^{-\int \tan x\, dx} = e^{-\log\sec x} = e^{\log(\sec x)^{-1}} = e^{\log\cos x} = \cos x$

∴ the solution is $y(IF) = \int (IF)Q\,dx \Rightarrow y.\cos x = \int \cos x(e^x\sec x)\,dx = \int e^x\,dx = e^x + c$

SECTION-C

18. Show that the points (1, –6), (5, 2), (7, 0), (–1, –4) are concyclic and find the equation of the circle on which they lie.

Sol: Let, A = (1, –6), B = (5, 2), C = (7, 0), D = (–1, –4)

Let, $S(x_1, y_1)$ be the centre of the circle $\Rightarrow SA = SB = SC$

Now, $SA = SB \Rightarrow SA^2 = SB^2 \Rightarrow (x_1-1)^2 + (y_1+6)^2 = (x_1-5)^2 + (y_1-2)^2$

$\Rightarrow (x_1^2 - 2x_1 + 1) + (y_1^2 + 12y_1 + 36) = (x_1^2 - 10x_1 + 25) + (y_1^2 - 4y_1 + 4)$

$\Rightarrow 8x_1 + 16y_1 + 8 = 0 \Rightarrow x_1 + 2y_1 + 1 = 0$ ....(1)

Also, $SB = SC \Rightarrow SB^2 = SC^2 \Rightarrow (x_1-5)^2 + (y_1-2)^2 = (x_1-7)^2 + (y_1-0)^2$

$\Rightarrow (x_1^2 - 10x_1 + 25) + (y_1^2 - 4y_1 + 4) = (x_1^2 - 14x_1 + 49) + y_1^2$

$\Rightarrow 4x_1 - 4y_1 - 20 = 0 \Rightarrow x_1 - y_1 - 5 = 0$ ......(2)

Solving (1) & (2) we get the centre $S(x_1, y_1)$

$(1) - (2) \Rightarrow 3y_1 + 6 = 0 \Rightarrow 3y_1 = -6 \Rightarrow y_1 = -2$

From (2) we get, $x_1 = 5 + y_1 = 5 - 2 = 3$

∴ The centre of the circle is $S(x_1,y_1) = (3,-2)$

Hence, radius $r = SA = \sqrt{(3-1)^2 + (-2+6)^2} = \sqrt{4+16} = \sqrt{20}$

∴ The equation of the circle with centre (3,–2) and radius $\sqrt{20}$ is

$(x-3)^2 + (y+2)^2 = 20 \Rightarrow (x^2 - 6x + 9) + (y^2 + 4y + 4) = 20 \Rightarrow x^2 + y^2 - 6x + 4y - 7 = 0$

Now, substituting D(–1,–4) in the above equation, we have

$(-1)^2 + (-4)^2 - 6(-1) + 4(-4) - 7 = 1 + 16 + 6 - 16 - 7 = 0$

∴ D(–1,–4) lies on the circle

∴ The given 4 points are concyclic.

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19. If the chord of contact of a point P with respect to the circle $x^2 + y^2 = a^2$ with centre O is cutting the circle at A,B such that ∠AOB = 90°, then show that P lies on the circle $x^2 + y^2 = 2a^2$.

Sol: Let, $P(x_1, y_1)$.

The chord of contact of P cuts the circle $S = x^2 + y^2 - a^2 = 0$ ....(1) in A and B such that

$\angle AOB = 90^o$. The chord of contact of $P(x_1,y_1)$ with respect to S=0 is $S_1 = xx_1 + yy_1 - a^2 = 0$......(2)

The equation to the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is obtained by homogenizing (1) using (2)

$\Rightarrow x^2 + y^2 - a^2\left(\frac{xx_1 + yy_1}{a^2}\right)^2 = 0 \Rightarrow a^2(x^2 + y^2) - (xx_1 + yy_1)^2 = 0$

$\Rightarrow x^2(a^2 - x_1^2) - 2x_1y_1xy + y^2(a^2 - y_1^2) = 0$

But, $\angle AOB = 90^o \Rightarrow$ coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow (a^2 - x_1^2) + (a^2 - y_1^2) = 0 \Rightarrow x_1^2 + y_1^2 = 2a^2$

Hence the point $P(x_1, y_1)$ lies on the circle $x^2 + y^2 = 2a^2$

20. Find the equation of the parabola whose focus is (–2, 3) and directrix is the line 2x + 3y – 4 = 0. Also find the length of the latusrectum and the equation of the axis of the parabola.

Sol: Let $P(x_1, y_1)$ be a point on the parabola

Given that focus S(–2, 3) and the equation of the directrix is $2x + 3y - 4 = 0$

Now, using the focus directrix property of the parabola we have SP = PM

$$\Rightarrow \sqrt{(x_1+2)^2+(y_1-3)^2} = \frac{|2x_1+3y_1-4|}{\sqrt{4+9}} \Rightarrow 13((x_1+2)^2+(y_1-3)^2) = (2x_1+3y_1-4)^2$$

$$\Rightarrow 13(x_1^2+4x_1+4+y_1^2-6y_1+9) = (2x_1+3y_1-4)^2$$

$$\Rightarrow 13(x_1^2+y_1^2+4x_1-6y_1+13) = (2x_1+3y_1-4)^2$$

$$\Rightarrow 13x_1^2+13y_1^2+52x_1-78y_1+169 = 4x_1^2+9y_1^2+16+12x_1y_1-16x_1-24y_1$$

$$\Rightarrow 9x_1^2-12x_1y_1+4y_1^2+68x_1-54y_1+153=0$$

$\therefore$ Equation of the parabola is $9x^2-12xy+4y^2+68x-54y+153=0$

Perpendicular distance from S(–2,3) to

the directrix 2x+3y–4=0 is $2a = \frac{|2(-2)+3.3-4|}{\sqrt{4+9}} = \frac{1}{\sqrt{13}}$

$\therefore$ Length of the latus rectum = $4a = 2(2a) = 2\left(\frac{1}{\sqrt{13}}\right) = \frac{2}{\sqrt{13}}$

We know that the axis is perpendicular to the directrix, and it passes through the focus.

Hence, equation of the directrix is taken as $3x - 2y + k = 0$

If this line passes through S(–2, 3) then $-6 - 6 + k = 0 \Rightarrow k=12$

$\therefore$ Equation of the axis of the given parabola is $3x - 2y + 12 = 0$

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21. Evaluate $\int \frac{1}{(1+\sqrt{x})\sqrt{(x-x^2)}}dx$

Sol: Put $x = t^2 \Rightarrow dx = 2t\,dt$.

Also $x-x^2 = t^2-t^4 = t^2(1-t^2)$

$$\therefore I = \int \frac{1}{(1+\sqrt{x})\sqrt{(x-x^2)}}dx = \int \frac{1}{(1+t)}\frac{(2t)dt}{\sqrt{t^2(1-t^2)}} = 2\int \frac{1}{(1+t)}\frac{dt}{\sqrt{(1-t^2)}} \quad .....(1)$$

Put $1+t = \frac{1}{z} \Rightarrow dt = -\frac{1}{z^2}dz$.

Also $t = \frac{1}{z}-1 = \frac{1-z}{z} \Rightarrow 1-t^2 = 1-\left(\frac{1-z}{z}\right)^2 = \frac{z^2-(1-2z+z^2)}{z^2} = \frac{2z-1}{z^2}$

$$\therefore I = 2\int \frac{1}{\left(\frac{1}{z}\right)\sqrt{\frac{2z-1}{z^2}}}\left(\frac{-1}{z^2}\right)dz = -2\int \frac{1}{\sqrt{2z-1}}dz = (-2).\frac{1}{2}.(2\sqrt{2z-1})+c = -2(\sqrt{2z-1})+c$$

$$=-2\left[\sqrt{2\left(\frac{1}{1+t}\right)-1}\right]+c=-2\left[\sqrt{\frac{2-1-t}{1+t}}\right]+c=-2\left[\sqrt{\frac{1-t}{1+t}}\right]+c=-2\left[\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\right]+c$$

$$=-2\left[\sqrt{\frac{\left(1-\sqrt{x}\right)^2}{(1+\sqrt{x})(1-\sqrt{x})}}\right]+c=\frac{-2(1-\sqrt{x})}{\sqrt{1-x}}+c=\frac{2(\sqrt{x}-1)}{\sqrt{1-x}}+c$$

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22. If $I_n = \int\cos^n x dx$, then show that $I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}I_{n-2}$ and hence deduce the value of $\int\cos^4 x dx$

Sol: Given that $I_n = \int\cos^n x dx = \int\cos^{n-1}x\,(\cos x)dx$.

$$I_n = \cos^{n-1}x\int\cos x dx - \int\left(\left(\frac{d}{dx}\cos^{n-1}x\int\cos x\,dx\right)\right)dx$$

$$= \cos^{n-1}x.\sin x - \int(n-1)\cos^{n-2}x\,(-\sin x)\sin x\,dx$$

$$= \cos^{n-1}x.\sin x + (n-1)\int\cos^{n-2}x\,(1-\cos^2 x)dx$$

$$= \cos^{n-1}x.\sin x + (n-1)\int\cos^{n-2}x dx - (n-1)\int\cos^n x dx$$

$$I_n = \cos^{n-1}x.\sin x + (n-1)I_n-2 - (n-1)I_n$$

$$I_n(1+n-1) = \cos^{n-1}x.\sin x + (n-1)I_{n-2}$$

$$\Rightarrow I_n = \frac{\cos^{n-1}x\sin x}{n} + \left(\frac{n-1}{n}\right)I_{n-2} \quad ....(1)$$

Now put n = 4 in the above formula we get, $I_4 = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}I_2$

Put n = 2 in the above formula $I_2 = \frac{1}{2}\cos x\sin x + \frac{1}{2}I_0$

$$\Rightarrow I_4 = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}I_0\right) \Rightarrow I_4 = \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}I_0$$

$$I_4 = \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + c \quad [\because I_0 = \int(\sin x)^0 dx = \int 1dx = x \Rightarrow I_0 = x]$$

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23. Evaluate $\int_0^1 x\,Tan^{-1}x\,dx$

Sol: Applying the "By Parts Rule", we have $\int_0^1 x\,Tan^{-1}x\,dx = \left[Tan^{-1}x.\frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1+x^2}\frac{x^2}{2}dx$

$$=\left[Tan^{-1}(1).\frac{1}{2}-0\right]-\frac{1}{2}\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{\pi}{4}.\frac{1}{2}-\frac{1}{2}\int_0^1 1dx+\frac{1}{2}\int_0^1\frac{1}{1+x^2}dx=\frac{\pi}{8}-\frac{1}{2}[x]_0^1+\frac{1}{2}\left[Tan^{-1}x\right]_0^1$$

$$=\frac{\pi}{8}-\frac{1}{2}+\frac{1}{2}\left[\mathrm{Tan}^{-1}(1)-\mathrm{Tan}^{-1}(0)\right]=\frac{\pi}{8}-\frac{1}{2}+\frac{1}{2}\left[\frac{\pi}{4}-0\right]$$

$$=\frac{\pi}{8}-\frac{1}{2}+\frac{\pi}{8}=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$

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24. Solve $\frac{dy}{dx}=\frac{3y-7x+7}{3x-7y-3}$

Sol: Given that, $\frac{dy}{dx}=\frac{3y-7x+7}{3x-7y-3}\Rightarrow\frac{dy}{dx}=\frac{-7x+3y+7}{3x-7y-3}$ ......(1)

Comparing, $\frac{-7x+3y+7}{3x-7y-3}$ with $\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ we have $\frac{a_1}{a_2}=\frac{-7}{3};\frac{b_1}{b_2}=\frac{3}{-7}\Rightarrow\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

Put, x = X+h, y = Y+k then $\frac{dy}{dx}=\frac{dY}{dX}$

$$\therefore(1)\Rightarrow\frac{dY}{dX}=\frac{-7(X+h)+3(Y+k)+7}{3(X+h)-7(Y+k)-3}=\frac{-7X+3Y+(-7h+3k+7)}{3X-7Y+(3h-7k-3)}\quad\text{......(2)}$$

Now choose h and k so that –7h + 3k +7 = 0, 3h – 7k – 3 = 0
Solving the above two equations we get h = 1, k = 0

Hence (2) becomes $\frac{dY}{dX}=\frac{-7X+3Y}{3X-7Y}$....(3) which is a homogeneous D.E in X & Y

Now, we take the substitution $Y=VX\Rightarrow\frac{dY}{dX}=V+X\frac{dV}{dX}$

From (3) $\Rightarrow V+X\frac{dV}{dX}=\frac{-7X+3VX}{3X-7VX}=\frac{-7+3V}{3-7V}$

$$\Rightarrow X\frac{dV}{dX}=\frac{-7+3V}{3-7V}-V=\frac{-7+3V-3V+7V^2}{3-7V}=\frac{7(V^2-1)}{3-7V}\Rightarrow\frac{7V-3}{V^2-1}dV=-7\frac{dX}{X}$$

(On resolving $\frac{7V-3}{V^2-1}$ into partial fractions we get $\frac{7V-3}{V^2-1}=\frac{5}{V+1}+\frac{2}{V-1}$)

$$\int\frac{7V-3}{(V^2-1)}dV=-7\int\frac{dX}{X}\Rightarrow\int\left[\frac{5}{V+1}+\frac{2}{V-1}\right]dV=-7\int\frac{dX}{X}$$

$$\Rightarrow 5\log(v+1)+2\log(v-1)=-7\log X+\log c$$

$$\Rightarrow 7\log X+5\log(v+1)+2\log(v-1)=\log c$$

$$\Rightarrow\log X^7+\log(V+1)^5+\log(V-1)^2=\log c\Rightarrow\log\left(X^7(V+1)^5(V-1)^2\right)=\log c$$

$$\Rightarrow X^7(V+1)^5(V-1)^2=c\Rightarrow X^7\left(\frac{Y}{X}+1\right)^5\left(\frac{Y}{X}-1\right)^2=c_1\Rightarrow(Y+X)^5(Y-X)^2=c_1$$

$$\Rightarrow[(y-0)+(x-1)]^5[(y-0)-(x-1)]^2=c_1\quad\Rightarrow(y+x-1)^5(y-x+1)^2=c$$