SOLVED PAPER-2

Time: 3 Hours **MATHS-1A** Max. Marks : 75

SECTION-A

I. Answer ALL the following Very Short Answer Questions: $10 \times 2 = 20$

1. $f: R \to R$ defined by $f(x)=\frac{2x+1}{3}$, then this function is injection or not? Justify.
2. Find the domain of the real function $\log(x^2-4x+3)$
3. Find the trace of $\begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$ **4.** If $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix then find the value of x.
5. Show that the triangle formed by the vectors $3\bar{i}+5\bar{j}+2\bar{k}, 2\bar{i}-3\bar{j}-5\bar{k}, -5\bar{i}-2\bar{j}+3\bar{k}$ is equilateral
6. Find the vector equation of the line passing through the points $2\bar{i}+\bar{j}+3\bar{k}, -4\bar{i}+3\bar{j}-\bar{k}$
7. If $|\bar{a}+\bar{b}|=|\bar{a}-\bar{b}|$ then find the angle between $\bar{a}$ and $\bar{b}$
8. Find the value of $\sin 34^\circ + \cos 64^\circ - \cos 4^\circ$
9. Find the period of $f(x) = \cos(3x+5)+7$ **10.** If $\sinh x = 3/4$ then find $\cosh 2x$ and $\sinh 2x$.

SECTION-B

II. Answer any FIVE of the following Short Answer Questions: $5 \times 4 = 20$

11. Show that the matrix $A=\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ is non-singular and find A^{-1}.
12. If the points whose position vectors are $3\bar{i}-2\bar{j}-\bar{k}, 2\bar{i}+3\bar{j}-4\bar{k}, -\bar{i}+\bar{j}+2\bar{k}, 4\bar{i}+5\bar{j}+\lambda\bar{k}$ are coplanar, then show that $\lambda=-\frac{146}{17}$
13. For any two vectors $\bar{a}$ and $\bar{b}$, show that $(1+|\bar{a}|)^2+(1+|\bar{b}|)^2=|1-\bar{a}.\bar{b}|^2+|\bar{a}+\bar{b}+\bar{a}\times\bar{b}|^2$
14. Show that $\sin\frac{\pi}{5}.\sin\frac{2\pi}{5}.\sin\frac{3\pi}{5}.\sin\frac{4\pi}{5}=\frac{5}{16}$ **15.** Solve $\cot^2 x-(\sqrt{3}+1)\cot x+\sqrt{3}=0$
16. Show that $2\sin^{-1}\left(\frac{3}{5}\right)-\cos^{-1}\left(\frac{5}{13}\right)=\cos^{-1}\left(\frac{323}{325}\right)$ **17.** If $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=3:5:7$ then, show that a:b:c = 6 : 5 : 4

SECTION-C

III. Answer any FIVE of the following Long Answer Questions: $5 \times 7 = 35$

18. If $f: A\to B$, $g: B\to C$ are two bijective functions then prove that $gof: A\to C$ is also a bijective function.
19. Using the principle of finite Mathematical Induction prove that 2.3+3.4+4.5+.... upto n terms $=\frac{n(n^2+6n+11)}{3}$ **20.** Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$
21. Solve the following equations by using Cramer's rule $2x-y+3z=9, x+y+z=6, x-y+z=2$
22. Prove that the smaller angle θ between any two diagonals of a cube is given by $\cos\theta = 1/3$
23. If A,B,C are angles in a triangle, then prove that $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$
24. In ΔABC prove that $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$

SOLUTIONS

SECTION-A

1. $f: R \to R$ defined by $f(x) = \frac{2x+1}{3}$, then this function is injection or not? Justify.

Sol: Let, $x_1, x_2 \in R$ be such that $f(x_1) = f(x_2)$

$\Rightarrow \frac{2x_1+1}{3} = \frac{2x_2+1}{3} \Rightarrow 2x_1+1 = 2x_2+1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$ $\therefore$ f is an injection.

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2. Find the domain of the real function $\log(x^2 - 4x + 3)$

Sol: $\log(x^2 - 4x + 3)$ is defined when $x^2 - 4x + 3 > 0$

$\Rightarrow (x-1)(x-3) > 0 \Rightarrow x \in (-\infty, 1) \cup (3, \infty)$ $\therefore$ Domain is $(-\infty, 1) \cup (3, \infty)$

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3. Find the trace of $\begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

Sol: Trace: The trace of a square matrix is the **sum of elements in the principal diagonal.**

$A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow \text{Tr}(A) = 1 + (-1) + 1 = 1$

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4. If $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix then find the value of x.

Sol: A is skew symmetric matrix $\Rightarrow A = -A^T$

Given $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix} \Rightarrow -A^T = -\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -x \\ -1 & 2 & 0 \end{bmatrix}$

Now $A = -A^T \Rightarrow x = 2$ [On equating the 3×2 elements in A and $-A^T$]

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5. Show that the triangle formed by the vectors $3\bar{i} + 5\bar{j} + 2\bar{k}, 2\bar{i} - 3\bar{j} - 5\bar{k}, -5\bar{i} - 2\bar{j} + 3\bar{k}$ is equilateral

Sol: Let $\bar{a} = 3\bar{i} + 5\bar{j} + 2\bar{k} \Rightarrow |\bar{a}| = \sqrt{9+25+4} = \sqrt{38}$

$\bar{b} = 2\bar{i} - 3\bar{j} - 5\bar{k} \Rightarrow |\bar{b}| = \sqrt{4+9+25} = \sqrt{38}$

$\bar{c} = -5\bar{i} - 2\bar{j} + 3\bar{k} \Rightarrow |\bar{c}| = \sqrt{25+4+9} = \sqrt{38}$

$\therefore |\bar{a}| = |\bar{b}| = |\bar{c}|$. Hence, the given vectors form an equilateral.

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6. Find the vector equation of the line passing through the points $2\bar{i} + \bar{j} + 3\bar{k}, -4\bar{i} + 3\bar{j} - \bar{k}$

Sol: Given points $A(\bar{a}) = 2\bar{i} + \bar{j} + 3\bar{k}$, $B(\bar{b}) = -4\bar{i} + 3\bar{j} - \bar{k}$

Vector equation of the line is $\bar{r} = (1-t)\bar{a} + t\bar{b}$, $t \in R$

$\therefore \bar{r} = (1-t)(2\bar{i} + \bar{j} + 3\bar{k}) + t(-4\bar{i} + 3\bar{j} - \bar{k}), t \in R$

7. If $|\bar{a}+\bar{b}|=|\bar{a}-\bar{b}|$ then find the angle between $\bar{a}$ and $\bar{b}$

Sol: Given that $|\bar{a}+\bar{b}|=|\bar{a}-\bar{b}| \Rightarrow |\bar{a}+\bar{b}|^2=|\bar{a}-\bar{b}|^2 \Rightarrow \bar{a}^2+\bar{b}^2+2\bar{a}.\bar{b}=\bar{a}^2+\bar{b}^2-2\bar{a}.\bar{b}$

$\Rightarrow 4\bar{a}.\bar{b}=0 \Rightarrow \bar{a}.\bar{b}=0 \Rightarrow \bar{a}\perp\bar{b}$ $\therefore$ Angle between $\bar{a}$ and $\bar{b}$ is 90°

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8. Find the value of $\sin 34^\circ+\cos 64^\circ-\cos 4^\circ$

Sol: $\sin 34^\circ+\cos 64^\circ-\cos 4^\circ=\sin 34^\circ(\cos 64^\circ-\cos 4^\circ)=\sin 34^\circ-2\sin\frac{64^\circ+4^\circ}{2}\sin\frac{64^\circ-4^\circ}{2}$

$(\because \cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2})$

$=\sin 34^\circ-2\sin 34^\circ-\sin 30^\circ$ $=\sin 34^\circ-2\sin 34^\circ.\frac{1}{2}$ $=\sin 34^\circ-\sin 34^\circ=0$

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9. Find the period of f(x) = cos(3x + 5) + 7

Sol: We know that period of coskx is $\frac{2\pi}{k}$ $\therefore$ Period of cos(3x + 5) + 7 is $2\pi/3$

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10. If sinhx = 3/4 then find cosh2x and sinh2x.

Sol: Given sinhx=3/4, then $\cosh x=\sqrt{\sinh^2 x+1}=\sqrt{\left(\frac{3}{4}\right)^2+1}=\sqrt{\frac{9}{16}+1}=\sqrt{\frac{9+16}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$

(i) $\cosh(2x)=\cosh^2 x+\sinh^2 x=\left(\frac{5}{4}\right)^2+\left(\frac{3}{4}\right)^2=\frac{25}{16}+\frac{9}{16}=\frac{34}{16}=\frac{17}{8}$

(ii) $\sinh(2x)=2\sinh x\cosh x=2\times\frac{3}{4}\times\frac{5}{4}=\frac{15}{8}$

SECTION-B

11. Show that the matrix $A=\begin{bmatrix}1&2&1\\3&2&3\\1&1&2\end{bmatrix}$ is non-singular and find A^{-1}.

Sol: Given $A=\begin{bmatrix}1&2&1\\3&2&3\\1&1&2\end{bmatrix}$

$\det A=1(4-3)-2(6-3)+1(3-2)=1-6+1=-4\neq 0 \Rightarrow$ A is non-singular

$$\text{Adj } A=\begin{bmatrix}\begin{vmatrix}2&3\\1&2\end{vmatrix} & -\begin{vmatrix}3&3\\1&2\end{vmatrix} & \begin{vmatrix}3&2\\1&1\end{vmatrix}\\ -\begin{vmatrix}2&1\\1&2\end{vmatrix} & \begin{vmatrix}1&1\\1&2\end{vmatrix} & -\begin{vmatrix}1&2\\1&1\end{vmatrix}\\ \begin{vmatrix}2&1\\2&3\end{vmatrix} & -\begin{vmatrix}1&1\\3&3\end{vmatrix} & \begin{vmatrix}1&2\\3&2\end{vmatrix}\end{bmatrix}^T=\begin{bmatrix}1&-3&1\\-3&1&1\\4&0&-4\end{bmatrix}^T \quad \therefore \text{Adj } A=\begin{bmatrix}1&-3&4\\-3&1&0\\1&1&-4\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj } A)=\frac{1}{-4}\begin{bmatrix}1&-3&4\\-3&1&0\\1&1&-4\end{bmatrix}=\begin{bmatrix}-1/4&3/4&-1\\3/4&-1/4&0\\-1/4&-1/4&1\end{bmatrix}$$

12. If the points whose position vectors are $3\bar{i}-2\bar{j}-\bar{k}, 2\bar{i}+3\bar{j}-4\bar{k}, -\bar{i}+\bar{j}+2\bar{k}, 4\bar{i}+5\bar{j}+\lambda\bar{k}$ are coplanar, then show that $\lambda=-\frac{146}{17}$

Sol: We take $\overline{OP}=3\bar{i}-2\bar{j}-\bar{k}, \overline{OQ}=2\bar{i}+3\bar{j}-4\bar{k}$,

$\overline{OR}=-\bar{i}+\bar{j}+2\bar{k}, \overline{OS}=4\bar{i}+5\bar{j}+\lambda\bar{k}$, where 'O' is the origin.

$\overline{PQ}=\overline{OQ}-\overline{OP}=(2\bar{i}+3\bar{j}-4\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=-\bar{i}+5\bar{j}-3\bar{k}$

$\overline{PR}=\overline{OR}-\overline{OP}=(-\bar{i}+\bar{j}+2\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=-4\bar{i}+3\bar{j}+3\bar{k}$

$\overline{PS}=\overline{OS}-\overline{OP}=(4\bar{i}+5\bar{j}+\lambda\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=\bar{i}+7\bar{j}+(\lambda+1)\bar{k}$

But $[\overline{PQ}\ \overline{PR}\ \overline{PS}]=0$ [Since P,Q,R,S are coplanar]

$$\Rightarrow\begin{vmatrix}-1 & 5 & -3\\ -4 & 3 & 3\\ 1 & 7 & \lambda+1\end{vmatrix}=0$$

$\Rightarrow (-1)[3(\lambda+1)-21] -5\ [-4(\lambda+1)-3]-3[(-28)-3]=0 \Rightarrow -1(3\lambda-18) -5(-4\lambda-7)-3(-31)=0$

$\Rightarrow -3\lambda+18+20\lambda+35+93=0 \Rightarrow -3\lambda+20\lambda+35+93+18=0 \Rightarrow -17\lambda=146 \Rightarrow \lambda=-146/17$

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13. For any two vectors $\bar{a}$ and $\bar{b}$, show that $(1+|\bar{a}|)^2+(1+|\bar{b}|)^2=|1-\bar{a}.\bar{b}|^2+|\bar{a}+\bar{b}+\bar{a}\times\bar{b}|^2$

Sol: Let, $(\bar{a},\bar{b})=\theta$

R.H.S. $|1-\bar{a}.\bar{b}|^2+|\bar{a}+\bar{b}+\bar{a}\times\bar{b}|^2=1+(\bar{a}.\bar{b})^2-2(\bar{a}.\bar{b})+|\bar{a}|^2+|\bar{b}|^2+|\bar{a}\times\bar{b}|^2+2(\bar{a}.\bar{b})$
$+2[\bar{b}.(\bar{a}\times\bar{b})]+2[(\bar{a}\times\bar{b}).\bar{a}]$

$=1+|\bar{a}|^2|b|^2\cos^2\theta-2(\bar{a}.\bar{b})+|\bar{a}|^2+|\bar{b}|^2+|\bar{a}\times\bar{b}|^2+2(\bar{a}.\bar{b})+0+0$

here, $\bar{b}(\bar{a}\times\bar{b})=0$ and $(\bar{a}\times\bar{b}).\bar{a}=0$

$=1+|\bar{a}|^2|\bar{b}|^2(1-\sin^2\theta)+|\bar{a}|^2+|\bar{b}|^2+|\bar{a}\times\bar{b}|^2$

$=1+|\bar{a}|^2|\bar{b}|^2-|\bar{a}|^2|\bar{b}|^2\sin^2\theta+|\bar{a}|^2+|\bar{b}|^2+|\bar{a}\times\bar{b}|^2$

$=1+|\bar{a}|^2|\bar{b}|^2-|\bar{a}\times\bar{b}|^2+|\bar{a}|^2+|\bar{b}|^2+|\bar{a}\times\bar{b}|^2$

$=1+|\bar{a}|^2+|\bar{b}|^2+|\bar{a}|^2|\bar{b}|^2$

$=(1+|\bar{a}|^2)(1+|\bar{b}|^2)=$ LHS.

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14. Show that $\sin\frac{\pi}{5}.\sin\frac{2\pi}{5}.\sin\frac{3\pi}{5}.\sin\frac{4\pi}{5}=\frac{5}{16}$

Sol: L.H.S.= $\sin\frac{\pi}{5}.\sin\frac{2\pi}{5}.\sin\frac{3\pi}{5}.\sin\frac{4\pi}{5}=\sin\frac{\pi}{5}.\sin\frac{2\pi}{5}.\sin\left(\pi-\frac{2\pi}{5}\right).\sin\left(\pi-\frac{\pi}{5}\right)$

$=\sin\frac{\pi}{5}.\sin\frac{2\pi}{5}.\sin\frac{2\pi}{5}.\sin\frac{\pi}{5}=\sin^2\frac{\pi}{5}.\sin^2\frac{2\pi}{5}=\sin^2 36^\circ\sin^2 72^\circ.$

$$\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2=\left(\frac{10-2\sqrt{5}}{16}\right)\left(\frac{10+2\sqrt{5}}{16}\right)=\frac{100-20}{16\times16}=\frac{80}{16\times16}=\frac{5}{16}=\text{R.H.S}$$

15. If $0 < x < \frac{\pi}{2}$ then solve $\cot^2 x - (\sqrt{3}+1)\cot x + \sqrt{3} = 0$

Sol: $\cot^2 x - (\sqrt{3}+1)\cot x + \sqrt{3} = 0 \Rightarrow \cot^2 x - \sqrt{3}\cot x - \cot x + \sqrt{3} = 0$

$\Rightarrow \cot x(\cot x - \sqrt{3}) - (\cot x - \sqrt{3}) = 0 \Rightarrow (\cot x - 1)(\cot x - \sqrt{3}) = 0$

$\Rightarrow \cot x - 1 = 0$ (or) $\cot x - \sqrt{3} = 0 \Rightarrow \cot x = 1$ (or) $\cot x = \sqrt{3}$

$\Rightarrow \tan x = 1$ (or) $\tan x = \frac{1}{\sqrt{3}}$

since $0 < x < \frac{\pi}{2}$, $\tan x = 1 = \tan\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$; $\tan x = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$

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16. Show that $2\mathrm{Sin}^{-1}\left(\frac{3}{5}\right) - \mathrm{Cos}^{-1}\left(\frac{5}{13}\right) = \mathrm{Cos}^{-1}\left(\frac{323}{325}\right)$

Sol: Let, $\sin^{-1}\left(\frac{3}{5}\right) = \theta \Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \cos\theta = \frac{4}{5} \Rightarrow 2\sin^{-1}\frac{3}{5} = 2\theta$

Also, $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25} \Rightarrow 2\theta = \sin^{-1}\frac{24}{25}$

Hence, the given problem reduces to $\sin^{-1}\frac{24}{25} - \cos^{-1}\frac{5}{13} = \cos^{-1}\frac{323}{325}$

Let, $\sin^{-1}\frac{24}{25} = \alpha \Rightarrow \sin\alpha = \frac{24}{25} \Rightarrow \cos\alpha = \frac{7}{25}$

$\cos^{-1}\frac{5}{13} = \beta \Rightarrow \cos\beta = \frac{5}{13} \Rightarrow \sin\beta = \frac{12}{13}$

Now $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{7}{25}\times\frac{5}{13} + \frac{24}{25}\times\frac{12}{13} = \frac{35+288}{325} = \frac{323}{325}$

$\Rightarrow \alpha - \beta = \mathrm{Cos}^{-1}\frac{323}{325}$. $\Rightarrow \mathrm{Sin}^{-1}\frac{24}{25} - \mathrm{Cos}^{-1}\frac{5}{13} = \mathrm{Cos}^{-1}\frac{323}{325}$ $\Rightarrow 2\mathrm{Sin}^{-1}\frac{3}{5} - \mathrm{Cos}^{-1}\frac{5}{13} = \mathrm{Cos}^{-1}\frac{323}{325}$.

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17. If $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3:5:7$ then show that a : b : c = 6 : 5 : 4

Sol: Given $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3:5:7 \Rightarrow \frac{\not{s}(s-a)}{\not{\Delta}} : \frac{\not{s}(s-b)}{\not{\Delta}} : \frac{\not{s}(s-c)}{\not{\Delta}} = 3:5:7$

$\Rightarrow (s-a):(s-b):(s-c) = 3:5:7$

We take $s-a = 3k$ ...(1), $s-b = 5k$ ....(2), $s-c = 7k$ .....(3), $k\neq 0$

(1)+(2)+(3) $\Rightarrow 3s-(a+b+c) = 3k+5k+7k \Rightarrow 3s-2s = 15k \Rightarrow s = 15k$

From (1), $s-a = 3k \Rightarrow 15k-a = 3k \Rightarrow a = 15k-3k = 12k$

From (2), $s-b = 5k \Rightarrow 15k-b = 5k \Rightarrow b = 15k-5k = 10k$

From (3), $s-c = 7k \Rightarrow 15k-c = 7k \Rightarrow c = 15k-7k = 8k$

$\therefore a:b:c = 12\not{k} : 10\not{k} : 8\not{k} = \not{12} : \not{10} : \not{8} = 6:5:4$

SECTION-C

18. If f : A→B, g : B→C are two bijective functions then prove that gof : A→C is also a bijective function.

Sol: Given that f,g are bijective functions. So f,g are both one one and onto functions.

(i) To prove that gof:A→C is one one

Let $(gof)(a_1)=(gof)(a_2)$, [for $a_1, a_2 \in A$]

$\Rightarrow g[f(a_1)]=g[f(a_2)]$

$\Rightarrow f(a_1)=f(a_2)$ ($\because$ g is one one)

$\Rightarrow a_1=a_2$ ($\because$ f is one one)

$\therefore$ gof: A→C is one one.

(ii) To prove that gof:A→C is onto.

Given f:A→B is onto, then f(a)=b........(1), [$\because$ for all $b\in B$ there exist $a\in A$ such that f(a)=b]

Given g:B→C is onto, then g(b)=c......(2), [$\because$ for all $c\in C$ there exist $b\in B$ such that g(b)=c]

Now (gof)(a) =g[f(a)]=g(b)=c, [From (1) and (2)]

$\therefore$ gof: A→C is onto. [$\because$ for all $c\in C$ we got some $a\in A$ such that gof(a)=c]

Hence, we proved that gof:A→C is a bijective function.

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19. Using the principle of finite Mathematical Induction prove that

2.3+3.4+4.5+.... upto n terms $= \dfrac{n(n^2+6n+11)}{3}$

Sol: The first terms of the series 2, 3, 4....are in A.P with a =2, d=1.

$\therefore T_n = a+(n-1)d = 2+(n-1)1= n+1$

The second terms of the series 3, 4, 5.....are obtained by adding 1 to the terms of first series.

$\therefore$ In the second series $T_n = (n+1)+1 = n+2$

Let S(n) : $2.3+3.4+..........+(n+1)(n+2) = \dfrac{n(n^2+6n+11)}{3}$

(a)L.H.S of S(1) = 2.3=6

R.H.S of S(1) $= \dfrac{1(1^2+6(1)+11)}{3} = \dfrac{1+6+11}{3} = \dfrac{18}{3} = 6$

$\therefore$ L.H.S of S(1) = R.H.S of S(1) ⇒ S(1) is true.

(b) Assume that S(k) is true for $k\in N$

S(k) : $2.3+3.4+..........+(k+1)(k+2) = \dfrac{k(k^2+6k+11)}{3}$......(1)

(c)Now we show that S(k+1)is true.

S(k+1) :$[2.3+3.4+..........+(k+1)(k+2)]+(k+2)(k+3) = \dfrac{(k+1)[(k+1)^2+6(k+1)+11]}{3}$

L.H.S of S(k+1) = $[2.3+3.4+..........+(k+1)(k+2)]+(k+2)(k+3)$

$$= \frac{k(k^2+6k+11)}{3} + (k+1)(k+2) = \frac{k(k^2+6k+11)+3(k+1)(k+2)}{3}$$

$$= \frac{(k^3+6k^2+11k)+3(k^2+5k+6)}{3} = \frac{(k^3+6k^2+11k)+3k^2+15k+18)}{3} = \frac{k^3+9k^2+26k+18}{3}........(2)$$

R.H.S of S(k+1) $= \dfrac{(k+1)[(k+1)^2+6(k+1)+11]}{3} = \dfrac{(k+1)[(k^2+2k+1)+6k+6+11]}{3}$

$$= \frac{(k+1)(k^2+8k+18)}{3} = \frac{(k^3+8k^2+18k+k^2+8k+18)}{3} = \frac{k^3+8k^2+26k+18}{3} \text{.........(3)}$$

From (2) & (3) L.H.S of S(k+1) = R.H.S of S(k+1)

$\therefore$ S(k+1) is true whenever S(k) is true.

Hence by the Principle of Mathematical Induction the given statement is true $\forall$ n$\in$N.

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20. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol: L.H.S $= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix}$ $(\because R_2 \Rightarrow R_2 - R_1$, $R_3 \Rightarrow R_3 - R_1)$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & (b-a)(b+a) & (b-a)(b^2+ba+a^2) \\ 0 & (c-a)(c+a) & (c-a)(c^2+ca+a^2) \end{vmatrix} \quad \left(\begin{array}{l} \because a^2-b^2=(a-b)(a-b) \\ a^3-b^3=(a-b)(a^2+ab+b^2) \end{array} \right)$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & a+c & a^2+ac+c^2 \end{vmatrix}$$

(Taking $(b-a)$ common from R_2 and $(c-a)$ common form R_3)

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & ac-ab+c^2-b^2 \end{vmatrix} \quad (\because R_3 \to R_3 - R_1)$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & (c-b)(a+b+c) \end{vmatrix} \quad \left(\begin{array}{l} \because ac-ab+c^2-b^2 \\ = a(c-b)+(c-b)(c+b) \\ = (c-b)(a+b+c) \end{array} \right)$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & 1 & (a+b+c) \end{vmatrix}$$

$= (b-a)(c-a)(c-b)1[(a+b)(a+b+c)-(a^2+ab+b^2)]$

$= (b-a)(c-a)(c-b)[(a+b)^2+c(a+b)-(a^2+ab+b^2)]$

$= (b-a)(c-a)(c-b)[(\cancel{a^2}+\cancel{b^2}+2ab)+(ca+cb)-\cancel{a^2}-ab-\cancel{b^2}]$

$= (a-b)(b-c)(c-a)(ab+bc+ca) = $ R.H.S

21. Solve the following equations by using Cramer's rule $2x - y + 3z = 9$, $x + y + z = 6$, $x - y + z = 2$

Sol: Given equations in the matrix equation form: AX = D, where

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$\Delta = \det A = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+1) + 1(1-1) + 3(-1-1) = 2(2)+1(0)+3(-2) = 4+0-6 = -2 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 9(1+1) + 1(6-2) + 3(-6-2) = 9(2) + 1(4) + 3(-8) = 18 + 4 - 24 = -2;$$

$$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2(6-2) - 9(1-1) + 3(2-6) = 2(4) - 9(0) + 3(-4) = 8 - 0 - 12 = -4;$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2(2+6) + 1(2-6) + 9(-1-1) = 2(8) + 1(-4) + 9(-2) = 16 - 4 - 18 = -6$$

By Cramer's rule,

$$x = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1; \quad y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2} = 3$$

$\therefore$ x=1, y=2, z=3 are the solutions of the given equations.

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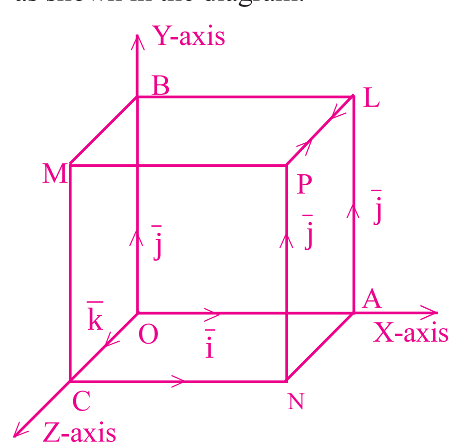
22. P.T the smaller angle θ between any two diagonals of a cube is given by $\cos\theta = 1/3$

Sol: Consider a unit cube with vertices O,A,B,C,L, M, N, P as shown in the diagram.

We take $\overline{OA} = \bar{i}, \overline{OB} = \bar{j}, \overline{OC} = \bar{k}$



Consider 2 diagonals $\overline{OP}$ and $\overline{CL}$

$\overline{OP} = \overline{OA} + \overline{AL} + \overline{LP} \quad = \bar{i} + \bar{j} + \bar{k}$ ......(1)

$\overline{CL} = \overline{CN} + \overline{NP} + \overline{PL} \quad = \bar{i} + \bar{j} - \bar{k}$ ....(2)

$$\therefore \cos\theta = \frac{\overline{OP}.\overline{CL}}{|\overline{OP}||\overline{CL}|}$$

$$= \frac{(1\times1) + (1\times1) + (1\times(-1))}{\sqrt{(1^2+1^2+1^2)\,(1^2+1^2+(-1)^2)}} = \frac{1+1-1}{\sqrt{(3)(3)}} = \frac{1}{3}$$

23. If A, B, C are angles in a triangle, then prove that $\sin^2A+\sin^2B-\sin^2C=2\sin A\sin B\cos C$

Sol: Given A,B,C are angles of a triangle, then A+B+C=180º

L.H.S = $\sin^2A + (\sin^2B - \sin^2C)$

$= \sin^2A+\sin(B+C)\sin(B-C) = \sin^2A+\sin(180^\circ-A)\sin(B-C)$

$= \sin^2A+\sin A\sin(B-C) = \sin A[\sin A+\sin(B-C)]$

$= \sin A[\sin(180^\circ-(B+C))+\sin(B-C)]= \sin A[\sin(B+C))+\sin(B-C)]$

$= \sin A(2\sin B\cos C)$ $\quad [\because (\sin(A+B)) + \sin(A-B)) = 2\sin A\cos B]$

$= 2\sin A\sin B\cos C$ = R.H.S

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24. In ΔABC prove that $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$

Sol: $\text{L.H.S.} = \sum\frac{r_1}{bc} = \sum\frac{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{(2R\sin B)(2R\sin C)}$

$$= \sum\frac{\cancel{4R}\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cancel{4R}(R\sin B.\sin C)} = \sum\frac{\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{R\sin B.\sin C} = \sum\frac{\sin\frac{A}{2}\cancel{\cos\frac{B}{2}}\cancel{\cos\frac{C}{2}}}{R(2\sin\frac{B}{2}\cancel{\cos\frac{B}{2}})(2\sin\frac{C}{2}\cancel{\cos\frac{C}{2}})}$$

$$= \sum\frac{\sin\frac{A}{2}}{4R\sin\frac{B}{2}\sin\frac{C}{2}} = \sum\frac{\sin\frac{A}{2}\sin\frac{A}{2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sum\frac{\sin^2\frac{A}{2}}{r} = \frac{1}{r}\sum\sin^2\frac{A}{2}$$

$$= \frac{1}{r}\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right) \quad = \frac{1}{r}\left(\left(1-\cos^2\frac{A}{2}\right)+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)$$

$$= \frac{1}{r}\left(1-(\cos^2\frac{A}{2}-\sin^2\frac{B}{2})+\sin^2\frac{C}{2}\right)$$

$$= \frac{1}{r}\left(1-(\cos\frac{A+B}{2}.\cos\frac{A-B}{2})+\sin^2\frac{C}{2}\right) \quad (\because \cos^2A-\sin^2B=\cos(A+B)\cos(A-B))$$

$$= \frac{1}{r}\left[\left(1-\cos(180^0-\frac{C}{2}).\cos\frac{A-B}{2}\right)+\sin^2\frac{C}{2}\right] \quad = \frac{1}{r}\left(1-\sin\frac{C}{2}.\cos\frac{A-B}{2}+\sin^2\frac{C}{2}\right)$$

$$= \frac{1}{r}\left(1-\sin\frac{C}{2}(\cos\frac{A-B}{2}-\sin\frac{C}{2})\right)$$

$$= \frac{1}{r}\left(1-\sin\frac{C}{2}(\cos\frac{A-B}{2}-\cos\frac{A+B}{2})\right) \quad \left[\therefore \sin\frac{C}{2}=\sin\left(90^\circ-\frac{A+B}{2}\right)=\cos\frac{A+B}{2}\right]$$

$$= \frac{1}{r}\left(1-\sin\frac{C}{2}(2\sin\frac{A}{2}\sin\frac{B}{2}\right) = \frac{1}{r}\left(1-(2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})\right)$$

$$= \frac{1}{r}\left(1-\frac{2(2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})}{2R}\right) \quad [\text{On multiplying \& dividing by } 2R]$$

$$= \frac{1}{r}\left(1-\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})}{2R}\right) = \frac{1}{r}\left(1-\frac{r}{2R}\right) = \frac{1}{r}-\frac{1}{\cancel{r}}.\frac{\cancel{r}}{2R} = \frac{1}{r}-\frac{1}{2R} = \text{R.H.S}$$