SOLVED PAPER-2

Time: 3 Hours

MATHS-2B

Max. Marks : 75

Section -A

I. Answer ALL the following Very Short Answer Questions: $10 \times 2 = 20$

1. Obtain the parametric equation of the circle $(x-3)^2 + (y-4)^2 = 8^2$
2. Find the equation of the normal at P(3,5) of the circle $S \equiv x^2 + y^2 - 10x - 2y + 6 = 0$
3. Find k if the pairs of circles are $x^2 + y^2 - 6x - 8y + 12 = 0$, $x^2 + y^2 - 4x + 6y + k = 0$ are orthogonal.
4. Find the coordinates of the point on the parabola $y^2 = 2x$ whose focal distance is 5/2.
5. If e, e_1 are the eccentricities of a hyperbola and its conjugate hyperbola, then prove that $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$
6. Evaluate $\int \sqrt{x} \log x \, dx$ on $(0, \infty)$
7. Evaluate $\int \frac{e^x}{e^x + 1} dx$
8. Evaluate $\int_1^5 \frac{dx}{\sqrt{2x-1}} dx$
9. Evaluate $\int_0^a \sqrt{a^2 - x^2} dx$
10. Find the order and degree of the differential equation $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/3}$

SECTION-B

II. Answer any FIVE of the following Short Answer Questions: $5 \times 4 = 20$

11. Find the equation of a circle which passes through (4, 1), (6, 5) and having the centre on $4x + 3y - 24 = 0$.
12. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$, $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then show that $f'g = fg'$.
13. Find the equations of the tangent and normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of latusrecta in the first quadrant
14. Prove that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$
15. Find the centre, eccentricity, foci, length of latus rectum and equations of the directrices of the Hyperbola $16y^2 - 9x^2 = 144$.
16. Evaluate $\int_{-3}^{+3} (9 - x^2)^{3/2} dx$
17. Solve $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$

SECTION-C

III. Answer any FIVE of the following Long Answer Questions: 5 × 7 = 35

18. Find the equation of the circle passing through the three points (1,2), (3,–4), (5,–6)

19. Show that the circles $x^2 + y^2 - 6x - 2y + 1 = 0$ and $x^2 + y^2 + 2x - 8y + 13 = 0$ touch each other. Find the point of contact and the equation of the common tangent at their point of contact.

20. Find the equation of the parabola whose axis is parallel to the y – axis and passing through the points (4, 5), (–2, 11), (–4, 21).

21. Evaluate $\int \frac{3\sin x + \cos x + 7}{\sin x + \cos x + 1} dx$

22. Evaluate $\int \frac{x^3 - 2x + 3}{x^2 + x - 2} dx$

23. Let AOB be the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with OA = a, OB = b. Then show that the area bounded between the chord AB and the arc AB of the ellipse is $\frac{(\pi - 2)ab}{4}$

24. Solve $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$

SOLUTIONS

Section -A

1. Obtain the parametric equation of the circle $(x-3)^2+(y-4)^2=8^2$

Sol: For the circle $(x-3)^2+(y-4)^2=8^2$

Centre C= $(-g,-f) = (3, 4)$ and radius $r = 8$

The parametric equations of the circle are given by is $x = -g + r\cos\theta$, $y = -f + r\sin\theta$,

$\Rightarrow x = 3 + 8\cos\theta, y = 4 + 8\sin\theta$

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2. Find the equation of the normal at P(3,5) of the circle $S \equiv x^2+y^2-10x-2y+6=0$

Sol: Equation of the normal at $P(x_1,y_1)$ on S=0 is $(x-x_1)(y_1+f)-(y-y_1)(x_1+g) = 0$

Hence equation of the normal at P(3,5) is $(x-3)(5-1)-(y-5)(3-5)=0$

$\Rightarrow(x-3)(4)-(y-5)(-2)=0 \quad \Rightarrow 4x-12+2y-10=0$

$\Rightarrow 4x+2y-22=0 \Rightarrow 2(2x+y-11)=0 \Rightarrow 2x+y-11=0$

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3. Find k if the pairs of circles are $x^2+y^2-6x-8y+12=0$, $x^2+y^2-4x+6y+k=0$ are orthogonal.

Sol: Here $g=-3, f=-4, c=12$ and $g'=-2, f'=3, c'=k$

The orthogonal condition is $2gg'+2ff'=c+c'$

$\Rightarrow 2(-3)(-2)+2(-4)(3)=12+k \Rightarrow 12-24=12+k \Rightarrow k=-24$

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4. Find the coordinates of the point on the parabola $y^2=2x$ whose focal distance is 5/2.

Sol: The focal distance of the required point $P(x_1, y_1)$, on the parabola $y^2=4ax$ is $SP = x_1 + a$

Now, $y_1^2 = 2x \Rightarrow 4a = 2 \Rightarrow a = \frac{2}{4} = \frac{1}{2}$

$\therefore$ Focal distance $x_1 + \frac{1}{2} = \frac{5}{2} \Rightarrow x_1 = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$

Now, $y_1^2 = 2x_1 \Rightarrow y_1^2 = 2(2) = 4 \Rightarrow y_1 = \pm 2$

$\therefore P(x_1,y_1) = (2, \pm 2)$

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5. If e, e_1 are the eccentricities of a hyperbola and its conjugate hyperbola, then Prove that $\frac{1}{e^2}+\frac{1}{e_1^2}=1$

Sol: e, e_1 are the eccentricities of the hyperbola $S = \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ and its

conjugate hyperbola $S' = \frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$ Then $e=\sqrt{\frac{a^2+b^2}{a^2}}, e_1=\sqrt{\frac{a^2+b^2}{b^2}}$

$\therefore \frac{1}{e^2}+\frac{1}{e_1^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}=1$

6. Evaluate $\int \sqrt{x}\log x\, dx$ on $(0,\infty)$

Sol: We take the first function u=logx and second function v = x

From the "By parts rule", we have $I=\log x.\int\sqrt{x}dx-\int\left(\int\sqrt{x}dx\right).d(\log x)dx$

$$=\log x.\frac{2}{3}x^{3/2}-\int\frac{2}{3}x^{3/2}.\frac{1}{x}dx=\frac{2}{3}x^{3/2}\log x-\frac{2}{3}\int\sqrt{x}dx$$

$$=\frac{2}{3}x^{3/2}\log x-\frac{2}{3}.\frac{2}{3}x^{3/2}+c=\frac{2}{3}x^{3/2}\log x-\frac{4}{9}x^{3/2}+c$$

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7. Evaluate $\int\frac{e^x}{e^x+1}dx$

Sol: Put $e^x+1=t \Rightarrow e^x dx = dt$ $\qquad \therefore I=\int\frac{dt}{t}=\log t+c=\log(e^x+1)+c$

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8. Evaluate $\int_1^5\frac{dx}{\sqrt{2x-1}}$

Sol: $\int_1^5\frac{dx}{\sqrt{2x-1}}=\left[\frac{1}{2}.2\sqrt{2x-1}\right]_1^5=\left[\sqrt{2x-1}\right]_1^5=\left[\sqrt{2(5)-1}-\sqrt{2(1)-1}\right]=\sqrt{9}-\sqrt{1}=3-1=2$

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9. Evaluate $\int_0^a\sqrt{a^2-x^2}dx$

Sol: Formula: $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a}+c$

$$\therefore \int_0^a\sqrt{a^2-x^2}dx=\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a}\right]_0^a=0+\frac{a^2}{2}\text{Sin}^{-1}(1)=\frac{a^2}{2}\left(\frac{\pi}{2}\right)=\frac{\pi a^2}{4}$$

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10. Find the order and degree of the differential equation $\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3}$

Sol: Given differential equation is $\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{3}}$

Cubing on both sides we get, $\left(\frac{d^2y}{dx^2}\right)^3=\left[1+\left(\frac{dy}{dx}\right)^2\right]^5$

In this, highest order derivative is $\frac{d^2y}{dx^2}$ $\qquad \therefore$ order = 2

Exponent of the highest order derivative $\frac{d^2y}{dx^2}$ is 3. Hence the degree is 3

SECTION-B

11. Find the equation of a circle which passes through (4,1), (6,5) and having the centre on 4x+3y–24=0.

Sol: Let $S(x_1,y_1)$ be the centre of the circle

Given that centre lies on $4x+3y-24=0 \Rightarrow 4x_1+3y_1-24=0$ .... (1)

A(4,1), B(6,5) are points on the circle $\Rightarrow SA=SB \Rightarrow SA^2=SB^2$.

$\Rightarrow (x_1-4)^2+(y_1-1)^2=(x_1-6)^2+(y_1-5)^2$

$\Rightarrow (x_1^2-8x_1+16)+(y_1^2-2y_1+1)=(x_1^2-12x_1+36)+(y_1^2-10y_1+25)$

$\Rightarrow 17-8x_1-2y_1=61-12x_1-10y_1 \Rightarrow 17-8x_1-2y_1-61+12x_1+10y_1=0 \Rightarrow 4x_1+8y_1-44=0$..... (2)

(1)–(2) $\Rightarrow -5y_1+20=0 \Rightarrow 5y_1=20 \Rightarrow y_1=4$

From (1), $4x_1+3(4)-24=0 \Rightarrow 4x_1-12=0 \Rightarrow 4x_1=12 \Rightarrow x_1=3$

Centre of the circle $S(x_1,y_1)=(3,4)$,

Since A=(4,1), Radius $r=SA=\sqrt{(3-4)^2+(4-1)^2}=\sqrt{1+9}=\sqrt{10}$

$\therefore$ the equation of the circle with centre (3,4) and radius $\sqrt{10}$ is $(x-3)^2+(y-4)^2=(\sqrt{10})^2$

$\Rightarrow (x^2+9-6x)+(y^2+16-8y)=10 \Rightarrow x^2+y^2-6x-8y+15=0$

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12. If the two circles $x^2+y^2+2gx+2fy=0$, $x^2+y^2+2g'x+2f'y=0$ touch each other, then show that $f'g=fg'$.

Sol: Both of the given equations do not contain the constant term.

$\therefore$ the circles pass through O(0,0).

If the circles touch each other, then O(0,0) and the centres $C_1=(-g,-f)$, $C_2=(-g',-f')$ become collinear

$\Rightarrow$ Area of $\Delta\ OC_1C_2=0$

$\Rightarrow \frac{1}{2}[(-g)(-f')+(f)(-g')]=0 \Rightarrow gf'-fg'=0 \Rightarrow fg'=f'g$.

O=(0,0), $A(x_1,y_1)$, $B(x_2,y_2)$
Area $=\frac{1}{2}\lvert(x_1y_2-x_2y_1)\rvert$

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13. Find the equations of the tangent and normal to the ellipse $9x^2+16y^2=144$ at the end of latusrecta in the first quadrant

Sol: $9x^2+16y^2=144 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1 \Rightarrow a^2=16, b^2=9$

$$e^2=\frac{a^2-b^2}{a^2}=\frac{16-9}{16}=\frac{7}{16} \Rightarrow e=\frac{\sqrt{7}}{4}$$

Positive end of latus rectum $L=\left(ae,\frac{b^2}{a}\right)=\left(4.\frac{\sqrt{7}}{4},\frac{9}{4}\right)=\left(\sqrt{7},\frac{9}{4}\right)$

Equation of the tangent at L is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \Rightarrow \frac{x\sqrt{7}}{16}+\frac{y}{9}\left(\frac{9}{4}\right)=1 \Rightarrow \sqrt{7}x+4y=16$

Equation of the normal at L is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2 \Rightarrow \frac{16x}{\sqrt{7}}-\frac{9y}{\frac{9}{4}}=16-9$

$\Rightarrow \frac{16x}{\sqrt{7}}-4y=7 \Rightarrow 16x-4\sqrt{7}y=7\sqrt{7}$

14. Prove that the condition for the line y = mx + c to be a tangent to the ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2 = a^2m^2 + b^2$

Sol: Let the line y = mx + c touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1,y_1)$

The equation of the tangent at $P(x_1,y_1)$ on S=0 is $S_1=0 \Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$

Comparing the above equation with y=mx+c i.e, mx−y+c=0, we have

$$\frac{x_1}{a^2m}=\frac{y_1}{b^2(-1)}=\frac{-1}{c} \Rightarrow x_1=\frac{-a^2m}{c}, y_1=\frac{b^2}{c}$$

But $P(x_1,y_1)$ lies on the line y=mx+c

$$\Rightarrow y_1=mx_1+c \Rightarrow \frac{b^2}{c}=m\left(\frac{-a^2m}{c}\right)+c \Rightarrow b^2=-a^2m^2+c^2 \Rightarrow c^2=a^2m^2+b^2$$

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15. Find the centre, eccentricity, foci, length of latus rectum and equations of the directrices of the Hyperbola $16y^2 - 9x^2 = 144$.

Sol: Equation of hyperbola is $16y^2 - 9x^2 = 144$

$$\Rightarrow \frac{16y^2}{144}-\frac{9x^2}{144}=1 \Rightarrow \frac{y^2}{9}-\frac{x^2}{16}=1 \Rightarrow \frac{x^2}{16}-\frac{y^2}{9}+1=0$$, which is in standard form II.

Here, $a^2 = 16, b^2 = 9$

(i) Centre C= (0,0) (ii) Eccentricity $e=\sqrt{\frac{a^2+b^2}{b^2}}=\sqrt{\frac{16+9}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$

(iii) Foci = (0, ±be) = (0, ±3(5/3)) = (0,±5)

(iv) Equation of directrices is y = ± b/e = ±3/(5/3)=±9/5 ⇒ 5y±9=0

(v) Length of latusrectum $=\frac{2a^2}{b}=\frac{2(16)}{3}=\frac{32}{3}$

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16. Evaluate $\int_{-3}^{+3}(9-x^2)^{3/2}\,dx$

Sol: Put $x = 3\sin\theta$,

Then, $dx = 3\cos\theta\, d\theta$; $x = -3,3 \Rightarrow \theta = -\pi/2, \pi/2$

$$\int_{-3}^{+3}(9-x^2)^{3/2}dx=\int_{-\pi/2}^{\pi/2}(9-9^2\theta)^{3/2}3\cos\theta d\theta$$

$$=81\int_{-\pi/2}^{\pi/2}\cos^4\theta d\theta=162\int_0^{\pi/2}\cos^4\theta d\theta =162\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=\frac{243\pi}{8}$$

17. Solve $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$

Sol: The given D.E is of the form variables separable.

$$\therefore \frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0 \Rightarrow \frac{dy}{dx}=-\frac{y^2+y+1}{x^2+x+1} \Rightarrow \frac{dy}{y^2+y+1}=-\frac{dx}{x^2+x+1}$$

$$\Rightarrow \int\frac{dy}{y^2+y+1}=-\int\frac{dx}{x^2+x+1} \Rightarrow \int\frac{dy}{y^2+y+\frac{1}{4}-\frac{1}{4}+1}=-\int\frac{dx}{x^2+x+\frac{1}{4}-\frac{1}{4}+1}$$

$$\Rightarrow \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}=-\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \Rightarrow \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=-\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \frac{1}{\frac{\sqrt{3}}{2}}\text{Tan}^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=-\frac{1}{\frac{\sqrt{3}}{2}}\text{Tan}^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c \Rightarrow \text{Tan}^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)=-\text{Tan}^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c_1$$

$\Rightarrow \text{Tan}^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\text{Tan}^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)=c_1$, which is the solution of the given D.E

SECTION-C

18. Find the equation of the circle passing through the three points (1,2), (3,–4), (5,–6)

Sol: Let the equation of the required circle be $S=x^2+y^2+2gx+2fy+c=0$

(1,2) lies on S=0 $\Rightarrow$ 1+4+2g(1)+2f(2)+c=0 $\Rightarrow$ 2g+4f+c+5=0 .....(1)

(3,–4) lies on S=0 $\Rightarrow$ 9+16+2g(3)+2f(–4)+c=0 $\Rightarrow$ 6g–8f+c+25=0 .....(2)

(5,–6) lies on S=0 $\Rightarrow$ 25+36+2g(5)+2f(–6)+c=0 $\Rightarrow$ 10g–12f+c+61=0 .....(3)

(2) – (1) $\Rightarrow$ 4g–12f+20=0 ....(4)

(3) – (2) $\Rightarrow$ 4g–4f+36=0.....(5)

(4) – (5) $\Rightarrow$ –8f–16=0 $\Rightarrow$–8f=16=0$\Rightarrow$f= –2

From (4), 4g–12(–2)+20=0 $\Rightarrow$4g+24= –20$\Rightarrow$ 4g = –44$\Rightarrow$g= –11

(1)$\Rightarrow$2(–11)+4(–2)+c+5=0$\Rightarrow$–22–8+k+5=0$\Rightarrow$c=25

Substituting the values of g= –11, f= –2 and k=25 in $x^2+y^2+2gx+2fy+c=0$, we get

$$x^2+y^2+2(-11)x+2(-2)y+25=0 \Rightarrow x^2+y^2-22x-4y+25=0$$

19. Show that the circles $x^2 + y^2 - 6x - 2y + 1 = 0$ and $x^2 + y^2 + 2x - 8y + 13 = 0$ touch each other. Find the point of contact and the equation of the common tangent at their point of contact.

Sol: For the circle $S \equiv x^2 + y^2 - 6x - 2y + 1 = 0$;

Centre $C_1 = (3,1)$, radius $r_1 = \sqrt{9+1-1} = 3$

$S^1 \equiv x^2 + y^2 + 2x - 8y + 13 = 0$;

Centre $C_2 = (-1,4)$, radius $r_2 = \sqrt{1+16-13} = 2$

$C_1C_2 = \sqrt{(3+1)^2 + (1-4)^2} = \sqrt{16+9} = \sqrt{25} = 5$

Also, $r_1 + r_2 = 3+2 = 5 = C_1C_2$.

$\therefore$ The circles touch each other externally.

Hence, the point of contact P divides C_1C_2 internally in the ratio $r_1 : r_2 = 3 : 2$

Point of contact $P = \left(\frac{3(-1)+2(3)}{5}, \frac{3(4)+2(1)}{5}\right) = \left(\frac{3}{5}, \frac{14}{5}\right)$

The equation of the common tangent to the circles $S=0$ and $S^1=0$ at the point of contact is given by $S-S^1=0 \Rightarrow (x^2+y^2-6x-2y+1)-(x^2+y^2+2x-8y+13) = 0$

$\Rightarrow (-6x-2x) - 2y + 8y + 1 - 13 = 0 \Rightarrow -8x + 6y - 12 = 0 \Rightarrow 4x - 3y + 6 = 0$

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20. Find the equation of the parabola whose axis is parallel to the y–axis and passing through the points (4, 5), (–2, 11), (–4, 21).

Sol : The equation of the parabola whose axis is parallel to the y-axis is $y = lx^2 + mx + n$

The parabola passes through $(4,5) \Rightarrow 5 = 16l + 4m + n$ ....(1)

The parabola passes through $(-2,11) \Rightarrow 11 = 4l - 2m + n$ ....(2)

The parabola passes through $(-4, 21) \Rightarrow 21 = 16l - 4m + n$ ....(3)

$(1) - (2) \Rightarrow 12l + 6m = -6$ .....(4)

$(3) - (2) \Rightarrow 12l - 2m = 10$ ....(5)

$(4) - (5) \Rightarrow 8m = -16 \Rightarrow m = -2$

Now, $(4) \Rightarrow 12l + 6(-2) = -6 \Rightarrow 12l = 6 \Rightarrow l = 1/2$

Put l,m in (1) $\Rightarrow 16\left(\frac{1}{2}\right) + 4(-2) + n = 5 \Rightarrow 8 - 8 + n = 5 \Rightarrow n = 5$

Substituting the values $l = 1/2$, $m = -2$ and $n = 5$ in $y = lx^2 + mx + n$ we get the equation of the required parabola as $y = \frac{1}{2}x^2 - 2x + 5 \Rightarrow x^2 - 2y - 4x + 10 = 0$

21. Evaluate $\int \frac{3\sin x + \cos x + 7}{\sin x + \cos x + 1} dx$

Sol: Put, $\tan\frac{x}{2} = t$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$

$$\therefore \int \frac{3\sin x + \cos x + 7}{\sin x + \cos x + 1} dx = \int \frac{3\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) + 7}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1}\left(\frac{2dt}{1+t^2}\right)$$

$$= \int \frac{6t + 1 - t^2 + 7 + 7t^2}{2t + 1 - t^2 + 1 + t^2}\left(\frac{2dt}{1+t^2}\right) = \int \frac{6t^2 + 6t + 8}{2t + 2}\left(\frac{2dt}{1+t^2}\right) = \int \frac{6t^2 + 6t + 8}{(t+1)(t^2+1)} dt$$

Let $\frac{6t^2 + 6t + 8}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt + C}{t^2 + 1} = \frac{A(t^2+1) + (Bt+C)(t+1)}{(t+1)(t^2+1)}$

$\Rightarrow A(t^2+1) + (Bt+C)(t+1) = 6t^2 + 6t + 8 \Rightarrow (A+B)t^2 + (B+C)t + (A+C) = 6t^2 + 6t + 8$

$\Rightarrow A + B = 6, B + C = 6, A + C = 8$;

Solving these, we get A=4, B=2, C=4

$$\therefore \int \frac{3\sin x + \cos x + 7}{\sin x + \cos x + 1} dx = \int \frac{6t^2 + 6t + 8}{(t+1)(t^2+1)} dt = \int \left[\frac{4}{t+1} + \frac{2t+4}{t^2+1}\right] dt$$

$= 4\log|t+1| + \log|t^2+1| + 4\text{Tan}^{-1} t + c$

$$= 4\log\left|1 + \tan\left(\frac{x}{2}\right)\right| + \log\left|1 + \tan^2\left(\frac{x}{2}\right)\right| + 4\text{Tan}^{-1}\left[\tan\left(\frac{x}{2}\right)\right] + c$$

$$= 4\log\left|1 + \tan\left(\frac{x}{2}\right)\right| - 2\log\left|\cos\left(\frac{x}{2}\right)\right| + 2x + c$$

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22. Evaluate $\int \frac{x^3 - 2x + 3}{x^2 + x - 2} dx$

Sol: Since the degree of numerator is greater than that of the denominator, on dividing the numerator with the denominator, we get $\frac{x^3 - 2x + 3}{x^2 + x - 2} = (x-1) + \frac{x+1}{x^2 + x - 2}$

$$\int \frac{x^3 - 2x + 3}{x^2 + x - 2} dx = \int (x-1) dx + \int \frac{(x+1)}{x^2 + x - 2} dx = \frac{(x-1)^2}{2} + \int \frac{(x+1)dx}{x^2 + x - 2} + c \text{....(1)}$$

Let $\frac{x+1}{x^2 + x - 2} = \frac{x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1) + B(x+2)}{(x+2)(x-1)}$

$\Rightarrow A(x-1) + B(x+2) = x+1$

Putting x =1 we get A(0) + B(1+2)=1+1 $\Rightarrow$ 3B =2 $\Rightarrow$ B=2/3

Putting x =–2 we get A(–2–1) +B(0) = –2+1 $\Rightarrow$ –3A= –1 $\Rightarrow$ A=1/3

$$\therefore \int \frac{x-1}{x^2 + x - 2} dx = \frac{1}{3}\int \frac{dx}{x+2} + \frac{2}{3}\int \frac{dx}{x-1} = \frac{1}{3}\log|x+2| + \frac{2}{3}\log|x-1| + k$$

$$= \log(x+2)^{1/3} + \log(x-1)^{2/3} + \log c_1 = \log|\, c_1 (x+2)^{1/3}(x-1)^{1/3}|$$

From (1), $\int \frac{x^3 - 2x + 3}{x^2 + x - 2} dx = \frac{(x-1)^2}{2} + \log|c_1(x+2)^{1/3}(x-1)^{2/3}| + c$

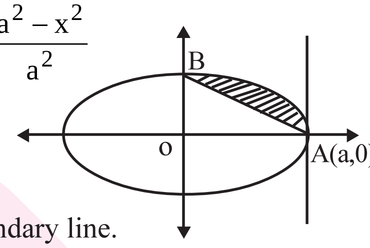
23. Let AOB be the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with OA = a, OB = b. Then S.T area bounded between the chord AB and the arc AB of the ellipse is $\frac{(\pi-2)ab}{4}$

Sol: Given that OA = a $\Rightarrow$ A=(a,0) and OB = b $\Rightarrow$ B = (0, b)

Equation of the line AB is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{y}{b}=1-\frac{x}{a} \Rightarrow \frac{y}{b}=\frac{a-x}{a} \Rightarrow y=\frac{b}{a}(a-x)$.....(1)

Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \Rightarrow \frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$

$\Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2) \Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$.....(2)



Here (2) is the upper boundary curve and (1) is the lower boundary line.

$\therefore$ Required area $= \int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}(a-x)\right]dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx - \frac{b}{a}\int_0^a (a-x)dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a}\right]_0^a - \frac{b}{a}\left[ax-\frac{x^2}{2}\right]_0^a$

$= \frac{b}{a}\left[0+\frac{a^2}{2}\text{Sin}^{-1}(1)-0\right]-\frac{b}{a}\left[a^2-\frac{a^2}{2}\right]=\frac{b}{a}.\frac{a^2}{2}.\frac{\pi}{2}-\frac{b}{a}\frac{a^2}{2}=\frac{\pi ab}{4}-\frac{ab}{2}=\frac{(\pi-2)ab}{4}$ Sq.units

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24. Solve $(x^2y-2xy^2)dx=(x^3-3x^2y)dy$

Sol: Given D.E is $(x^2y-2xy^2)dx=(x^3-3x^2y)dy$

$\Rightarrow \frac{dy}{dx}=\frac{x^2y-2xy^2}{x^3-3x^2y}$. This is a homogeneous D.E

Put, $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore (1) \Rightarrow v+x\frac{dv}{dx}=\frac{2vx^3-2v^2x^3}{x^3-3vx^3}=\frac{(2v-2v^2)x^3}{(1-3v)x^3} \Rightarrow x\frac{dv}{dx}=\frac{2v-2v^2}{1-3v}-v=\frac{2v-2v^2-v+3v^2}{1-3v}=\frac{v^2+v}{1-3v}$

$\Rightarrow \frac{1-3v}{v(1+v)}dv=\frac{dx}{x} \Rightarrow \int\frac{1-3v}{v(1+v)}dv=\int\frac{dx}{x} \Rightarrow \int\left(\frac{1}{v}-\frac{4}{1+v}\right)dv=\int\frac{dx}{x}$

$\Rightarrow \log v-4\log(1+v)=\log x+\log c \Rightarrow \log\frac{v}{(1+v)^4}=\log cx \Rightarrow \frac{v}{(1+v)^4}=cx \Rightarrow v=cx(1+v)^4$

$\Rightarrow \frac{y}{x}=cx\left(1+\frac{y}{x}\right)^4 \Rightarrow \frac{y}{x}=cx\frac{(x+y)^4}{x^4} \Rightarrow x^2y=c(x+y)^4$