SOLVED PAPER-1

Time: 3 Hours **MATHS- IB** Max. Marks : 75

(Board of Intermediate Education -Model Paper)

SECTION -A

I. Answer ALL the following Very Short Answer Questions: $10 \times 2 = 20$

1. Find the value of x, if the slope of the line passing through (2,5) and (x,3) is 2.
2. Transform the equation $x + y + 1 = 0$ into Normal form
3. Show that the points (1,2,3), (2,3,1) and (3,1,2) form an equilateral triangle.
4. Find the angle between the planes $2x-y+z=6$ and $x+y+2z=7$
5. Show that $\underset{x\to 0+}{Lt}\left(\frac{2|x|}{x}+x+1\right)=3.$ **6.** Evaluate $\underset{x\to 0}{Lt}\frac{e^{3+x}-e^3}{x}$
7. If $f(x) = a^x.e^{x^2}$ then find $f'(x)$ **8.** Find the derivative of $\log(\sin(\log x))$
9. Find the approximate value of $\sqrt[3]{65}$ **10.** Verify Rolle's theorem for the function $y=f(x)=x^2+4$ on $[-3,3]$

SECTION-B

II. Answer any FIVE of the following Short Answer Questions: $5 \times 4 = 20$

11. A(2,3) and B(−3,4) be two given points. Find the equation of the locus of P so that the area of the triangle PAB is 8.5 sq.units.
12. Find the transformed equation of $x^2+2\sqrt{3}xy-y^2=2a^2$, when the axes are rotated through an angle $\pi/6$.
13. Find the points on the line $3x-4y-1=0$ which are at a distance of 5 units from the point (3, 2).
14. Show that $f(x)=\begin{cases}\frac{\cos ax-\cos bx}{x^2} & \text{if } x\neq 0\\ \frac{1}{2}(b^2-a^2) & \text{if } x=0\end{cases}$, is continuous at 0.
15. Find the derivative of sin 2x from the first principle.
16. A Particle is moving in a straight line so that after t seconds its distance s (in cms) from a fixed point on the line is given by $s = f(t) = 8t + t^3$. Find (i) the velocity at time $t = 2$ sec (ii) the initial velocity (iii) acceleration at $t = 2$sec.
17. Show that the tangent at any point θ on the curve $x = c\sec\theta$, $y = c\tan\theta$ is $y\sin\theta = x-c\cos\theta$.

SECTION-C

III. Answer any FIVE of the following Long Answer Questions: $5 \times 7 = 35$

18. Find the equation of the straight lines passing through the point (1, 2) and making an angle of 60º with the line $\sqrt{3}x+y+2=0$
19. Prove that the area of the triangle formed by the pair of lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is $\frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$
20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line $x + 2y = k$ are mutually perpendicular.
21. If a ray makes angle α,β,γ,δ with the four diagonals of a cube then show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = 4/3$. **22.** If $x=\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}$ then find $\frac{dy}{dx}$
23. At any point t on the curve $x=a(t+\sin t), y=a(1-\cos t)$, find the lengths of tangent and normal
24. A wire of length l is cut into two parts which are bent respectively in the form of a Square and a circle. What are the lengths of pieces of wire so that the sum of areas is least?

SOLUTIONS

SECTION-A

1. Find the value of x, if the slope of the line passing through (2,5) and (x,3) is 2.

Sol: Slope of the line joining (2, 5) and (x, 3) is $m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-5}{x-2}=\frac{-2}{x-2}$

Given that, slope of the line m = 2

$\therefore \frac{-2}{x-2}=2 \Rightarrow x-2=\frac{-2}{2}=-1 \Rightarrow x=-1+2=1$

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2. Transform the equation x+y+1=0 into (i) slope intercept form (ii) intercept form (iii) Normal form

Sol: (i) **The slope intercept form is y=mx+c**

$\therefore x+y+1=0 \Rightarrow y=-x-1 \Rightarrow y=(-1)x+(-1)$

(ii) **The intercept form is** $\frac{x}{a}+\frac{y}{b}=1$

$\therefore x+y+1=0 \Rightarrow x+y=-1 \Rightarrow \frac{x}{(-1)}+\frac{y}{(-1)}=1$

(iii) **The normal form is xcosα+ysinα=p**

The given equation is $x+y+1=0 \Rightarrow x+y=-1 \Rightarrow -x-y=1$

Dividing by $\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$, we have

$-\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y=\frac{1}{\sqrt{2}} \Rightarrow x\cos 225^\circ+y\sin 225^\circ=\frac{1}{\sqrt{2}}$

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3. Show that the points (1,2,3), (2,3,1) and (3,1,2) form an equilateral triangle.

Sol: Let A=(1,2,3), B=(2,3,1) and C=(3,1,2)

$\Rightarrow AB^2=(1-2)^2+(2-3)^2+(3-1)^2=1+1+4=6$

$BC^2=(2-3)^2+(3-1)^2+(1-2)^2=1+4+1=6$

$CA^2=(3-1)^2+(1-2)^2+(2-3)^2=4+1+1=6$

$\therefore AB^2=BC^2=CA^2 \Rightarrow AB=BC=CA$

Hence, A,B,C form an equilateral triangle.

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4. Find the angle between the planes 2x–y+z=6 and x+y+2z=7

Sol : If θ is the angle between the planes then $\cos\theta=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}}$

$\therefore \cos\theta=\frac{|2(1)-1(1)+1(2)|}{\sqrt{4+1+1}\sqrt{1+1+4}}=\frac{3}{6}=\frac{1}{2}=\cos\frac{\pi}{3}$

$\therefore$ the angle between the given planes is $\frac{\pi}{3}$

5. Show that $\underset{x\to 0+}{\mathrm{Lt}}\left(\frac{2|x|}{x}+x+1\right)=3.$

Sol: $x \to 0+ \Rightarrow x>0$

$\therefore |x| = x$

Hence, $\underset{x\to 0+}{\mathrm{Lt}}\left(\frac{2|x|}{x}+x+1\right)=\underset{x\to 0+}{\mathrm{Lt}}\left(\frac{2x}{x}+x+1\right)=2+0+1=3$

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6. Evaluate $\underset{x\to 0}{\mathrm{Lt}}\frac{e^{3+x}-e^{3}}{x}$

Sol: $\underset{x\to 0}{\mathrm{Lt}}\frac{e^{3+x}-e^{3}}{x}=\underset{x\to 0}{\mathrm{Lt}}\frac{e^{3}(e^{x}-1)}{x}=e^{3}\underset{x\to 0}{\mathrm{Lt}}\frac{e^{x}-1}{x}=e^{3}(1)=e^{3}$

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7. If $f(x) = a^x.e^{x^2}$ **then find f'(x)**

Sol: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$, where u,v are functions of x

Given $f(x) = a^x.e^{x^2}$

$\Rightarrow f'(x)=(a^{x})\frac{d}{dx}(e^{x^{2}})+(e^{x^{2}})\frac{d}{dx}(a^{x})=a^{x}.e^{x^{2}}.2x+e^{x^{2}}.a^{x}.\log a=a^{x}.e^{x^{2}}(2x+\log a)$

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8. Find the derivative of log(sin(logx))

Sol : $\frac{d}{dx}\log(\sin(\log x))=\frac{1}{\sin(\log x)}\frac{d}{dx}\sin(\log x)=\frac{1}{\sin(\log x)}\cos(\log x).\frac{d}{dx}\log x$

$=\frac{\cos(\log x)}{\sin(\log x)}\frac{1}{x}=\frac{\cot(\log x)}{x}$

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9. Find the approximate value of $\sqrt[3]{65}$

Sol: $\sqrt[3]{65}=\sqrt[3]{64+1}$

$\therefore$ known value x = 64 and $\Delta x=1$.

Let $f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}\Rightarrow f'(x)=\frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3x^{2/3}}$

$\therefore$ approximate value is given by $f(x+\Delta x)=[f(x)+f'(x)\Delta x]_{\text{at known } x}$

$\therefore \sqrt[3]{65}\cong\sqrt[3]{x}+\frac{1}{3x^{2/3}}\Delta x=\sqrt[3]{64}+\frac{1}{3(64)^{2/3}}(1)$

$=4+\frac{1}{3(4^{3})^{\frac{2}{3}}}(1)=4+\frac{1}{3(4^{2})}=4+\frac{1}{3(16)}=4+\frac{1}{48}=\frac{192+1}{48}=\frac{193}{48}=4.0208$

10. Verify Rolle's theorem for the function $y=f(x)=x^2+4$ on $[-3,3]$

Sol : Being a Polynomial function, the given function f(x) is continuous and differentiable on R

$\therefore$ $f(x) = x^2+4$ is (i) continuous on $[-3,3]$ and (ii) differentiable in $(-3,3)$

Now, $f(-3) = (-3)^2+4 = 9+4 = 13$

$f(3) = 3^2+4 = 9+4 = 13$

$\therefore$ $f(-3) = f(3)$

Hence, f(x) satisfies all the 3 conditions of Rolle's theorem.

$\therefore$ By Rolle's theorem, there exists $c\in(-3,3)$ such that $f'(c)=0$

Now, $f(x) = x^2+4 \Rightarrow f'(x) = 2x$

Hence, $f'(c)=0 \Rightarrow 2c=0 \Rightarrow c=0$

Here, $c=0\in(-3,3)$

Thus, Rolle's theorem is verified.

SECTION-B

11. A(2,3) and B(–3,4) be two given points. Find the equation of the locus of P so that the area of the triangle PAB is 8.5 sq.units.

Sol : Given that A=(2,3), B=(–3,4) and P(x, y) be a point on the locus.

From the given condition, area of ΔPAB=8.5 sq.units

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x-2 & x+3 \\ y-3 & y-4 \end{vmatrix} = 8.5 \Rightarrow |(x-2)(y-4)-(y-3)(x+3)| = 2(8.5)$$

$$\Rightarrow |xy-4x-2y+8-(xy+3y-3x-9)| = 17 \Rightarrow |-x-5y+17| = 17$$

$$\Rightarrow -x-5y+17 = 17 \text{ (or) } -x-5y+17 = -17$$

$$\Rightarrow x+5y = 0 \text{ (or) } x+5y = 34$$

$$\Rightarrow (x+5y)(x+5y-34) = 0$$

$\therefore$ The equation of locus of P(x,y) is $(x+5y)(x+5y-34)=0$

Hint

If $A=(x_1,y_1)$, $B=(x_2,y_2)$, $C=(x_3,y_3)$ then area

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1-x_2 & x_1-x_3 \\ y_1-y_2 & y_1-y_3 \end{vmatrix}$$

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12. Find the transformed equation of $x^2+2\sqrt{3}xy-y^2=2a^2$, when the axes are rotated through an angle $\pi/6$.

Sol : The angle of rotation is $\theta=\pi/6=30^o$

$$x = X\cos\theta - Y\sin\theta \Rightarrow x = X\cos 30^o - Y\sin 30^o = X\left(\frac{\sqrt{3}}{2}\right) - Y\left(\frac{1}{2}\right) = \frac{\sqrt{3}X-Y}{2}$$

$$y = Y\cos\theta + X\sin\theta \Rightarrow y = Y\cos 30^o + X\sin 30^o = Y\left(\frac{\sqrt{3}}{2}\right) + X\left(\frac{1}{2}\right) = \frac{\sqrt{3}Y+X}{2}$$

$\therefore$ the transformed equation of $x^2+2\sqrt{3}xy-y^2 = 2a^2$ is

$$\left(\frac{\sqrt{3}X-Y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}X-Y}{2}\right)\left(\frac{\sqrt{3}Y+X}{2}\right) - \left(\frac{\sqrt{3}Y+X}{2}\right)^2 = 2a^2$$

$$\Rightarrow (\sqrt{3}X-Y)^2 + 2\sqrt{3}(\sqrt{3}X-Y)(\sqrt{3}Y+X) - (\sqrt{3}Y+X)^2 = 4(2a^2)$$

$$\Rightarrow (3X^2-2\sqrt{3}XY+Y^2) + 2\sqrt{3}(3XY+\sqrt{3}X^2-\sqrt{3}Y^2-XY) - (3Y^2+2\sqrt{3}XY+X^2) = 8a^2$$

$$\Rightarrow X^2[3+6-1] + XY(-2\sqrt{3}+6\sqrt{3}-2\sqrt{3}-2\sqrt{3}) + Y^2(1-6-3) = 8a^2 \Rightarrow X^2-Y^2 = a^2$$

This transformed equation can be written as $x^2-y^2=a^2$.

13. Find the points on the line 3x–4y–1=0 which are at a distance of 5 units from the point (3, 2).

Sol: We consider the parametric equations of the line $(x,y)=(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

Here, $(x_1,y_1)=(3, 2)$, $|r|=5$,

slope of the line 3x–4y–1=0 is $-\left(\frac{3}{-4}\right)=\frac{3}{4}$

$\Rightarrow m=\tan\theta=\frac{3}{4} \Rightarrow \cos\theta=\frac{4}{5},\ \sin\theta=\frac{3}{5}$

$\therefore$ the required points are given by $(x,y)=(x_1+r\cos\theta, y_1+r\sin\theta)$ and $(x_1-r\cos\theta, y_1-r\sin\theta)$

So, the required points are $\left(3+5\left(\frac{4}{5}\right),\ 2+5\left(\frac{3}{5}\right)\right)=(3+4, 2+3)=(7,5)$

and $\left(3-5\left(\frac{4}{5}\right),\ 2-5\left(\frac{3}{5}\right)\right)=(3-4, 2-3)=(-1,-1)$

$\therefore$ The required points are (7,5) and (–1,–1)

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14. Show that $f(x)=\begin{cases} \dfrac{\cos ax - \cos bx}{x^2} & \text{if } x \neq 0 \\ \dfrac{1}{2}(b^2-a^2) & \text{if } x=0 \end{cases}$**, is continuous at 0.**

Sol: Given that $f(0)=\frac{1}{2}(b^2-a^2)$

$$\underset{x\to 0}{Lt}\frac{\cos ax-\cos bx}{x^2}=\underset{x\to 0}{Lt}\frac{2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{bx-ax}{2}\right)}{x^2}$$

$$=2\underset{x\to 0}{Lt}\left(\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\frac{\sin\left(\frac{b-a}{2}\right)x}{x}\right)=2\left(\underset{x\to 0}{Lt}\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\right)\left(\underset{x\to 0}{Lt}\frac{\sin\left(\frac{b-a}{2}\right)x}{x}\right)$$

$$=2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)=\frac{b^2-a^2}{2} \qquad \left(\because \underset{x\to 0}{Lt}\frac{\sin kx}{x}=k\right)$$

$\therefore \underset{x\to 0}{Lt} f(x)=f(0)\cdot$

Hence f(x) is continuous at x=0

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15. Find the derivative of sin2x from the first principle.

Sol: Let f(x)=sin2x $\Rightarrow f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

$$=\underset{h\to 0}{Lt}\frac{1}{h}(\sin 2(x+h)-\sin 2x)=\underset{h\to 0}{Lt}\frac{1}{h}(\sin(2x+2h)-\sin 2x)$$

$$=\underset{h\to 0}{Lt}\frac{1}{h}\left(2\cos\left(\frac{2x+2h+2x}{2}\right)\sin\left(\frac{2x+2h-2x}{2}\right)\right)=2\underset{h\to 0}{Lt}\frac{1}{h}\cos(2x+h)\sinh$$

$$=2\underset{h\to 0}{Lt}\cos(2x+h).\underset{h\to 0}{Lt}\frac{\sinh}{h}=2\cos(2x+0)(1)=2\cos 2x$$

STAR-Q

16. A Particle is moving in a straight line so that after t seconds its distance s (in cms) from a fixed point onthe line is given by $s=f(t) = 8t+t^3$. Find (i) the velocity at time t=2 sec (ii) the initial velocity (iii) acceleration at t=2sec.

Sol: The distance-time relation is given by $s=f(t) = 8t+t^3$ ....(1)

$\therefore$ Velocity $v = \dfrac{ds}{dt} = 8+3t^2$ ....(2)

Acceleration $a = \dfrac{dV}{dt} = 6t$ ....(3)

(i) From (2) , the velocity at t = 2 is 8+3(4) = 20 cm/sec

(ii) From (1), the initial velocity at t=0 is 8(0)+0 = 8 cm/sec

(iii) From (3), the acceleration at t=2 is 6(2) = 12 cm/sec^2.

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17. S.T the tangent at any point θ on the curve $x=c\sec\theta$, $y=c\tan\theta$ is $y\sin\theta=x-c\cos\theta$.

Sol: Slope of the tangent at any point θ(csecθ, ctanθ) on the curve is

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\frac{d}{d\theta}(c\tan\theta)}{\frac{d}{d\theta}(c\sec\theta)} = \frac{c\sec^2\theta}{c\sec\theta\tan\theta} = \text{cosec}\,\theta$$

$\therefore$ The equation of the tangent with slope cosecθ at θ(csecθ, ctanθ) is $y-c\tan\theta=\text{cosec}\,\theta(x-c\sec\theta)$

$$\Rightarrow y - c\frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta}\left(x-\frac{c}{\cos\theta}\right) \Rightarrow \frac{y\cos\theta - c\sin\theta}{\cancel{\cos\theta}} = \frac{1}{\sin\theta}\left(\frac{x\cos\theta - c}{\cancel{\cos\theta}}\right)$$

$$\Rightarrow y\sin\theta\cos\theta - c\sin^2\theta = x\cos\theta - c \Rightarrow x\cos\theta - y\sin\theta\cos\theta - c(1-\sin^2\theta) = 0$$

$$\Rightarrow x\cos\theta - y\sin\theta\cos\theta - c(\cos^2\theta) = 0 \Rightarrow \cos\theta(x - y\sin\theta - c\cos\theta) = 0$$

$$\Rightarrow x - y\sin\theta - c\cos\theta = 0 \Rightarrow y\sin\theta = x - c\cos\theta.$$

SECTION-C

18. Find the equation of the straight lines passing through the point (1,2) and making an angle of 60° with the line $\sqrt{3}x + y + 2 = 0$

Sol: Let slope of the required line be m

The slope of the given line $\sqrt{3}x + y + 2 = 0$ is $-\dfrac{a}{b} = -\dfrac{\sqrt{3}}{1} = -\sqrt{3}$

The angle between the lines is 60° $\Rightarrow \tan 60^\circ = \left|\dfrac{m-(-\sqrt{3})}{1+m(-\sqrt{3})}\right|$

$$\Rightarrow \sqrt{3} = \frac{|m+\sqrt{3}|}{|1-\sqrt{3}m|} \Rightarrow 3(1-\sqrt{3}m)^2 = (m+\sqrt{3})^2 \Rightarrow 3(1-2\sqrt{3}m+3m^2) = m^2 + 2\sqrt{3}m + 3$$

$$\Rightarrow 3 - 6\sqrt{3}m + 9m^2 = m^2 + 2\sqrt{3}m + 3$$

$$\Rightarrow 8m^2 - 8\sqrt{3}m = 0 \Rightarrow 8m(m-\sqrt{3}) = 0 \Rightarrow m = 0 \text{ (or) } m = \sqrt{3}$$

$\therefore$ The slope of the required line is 0 (or) $\sqrt{3}$.

The equation of the horizontal line passing through the point (1, 2) with slope 0 is $y = 2$ [$\because$ y=k]

Also, the equation of the other line passing through (1,2) with slope $\sqrt{3}$ is

$$y - 1 = \sqrt{3}(x-2) \Rightarrow y - 1 = \sqrt{3}x - 2\sqrt{3} \Rightarrow \sqrt{3}x - y + (1-2\sqrt{3}) = 0$$

STAR-Q

19. Prove that the area of the triangle formed by the pair of lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is $\dfrac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$

Sol : Let the 2 lines of $ax^2+2hxy+by^2=0$ be $y=m_1x$ and $y=m_2x$.

Then $ax^2+2hxy+by^2=(m_1x-y)(m_2x-y)$

$$\Rightarrow m_1+m_2=-\frac{2h}{b},\quad m_1m_2=\frac{a}{b}$$

Let A be the point of intersection of $lx+my+n=0$, $m_1x-y=0$

$lx+my+n=0$

$m_1x-y+0=0$

$$\Rightarrow \frac{x}{m(0)-(-1)(n)}=\frac{y}{n(m_1)-l(0)}=\frac{1}{l(-1)-mm_1}$$

$$\Rightarrow \frac{x}{n}=\frac{y}{nm_1}=\frac{1}{-l-mm_1}\Rightarrow A=\left(\frac{-n}{l+mm_1},\frac{-nm_1}{l+mm_1}\right)$$

Similarly, by solving $lx+my+n=0$, $m_2x-y=0$ we get $B=\left(\dfrac{-n}{l+mm_2},\dfrac{-nm_2}{l+mm_2}\right)$

The area of the triangle with vertices, $O(0,0), A(x_1,y_1), B(x_2,y_2)$ is $\Delta=\dfrac{1}{2}|x_1y_2-x_2y_1|$

$$\therefore \text{Area of } \Delta OAB=\frac{1}{2}\left|\left(\frac{-n}{l+mm_1}\right)\left(\frac{-nm_2}{l+mm_2}\right)-\left(\frac{-n}{l+mm_2}\right)\left(\frac{-nm_1}{l+mm_1}\right)\right|$$

$$=\frac{1}{2}\left|\frac{n^2m_2-n^2m_1}{(l+mm_1)(l+mm_2)}\right|=\frac{1}{2}\left|\frac{n^2(m_2-m_1)}{l^2+lmm_2+lmm_1+m^2m_1m_2}\right|$$

$$=\frac{1}{2}\frac{n^2\sqrt{(m_1+m_2)^2-4m_1m_2}}{|l^2+lm(m_1+m_2)+m^2(m_1m_2)|}\quad\left[\because (a-b)^2=(a+b)^2-4ab\right]$$

$$=\frac{1}{2}\frac{n^2\sqrt{\frac{4h^2}{b^2}-4\frac{a}{b}}}{\left|l^2+lm\left(\frac{-2h}{b}\right)+m^2\left(\frac{a}{b}\right)\right|}=\frac{1}{2}\frac{n^2\sqrt{\frac{4h^2-4ab}{b^2}}}{\left|\frac{bl^2-2hlm+am^2}{b}\right|}$$

$$=\frac{1}{2}\frac{n^2\sqrt{4h^2-4ab}}{|am^2-2hlm+bl^2|}=\frac{1}{2}\frac{2\left(n^2\sqrt{h^2-ab}\right)}{|am^2-2hlm+bl^2|}=\frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}\text{ sq. units}$$

20. Find the value of k, if the lines joining the origin with the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line x+2y=k are mutually perpendicular.

Sol : The given line is x+2y=k $\Rightarrow \frac{x+2y}{k}=1$ ...(1)

Now, we homogenise the equation $2x^2-2xy+3y^2+2x-y-1=0$ using (1)

$\Rightarrow 2x^2-2xy+3y^2+2x(1)-y(1)-(1^2)=0$

$\Rightarrow 2x^2-2xy+3y^2+2x\left(\frac{x+2y}{k}\right)-y\left(\frac{x+2y}{k}\right)-\frac{(x+2y)^2}{k^2}=0,$ from(1)

$\Rightarrow k^2(2x^2-2xy+3y^2)+k(2x^2+4xy)-k(xy+2y^2)-(x^2+4xy+4y^2)=0$

If the above pair of lines are perpendicular, then sum of coefficients of x^2 and y^2 is zero

$\Rightarrow 2k^2+3k^2+2k-2k-1-4=0$

$\Rightarrow 5k^2-5=0 \Rightarrow 5(k^2-1)=0 \Rightarrow k^2-1=0 \Rightarrow k^2=1 \Rightarrow k=\pm 1$

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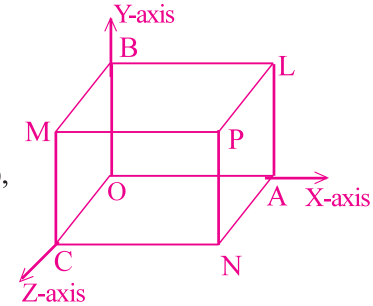
21. If a ray makes angle α,β,γ,δ with the four diagonals of a cube then show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = 4/3$.

Sol : Let one of the vertices of the cube coincides with the origin O(0,0,0) and edges coincide with the coordinate axes.

Let A,B,C be the vertices of the cube on the x-axis, y-axis, z-axis so that OA=OB=OC=a

The vertices of the cube on xy-plane, yz-plane, zx-plane be L,M,N and P be the remaining vertex in the space.

The 4 diagonals of the cube are OP, CL, NB, AM



The coordinate of the vertices are A(a,0,0), B(0,a,0), C(0,0,a), L(a,a,0), M(0,a,a), N(a,0,a) and P(a,a,a)

The d.r's of OP=(a–0, a–0, a–0) = (a,a,a)

The d.r's of CL=(a–0, a–0, 0–a)=(a,a,–a)

The d.r's of NB=(0–a, a–0, 0–a) = (–a, a, –a)

The d.r's of AM=(0–a, a–0, a–0)=(–a,a,a)

Let (l,m,n) be the d.c's of the given ray, then we have $l^2+m^2+n^2=1$

The ray makes angles α,β,γ,δ with OP,CL,NB,AM

$$\Rightarrow \cos\alpha=\frac{al+am+an}{\sqrt{(a^2+a^2+a^2)(l^2+m^2+n^2)}}=\frac{a(l+m+n)}{\sqrt{3a^2(1)}}=\frac{a(l+m+n)}{\sqrt{3}a} \Rightarrow \cos\alpha=\frac{l+m+n}{\sqrt{3}}$$

$$\text{Similarly, } \cos\beta=\frac{l+m-n}{\sqrt{3}},\ \cos\gamma=\frac{-l+m-n}{\sqrt{3}},\ \cos\delta=\frac{-l+m+n}{\sqrt{3}}$$

$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$

$=\frac{1}{3}((l+m+n)^2+(l+m-n)^2+(-l+m-n)^2+(-l+m+n)^2)$

$=\frac{1}{3}((l+m+n)^2+(l+m-n)^2+(l-m+n)^2+(l-m-n)^2)$

$=\frac{1}{3}[((l+m)+n)^2+((l+m)-n)^2]+[((l-m)+n)^2+((l-m)-n)^2]$

$=\frac{1}{3}[2((l+m)^2+n^2)+2((l-m)^2+n^2)]=\frac{2}{3}[(l+m)^2+(l-m)^2+2n^2]$

$=\frac{2}{3}[2(l^2+m^2)+2n^2]=\frac{4}{3}(l^2+m^2+n^2)=\frac{4}{3}(1)=\frac{4}{3}$

22. If $x=\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}$ then find $\frac{dy}{dx}$

Sol: $x=\frac{3at}{1+t^3}\Rightarrow\frac{dx}{dt}=\frac{(1+t^3)3a-3at(3t^2)}{(1+t^3)^2}=\frac{3a(1+t^3-3t^3)}{(1+t^3)^2}=\frac{3a(1-2t^3)}{(1+t^3)^2}$

$y=\frac{3at^2}{(1+t^3)}\Rightarrow\frac{dy}{dt}=\frac{(1+t^3)(6at)-3at^2(3t^2)}{(1+t^3)^2}=\frac{3at(2+2t^3-3t^3)}{(1+t^3)^2}=\frac{3at(2-t^3)}{(1+t^3)^2}$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{3a\,t(2-t^3)}{(1+t^3)^2}\cdot\frac{(1+t^3)^2}{3a(1-2t^3)}=\frac{t(2-t^3)}{1-2t^3}$$

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23. At any point t on the curve x=a(t+sint),y=a(1–cost), find the lengths of tangent, normal, subtangent and subnormal.

Sol: Given that x = a(t+sin t), y = a(1–cost)

$$m=\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\frac{d}{dt}(a(1-\cos t))}{\frac{d}{dt}(a(t+\sin t))}=\frac{a\sin t}{a(1+\cos t)}=\frac{\sin t}{1+\cos t}=\frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}=\tan\frac{t}{2}$$

Hence, we have m= $\tan\frac{t}{2}$; also y = a(1–cos t)

(i) Length of the tangent

$$=\left|\frac{y\sqrt{1+m^2}}{m}\right|=\frac{a(1-\cos t)\sqrt{1+\tan^2\frac{t}{2}}}{\tan\frac{t}{2}}=\left|\frac{a.2\sin^2\frac{t}{2}\sec\frac{t}{2}}{\tan\frac{t}{2}}\right|=\left|a.2\sin\frac{t}{2}\right|=\left|2a\sin\frac{t}{2}\right|$$

(ii) Length of the normal

$$=\left|y\sqrt{1+m^2}\right|=\left|a(1-\cos t)\sqrt{1+\tan^2\frac{t}{2}}\right|=\left|a.2\sin^2\frac{t}{2}.\sec\frac{t}{2}\right|$$

$$=\left|2a\sin^2\frac{t}{2}.\frac{1}{\cos\frac{t}{2}}\right|=\left|2a\sin\frac{t}{2}.\tan\frac{t}{2}\right|$$

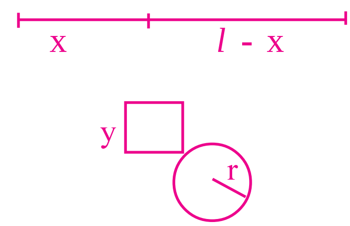
(iii)Length of the subtangent $=\left|\frac{y}{m}\right|=\left|\frac{a(1-\cos t)}{\tan\frac{t}{2}}\right|=\left|\frac{a.2\sin^2\frac{t}{2}}{\tan\frac{t}{2}}\right|=\left|a.2\sin\frac{t}{2}.\cos\frac{t}{2}\right|=|a\sin t|$

(iv) Length of the subnormal $=|y.m|=\left|a(1-\cos t)\tan\frac{t}{2}\right|=\left|a(2\sin^2\frac{t}{2})\tan\frac{t}{2}\right|=\left|2a\sin^2\frac{t}{2}\tan\frac{t}{2}\right|$

STAR-Q

24. A wire of length l is cut into two parts which are bent respectively in the form of a Square and a circle. What are the lengths of pieces of wire so that the sum of areas is least?

Sol: Let a part of x of 'l' be reduced into a square of side y and the remaining part (l–x) is made into a circle of radius'r'.



$\Rightarrow$ Perimeter of the square 4y=x

The circumference of the circle is $2\pi r=(l-x)$ ....(1)

The sum of the areas of the square and the circle is

$$A = y^2 + \pi r^2 = \frac{x^2}{16} + \pi\frac{(l-x)^2}{4\pi^2} \text{ [from (1)]}$$

$$\Rightarrow A(x) = \frac{x^2}{16} + \frac{(l-x)^2}{4\pi} \Rightarrow A'(x) = \frac{2}{16}x + \frac{2(l-x)}{4\pi}(-1) = \frac{x}{8} - \frac{(l-x)}{2\pi} ....(2)$$

The extreme values of A(x) are attained when A'(x)=0

$$\Rightarrow \frac{x}{8} - \frac{(l-x)}{2\pi} = 0 \Rightarrow \frac{x}{8} = \frac{(l-x)}{2\pi} \Rightarrow \frac{x}{4} = \frac{(l-x)}{\pi} \Rightarrow x\pi = 4l - 4x \Rightarrow x(\pi+4) = 4l \Rightarrow x = \frac{4l}{(\pi+4)}$$

$$\text{Also, } l - x = l - \frac{4l}{\pi+4} = \frac{\pi l}{\pi+4}$$

$$\text{From (2), } A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0 \Rightarrow A(x) \text{ has a minimum value}$$

$\therefore$ the length of the piece of wire which form a square is $\frac{4l}{\pi+4}$ and circle is $\frac{\pi l}{\pi+4}$

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