SOLVED PAPER -1

Time: 3 Hours

MATHS-2B

Max. Marks : 75

(Board of Intermediate Education Model Paper)

Section -A

I. Answer ALL the following Very Short Answer Questions: 10 × 2 = 20

1. Find the power of the point P(–1,1) with respect to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$
2. State the necessary and sufficient condition for $lx + my + n = 0$ to be a normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.
3. Find the angle between the circles $x^2 + y^2 - 12x - 6y + 41 = 0$ and $x^2 + y^2 + 4x + 6y - 59 = 0$
4. Find the equation of the parabola whose focus is (1, –7) and vertex is (1, –2).
5. Find the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
6. Evaluate $\int \frac{1}{(x+3)\sqrt{x+2}}dx$
7. Evaluate $\int \frac{\sin^4 x}{\cos^6 x}dx$
8. Evaluate $\int_0^1 \frac{x^2}{x^2+1}dx$
9. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x}dx$
10. Find the order and degree to the differential equation $\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right]^{\frac{6}{5}} = 6y$

SECTION-B

II. Answer any FIVE of the following Short Answer Questions: 5 × 4 = 20

11. Find the pole of the line $3x+4y-45=0$ w.r.t the circle $x^2+y^2-6x-8y+5=0$
12. Find the equation of the circle passing through the points of intersection of the circles $x^2+y^2-8x-6y+21=0$, $x^2+y^2-2x-15=0$ and (1,2)
13. Find the length of major axis, minor axis,latusrectum, eccentricity of the ellipse of $9x^2+16y^2=144$

14. Show that the point of intersection of the perpendicular to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a>b) lies on a circle.
15. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are (i) Parallel to (ii) Perpendicular to the line $y = x-7$
16. Find the reduction formula for $\int_0^{\pi/2} \sin^n x\,dx$
17. Solve $(1 + y^2)\,dx = (\tan^{-1}y - x)\,dy$

SECTION-C

III. Answer any FIVE of the following Long Answer Questions: 5 × 7 = 35

18. Show that the points (1, 1), (–6, 0), (–2, 2) and (–2, –8) are concyclic.
19. Find the direct common tangents to the circles $x^2+y^2+22x-4y-100=0$, $x^2+y^2-22x+4y+100=0$
20. If y_1, y_2, y_3 are the y-coordinates of the vertices of the triangle inscribed in the parabola $y^2 = 4ax$ then show that the area of the triangle is $\frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$ square units.
21. Evaluate $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx$
22. Evaluate $\int \frac{dx}{(1+x)\sqrt{3+2x-x^2}}$
23. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$
24. Solve $\frac{dy}{dx}=\frac{2x+y+3}{2y+x+1}$

SOLUTIONS

Section -A

1. Find the power of the point P(–1,1) with respect to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$

Sol: Here $P(x_1, y_1) = (-1, 1)$ and $S \equiv x^2 + y^2 - 6x + 4y - 12$

$\therefore$ Power of $P(x_1, y_1)$ is $S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$=(-1)^2 + 1^2 - 6(-1) + 4(1) - 12 = 1 + 1 + 6 + 4 - 12 = 0$

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2. State the necessary and sufficient condition for lx+my+n=0 to be a normal to the circle $x^2+y^2+2gx+2fy+c=0$.

Sol: The line $lx+my+n=0$ is a normal to the circle $x^2+y^2+2gx+2fy+c=0$

$\Leftrightarrow$ Its center $(-g,-f)$ lies on the line $lx+my+n=0$

$\Leftrightarrow l(-g)+m(-f)+n=0 \Leftrightarrow lg+mf = n$

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3. Find the angle between the circles $x^2+y^2-12x-6y+41=0$ and $x^2+y^2+4x+6y-59=0$

Sol: Equations of the given circles are $x^2+y^2-12x-6y+41=0$, $x^2+y^2+4x+6y-59=0$

Centre $C_1=(6,3)$, $C_2=(-2,-3)$

$d = C_1C_2 = \sqrt{(6+2)^2+(3+3)^2} = \sqrt{64+36} = \sqrt{100} = 10$

$r_1 = \sqrt{36+9-41} = \sqrt{45-41} = \sqrt{4} = 2$; $r_2 = \sqrt{4+9+59} = \sqrt{13+59} = \sqrt{72} = 6\sqrt{2}$

If θ is the angle between the given circles,then $\cos\theta = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$

$$\Rightarrow \cos\theta = \frac{10^2 - 2^2 - (6\sqrt{2})^2}{2(2)(6\sqrt{2})} = \frac{100-4-72}{2\times 2\sqrt{72}} = \frac{24}{4\times 6\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45^\circ$$

Angle between the circles is $\theta = 45^\circ = \dfrac{\pi}{4}$

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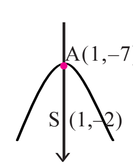
4. Find the equation of the parabola whose focus is (1, –7) and vertex is (1, –2).

Sol: The vertex A = (1, –2), focus S=(1, –7). Here, the x-coordinates of A, S are equal

$\therefore$ the axis is parallel to the y-axis.

Also, the parabola is vertically downward. ($\because$ focus S lies below the vertex A)



Now, $a = AS = \sqrt{(1-1)^2+(-2+7)^2} = \sqrt{25} = 5$

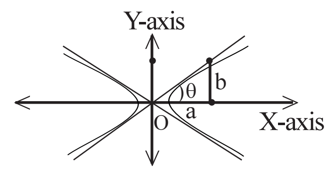
$\therefore$ the equation of the parabola with vertex $(h, k)=(1,-2)$ is $(x-h)^2 = -4a(y-k)$

$\Rightarrow (x-1)^2 = -4(5)(y-(-2)) \Rightarrow (x-1)^2 = -20(y+2)$

5. Find the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Sol: The asymptotes of S=0 are $y=\pm\frac{b}{a}x$

They are equally inclined at an angle θ to the x-axis.



Here, $\tan\theta=\frac{b}{a}$ (or) $\theta=\text{Tan}^{-1}\frac{b}{a}$

Now, $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{\frac{a^2+b^2}{a^2}}=e$

$\therefore \sec\theta=e\Rightarrow\theta=\text{Sec}^{-1}e$

Hence, the angle between the 2 asymptotes is $2\theta=2\text{Sec}^{-1}(e)$ (or) 2Tan^{-1} b/a

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6. Evaluate $\int\frac{1}{(x+3)\sqrt{x+2}}dx$

Sol: Put $\sqrt{x+2}=t$, then $\frac{1}{2\sqrt{x+2}}dx=dt$

Also, $x+2=t^2\Rightarrow x+3=t^2+1$

$\therefore I=\int\frac{1}{(x+3)\sqrt{x+2}}dx=\int\frac{2}{t^2+1}dt=2\text{Tan}^{-1}(t)+c\Rightarrow 2\text{Tan}^{-1}(\sqrt{x+2})+c$

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7. Evaluate $\int\frac{\sin^4 x}{\cos^6 x}dx$

Sol: $\int\frac{\sin^4 x}{\cos^6 x}dx=\int\frac{\sin^4 x}{\cos^4 x}\frac{1}{\cos^2 x}dx=\int\tan^4 x.\sec^2 x\,dx$

Put $\tan x=t\Rightarrow\sec^2x\,dx=dt$

$\therefore I=\int t^4\,dt=\frac{t^5}{5}+c=\frac{\tan^5 x}{5}+c$

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8. Evaluate $\int_0^1\frac{x^2}{x^2+1}dx$

Sol: $\int_0^1\frac{x^2}{1+x^2}dx=\int_0^1\frac{1+x^2-1}{1+x^2}dx=\int_0^1\left(\frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}\right)dx$

$=\int_0^1\left(1-\frac{1}{1+x^2}\right)dx=\left[x-\text{Tan}^{-1}x\right]_0^1=(1-\text{Tan}^{-1}1)-(0-\text{Tan}^{-1}0)=1-\frac{\pi}{4}$

9. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx$

Sol: We know $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx \quad ....(1)$$

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right) - \cos^2\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right) + \cos^3\left(\frac{\pi}{2}-x\right)} dx = \int_0^{\pi/2} \frac{\cos^2 x - \sin^2 x}{\cos^3 x + \sin^3 x} dx \quad ....(2)$$

Adding (1) and (2) we get $2I = \int_0^{\pi/2} \frac{0}{\cos^3 x + \sin^3 x} dx = 0 \Rightarrow I = 0$

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10. Find the order and degree to the differential equation $\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right]^{\frac{6}{5}} = 6y$

Sol: The given D.E is $\left(\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right)^{6/5} = 6y \Rightarrow \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = (6y)^{5/6}$

Here, the highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ order =2

The exponent of $\frac{d^2y}{dx^2}$ is 1 $\therefore$ degree=1

SECTION-B

11. Find the pole of the line 3x+4y–45=0 w.r.t the circle $x^2+y^2-6x-8y+5=0$

Sol: Let $P(x_1,y_1)$ be the pole of $3x + 4y - 45 = 0$ w.r.to the circle $x^2 + y^2 - 6x - 8y + 5 = 0$

$\therefore$ the polar of $P(x_1, y_1)$ w.r.to $S = x^2 + y^2 - 6x - 8y + 5 = 0$ is $S_1 = 0$

$\Rightarrow x_1x + y_1y - 3(x_1+x) - 4(y_1+y) + 5 = 0 \quad \Rightarrow (x_1-3)x + (y_1-4)y - 3x_1 - 4y_1 + 5 = 0$

Comparing the above equation with the given line 3x+4y–45=0, we have

$\frac{x_1-3}{3} = \frac{y_1-4}{4} = \frac{-3x_1-4y_1+5}{-45}$ =k (say) $\quad \Rightarrow x_1-3=3k \Rightarrow x_1=3+3k;\ y_1-4=4k \Rightarrow y_1=4+4k$

$\therefore -3x_1-4y_1+5=-45k$

$\Rightarrow -3(3+3k)-4(4+4k)+5=-45 \Rightarrow -9-9k-16-16k+5=-45k$

$\Rightarrow 20k=20 \Rightarrow k=1$

$\Rightarrow x_1=3+3k=3+3(1)=6;\qquad y_1=4+4k=4+4(1)=8 \qquad \therefore$ Pole is $(x_1,y_1)=(6,8)$

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12. Find the equation of the circle passing through the points of intersection of the circles $x^2+y^2-8x-6y+21=0$, $x^2+y^2-2x-15=0$ and (1,2)

Sol: The equation of circle passing through the points of intersection of the given circles is

$(x^2+y^2-8x-6y+21)+\lambda(x^2+y^2-2x-15)=0$...(1), $\lambda\in R$

If it passes through (1,2) then $(1^2+2^2-8(1)-6(2)+21)+\lambda(1^2+2^2-2(1)-15)=0$

$\Rightarrow (1+4-8-12+21)+\lambda(1+4-2-15)=0$

$\Rightarrow 6+\lambda(-12)=0 \Rightarrow 6-12\lambda=0 \Rightarrow 12\lambda=6 \Rightarrow \lambda=6/12=1/2$

Hence, from (1), the equation of the required circle is $(x^2+y^2-8x-6y+21)+\frac{1}{2}(x^2+y^2-2x-15)=0$

$\Rightarrow 2(x^2+y^2-8x-6y+21)+1(x^2+y^2-2x-15)=0$

$\Rightarrow (2x^2+2y^2-16x-12y+42)+(x^2+y^2-2x-15)=0$

$\Rightarrow 3(x^2+y^2)-18x-12y+27=0$

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13. Find the length of major axis, minor axis, latusrectum, eccentricity of the ellipse of $9x^2+16y^2=144$

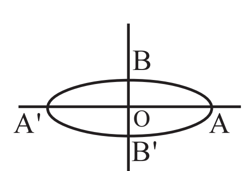
Sol: Equation of ellipse is $9x^2+16y^2=144 \Rightarrow \frac{9x^2}{144}+\frac{16y^2}{144}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$.

Here, $a^2=16 \Rightarrow a=4$, $b^2=9 \Rightarrow b=3 \Rightarrow a>b$.

Hence the ellipse is horizontal

(i) Length of Major axis 2a=2(4)=8

(ii) Length of Minor axis 2b = 2(3)=6



(iii) Length of latusrectum $\frac{2b^2}{a}=\frac{2(9)}{4}=\frac{9}{2}$

(iv) Eccentricity $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}$

14. Show that the point of intersection of the perpendicular to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a>b) lies on a circle.

Sol: Let the equation of the ellipse be $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

Let $P(x_1,y_1)$ be a point on the locus

The equation of the tangent with slope m to the ellipse S=0 is $y=mx\pm\sqrt{a^2m^2+b^2}$

If it pass through $P(x_1,y_1)$, then $y_1=mx_1\pm\sqrt{a^2m^2+b^2} \Rightarrow y_1-mx_1=\pm\sqrt{a^2m^2+b^2}$

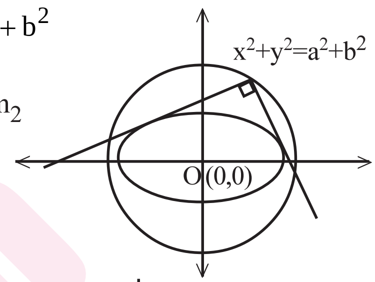
$\Rightarrow (y_1-mx_1)^2=a^2m^2+b^2 \Rightarrow y_1^2+m^2x_1^2-2x_1y_1m=a^2m^2+b^2$

$\Rightarrow (x_1^2-a^2)m^2-2x_1y_1m+(y_1^2-b^2)=0 \ldots.(1)$

(1) is a quadratic equation in m and its roots be taken as m_1,m_2

Here, m_1,m_2 represent the slopes of the tangents.

If the tangents intersect at right angle then $m_1m_2=-1$



From (1), product of roots $\frac{y_1^2-b^2}{x_1^2-a^2}=-1$

Product of roots of $ax^2+bx+c=0$ is c/a

$\Rightarrow y_1^2-b^2=-(x_1^2-a^2)=-x_1^2+a^2 \Rightarrow x_1^2+y_1^2=a^2+b^2$

$\therefore$ the equation of locus of $P(x_1,y_1)$ is $x^2+y^2=a^2+b^2$, which is a circle called director circle.

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15. Find the equations of the tangents to the hyperbola $3x^2-4y^2=12$ which are (i) Parallel to (ii) Perpendicular to the line y = x–7

Sol: $3x^2-4y^2=12 \Rightarrow \frac{x^2}{4}-\frac{y^2}{3}=1 \Rightarrow a^2=4, b^2=3$

The slope of the given line y = x–7 is m =1

The equation of the tangent with slope m to the hyperbola S=0 is $y=mx\pm\sqrt{a^2m^2-b^2}$

(i) Equation of the tangent is $y=mx\pm\sqrt{a^2m^2-b^2} \Rightarrow y=1.x\pm\sqrt{4(1)^2-3}=x\pm1 \Rightarrow x-y\pm1=0$

(ii) Slope of the normal is –1 ($\because$ slope of tangent is 1)

$\therefore$ equation of the normal is $y=(-1)x\pm\sqrt{4(1)^2-3}=-x\pm1 \Rightarrow x+y\pm1=0$

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16. Find the reduction formula for $\int_0^{\pi/2}\sin^n x\,dx$

Sol: Given that $I_n=\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\sin^{n-1}x(\sin x)dx$

Now, we apply the process of 'Integration by parts' by taking $u=\sin^{n-1}x$, v=sinx

$\therefore I_n=[\sin^{n-1}x(-\cos x)]_0^{\pi/2}-\int_0^{\pi/2}(n-1)\sin^{n-2}x\cos x(-\cos x)dx=0+(n-1)\int_0^{\pi/2}\sin^{n-2}x(1-\sin^2x)dx$

$=(n-1)\left[\int_0^{\pi/2}\sin^{n-2}x\,dx-\int_0^{\pi/2}\sin^n x\,dx\right]=(n-1)I_{n-2}-(n-1)I_n$

$\Rightarrow I_n(1+(n-1))=(n-1)I_{n-2} \Rightarrow I_n n=(n-1)I_{n-2}$ $\qquad \therefore I_n=\left(\frac{n-1}{n}\right)I_{n-2}$

17. Solve: $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

Sol: The given equation is $(1+y^2)dx = (\text{Tan}^{-1}y - x)dy$

$$\Rightarrow \frac{dx}{dy} = \frac{\text{Tan}^{-1}y}{1+y^2} - \frac{x}{1+y^2} \Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\text{Tan}^{-1}y}{1+y^2}$$

The above equation is in the form $\frac{dx}{dy} + P(y)x = Q(y)$, which is linear D.E in x.

Here, $P = \frac{1}{1+y^2}$, $Q = \frac{\text{Tan}^{-1}y}{1+y^2}$; $\int Pdy = \int \frac{1}{1+y^2}dy = \text{Tan}^{-1}y.$

$\therefore$ I.F $= e^{\int Pdy} = e^{\text{Tan}^{-1}y}$ $\therefore$ The solution is $x.(\text{I.F}) = \int Q.(\text{I.F})dy$

$\Rightarrow xe^{\text{Tan}^{-1}y} = \int \frac{\text{Tan}^{-1}y}{1+y^2} e^{\text{Tan}^{-1}y}dy = \int te^t dt$, where $t = \text{Tan}^{-1}y$ and $dt = \frac{1}{1+y^2}dy$

$= te^t - \int 1.e^t dt = (t-1)e^t + c = (\text{Tan}^{-1}y - 1)e^{\text{Tan}^{-1}y} + c$

$\Rightarrow xe^{\text{Tan}^{-1}y} = (\text{Tan}^{-1}y - 1)e^{\text{Tan}^{-1}y} + c \Rightarrow x = \text{Tan}^{-1}y - 1 + ce^{-\text{Tan}^{-1}y}$

SECTION-C

18. Show that the points (1, 1), (–6, 0), (–2, 2) and (–2, –8) are concyclic.

Sol: Let A=(1,1), B=(–6,0), C=(–2,2), D=(–2,–8)

Let $S(x_1,y_1)$ be the centre of the circle $\Rightarrow$ SA=SB=SC

Now, $SA = SB \Rightarrow SA^2 = SB^2 \Rightarrow (x_1-1)^2 + (y_1-1)^2 = (x_1+6)^2 + (y_1-0)^2$

$\Rightarrow (x_1^2 - 2x_1 + 1) + (y_1^2 - 2y_1 + 1) = (x_1^2 + 12x_1 + 36) + y_1^2$

$\Rightarrow 14x_1 + 2y_1 + 34 = 0 \Rightarrow 7x_1 + y_1 + 17 = 0$ ....(1)

Also, $SB = SC \Rightarrow SB^2 = SC^2 \Rightarrow (x_1+6)^2 + (y_1-0)^2 = (x_1+2)^2 + (y_1-2)^2$

$\Rightarrow (x_1^2 + 12x_1 + 36) + y_1^2 = (x_1^2 + 4x_1 + 4) + (y_1^2 - 4y_1 + 4)$

$\Rightarrow 8x_1 + 4y_1 + 28 = 0 \Rightarrow 2x_1 + y_1 + 7 = 0$ ......(2)

Solving (1) & (2) we get the centre $S(x_1,y_1)$

$(1)-(2) \Rightarrow 5x_1 + 10 = 0 \Rightarrow x_1 = -2$

From (1), $7(-2) + y_1 + 17 = 0 \Rightarrow y_1 = -3$

$\therefore$ the centre of the circle is $S(x_1, y_1) = (-2,-3)$

Hence, radius $r = SA = \sqrt{(1+2)^2 + (1+3)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$\therefore$ the equation of the circle with centre $(-2,-3)$ and radius 5 is

$(x+2)^2 + (y+3)^2 = 5^2 \Rightarrow (x^2+4x+4) + (y^2+6y+9) = 25 \Rightarrow x^2 + y^2 + 4x + 6y - 12 = 0$

Now, substituting D(–2,–8) in the above equation, we have

$(-2)^2+(-8)^2+4(-2)+6(-8)-12=4+64-8-48-12=68-68=0$

$\therefore$ D(–2,–8) lies on the circle $\therefore$ the given 4 points are concyclic.

19. Find the direct common tangents to the circles $x^2 + y^2 + 22x - 4y - 100 = 0$, $x^2 + y^2 - 22x + 4y + 100 = 0$

Sol: For the circle $x^2+y^2+22x-4y-100=0$, Centre $C_1=(-11,2)$, Radius $r_1 \sqrt{121+4+100}=\sqrt{225}=15$

For the circle $x^2+y^2-22x+4y+100=0$, Centre $C_2=(11,-2)$, Radius $r_2=\sqrt{121+4-100}=\sqrt{25}=5$

Now, $C_1C_2 = \sqrt{(-11-11)^2+(2+2)^2}=\sqrt{484+16}=\sqrt{500}=10\sqrt{5}$

$r_1+r_2 = 15+5 = 20$. Hence $C_1C_2>r_1+r_2$ Also, $r_1:r_2 = 15:5 = 3:1$

$\therefore$ External centre of similitude E divides $\overline{C_1C_2}$ in the ratio 3:1 externally.

$\Rightarrow E = \left(\frac{3(11)-(1)(-11)}{3-1}, \frac{3(-2)-1(2)}{3-1}\right) = (22,-4)$

Now, the equation of tangent through (22,–4) with slope 'm' is

$(y+4) = m(x-22) \Rightarrow mx-y-22m-4 = 0$....(1)

The perpendicular distance from $C_2(11,-2)$ to (1) is $r_2=5$

$\Rightarrow \frac{|11m+2-22m-4|}{\sqrt{m^2+1}}=5 \Rightarrow \frac{|-11m-2|}{\sqrt{m^2+1}}=5 \Rightarrow |11m+2|=5\sqrt{m^2+1}$

$\Rightarrow (11m+2)^2=25(m^2+1) \Rightarrow 121m^2+4+44m=25m^2+25 \Rightarrow 96m^2+44m-21=0$

$\Rightarrow 96m^2+72m-28m-21=0 \Rightarrow 24m(4m+3)-7(4m+3)=0$

$\Rightarrow (24m-7)(4m+3)=0 \Rightarrow m=\frac{7}{24}, \frac{-3}{4}$

$\therefore$ Equation of the tangent is $(y+4)=\frac{7}{24}(x-22) \Rightarrow 24(y+4)=7(x-22) \Rightarrow 7x-24y-250=0$

And $(y+4)=\frac{-3}{4}(x-22) \Rightarrow 4(y+4)=-3(x-22) \Rightarrow 3x+4y-50=0$

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20. If y_1, y_2, y_3 are the y-coordinates of the vertices of the triangle inscribed in the parabola $y^2 = 4ax$ then show that the area of the triangle is $\frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$ square units.

Sol: The vertices of the triangle are $P(x_1,y_1)=(at_1^2, 2at_1)$, $Q(x_2,y_2)=(at_2^2,2at_2)$, $R(x_3,y_3)=(at_3^2, 2at_3)$

$$\text{Area of } \Delta PQR = \frac{1}{2}|at_1^2(2at_2-2at_3)+at_2^2(2at_3-2at_1)+at_3^2(2at_1-2at_2)|$$

$$=\frac{1}{2}.2a^2|t_1^2(t_2-t_3)+t_2^2(t_3-t_1)+t_3^2(t_1-t_2)|$$

$$=a^2|(t_1-t_2)(t_2-t_3)(t_3-t_1)|$$

$$=\frac{a^2}{(2a)(2a)(2a)}|(2at_1-2at_2)(2at_2-2at_3)(2at_3-2at_1)|$$

$$=\frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)| \text{ Sq.Units}$$

21. Evaluate $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx$

Sol: Let $(9\cos x - \sin x) = A\frac{d}{dx}(4\sin x + 5\cos x) + B(4\sin x + 5\cos x)$

$\Rightarrow (9\cos x - \sin x) = A(4\cos x - 5\sin x) + B(4\sin x + 5\cos x)$

Equating the coefficients of cosx, we get 4A + 5B = 9 ....(1)

Equating the coefficients of sinx, we get –5A + 4B = –1 ....(2)

(1) × 4 ⇒ 16A+20B = 36 .....(3)

(2) × 5 ⇒ –25A+20B = –5 ....(4)

(3) –(4) ⇒ $41A = 41 \Rightarrow A = 1$

From (1), $5B = 9 - 4A = 9 - 4(1) = 5 \Rightarrow B = 1$

$\therefore I = \int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx = \int \frac{1(4\cos x - 5\sin x) + 1(4\sin x + 5\cos x)}{4\sin x + 5\cos x}dx$

$= \int \frac{(4\cos x - 5\sin x)}{4\sin x + 5\cos x}dx + \int \frac{4\sin x + 5\cos x}{4\sin x + 5\cos x}dx = \log|4\sin x + 5\cos x| + x + c$

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22. Evaluate $\int \frac{1}{(1+x)\sqrt{3+2x-x^2}}dx$

Sol: Put $1 + x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$. Also $x = \frac{1}{t} - 1 = \frac{1-t}{t}$

$\therefore 3 + 2x - x^2 = 3 + 2\left(\frac{1-t}{t}\right) - \left(\frac{1-t}{t}\right)^2 = 3 + 2\left(\frac{1}{t} - 1\right) - \left(\frac{1 - 2t + t^2}{t^2}\right)$

$= 3 + \frac{2}{t} - 2 - \left(\frac{1}{t^2} - \frac{2}{t} + 1\right) = 3 + \frac{2}{t} - 2 - \frac{1}{t^2} + \frac{2}{t} - 1 = \frac{4}{t} - \frac{1}{t^2} = \frac{4t-1}{t^2}$

$\therefore I = \int \frac{1}{(1+x)\sqrt{3+2x-x^2}}dx = \int \frac{1}{\frac{1}{t}\sqrt{\frac{4t-1}{t^2}}}\left(\frac{-1}{t^2}\right)dt$

$= -\int \frac{dt}{\sqrt{4t-1}} = \frac{-1}{4} \times 2\sqrt{4t-1} + c = -\frac{1}{2}\sqrt{\frac{4}{1+x} - 1} + c = -\frac{1}{2}\sqrt{\left[\frac{3-x}{1+x}\right]} + c$

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23. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Sol: Put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$. Also, $x = 0 \Rightarrow \theta = 0; x = 1 \Rightarrow \theta = \frac{\pi}{4}$

Now, $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

$\therefore \int_0^1 \frac{\log(1+x)}{1+x^2}dx = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta d\theta = \int_0^{\pi/4} \log(1+\tan\theta)d\theta$

$$\therefore I=\int_0^{\pi/4}\log[1+\tan\theta]d\theta=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta=\int_0^{\pi/4}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta=\int_0^{\pi/4}\log\left[\frac{(1+\tan\theta)+(1-\tan\theta)}{1+\tan\theta}\right]d\theta=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$=\int_0^{\pi/4}[\log 2-\log(1+\tan\theta)]\,d\theta=\int_0^{\pi/4}\log 2d\theta-\int_0^{\pi/4}\log(1+\tan\theta)d\theta=\int_0^{\pi/4}\log 2d\theta-I$$

$$\Rightarrow I+I=2I=\int_0^{\pi/4}\log 2d\theta=\log 2[\theta]_0^{\pi/4}=\frac{\pi}{4}\log 2\Rightarrow I=\frac{\pi}{8}\log 2$$

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24. Solve $\frac{dy}{dx}=\frac{2x+y+3}{2y+x+1}$

Sol: Given D.E is $\frac{dy}{dx}=\frac{2x+y+3}{2y+x+1}$ ...(1) Put x = X+h, y = Y+k $\therefore\frac{dy}{dx}=\frac{dY}{dX}$

Hence (1) becomes $\frac{dY}{dX}=\frac{2X+Y+2h+k+3}{2Y+X+2k+h+1}$......(2)

Now choose h and k sothat 2h+k+3=0, h+2k+1=0

h = –5/3 , k= 1/3

Hence (2) becomes $\frac{dY}{dX}=\frac{2X+Y}{2Y+X}$....(3) which is a homogeneous differential equation in X & Y

Now, we take the substitution $Y=VX\Rightarrow\frac{dY}{dX}=V+X\frac{dv}{dx}$

Hence (3) becomes $V+X\frac{dV}{dX}=\frac{2+V}{2V+1}\Rightarrow X\frac{dV}{dX}=\frac{2+V}{2V+1}-V$

$$\Rightarrow x\frac{dV}{dX}=\frac{2(1-V^2)}{2V+1}\Rightarrow\frac{2V+1}{(1+V)(1-V)}dV=2\frac{dX}{X}\Rightarrow\frac{3}{2(1-V)}dV-\frac{1}{2(1+V)}dV=2\frac{dX}{X}$$

Integrating on both sides, we get $\int\frac{3}{2(1-v)}dV-\int\frac{1}{2(1+V)}dV=\int\frac{2}{X}dX$

$$\therefore -\frac{3}{2}\log(1-V)-\frac{1}{2}\log(1+V)=2\log X-\frac{1}{2}\log c$$

$$\Rightarrow 3\log(1-V)+\log(1+V)+4\log X=\log c\Rightarrow\log\left[(1-V)^3(1+V)X^4\right]=\log c$$

$$\Rightarrow X^4(1-V)^3(1+V)=c\Rightarrow x^4\left(1-\frac{Y}{X}\right)^3\left(1+\frac{Y}{X}\right)=c\Rightarrow(X+Y)(X-Y)^3=c$$

$$\Rightarrow\left(x+\frac{5}{3}+y-\frac{1}{3}\right)\left(x+\frac{5}{3}-y+\frac{1}{3}\right)^3=c\Rightarrow\left(x+y+\frac{4}{3}\right)(x-y+2)^3=c$$