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DIGITAL MATERIAL

HYPERBOLIC FUNCTIONS-INDEX

9. HYPERBOLIC FUNCTIONS

1. INTRODUCTION PAGE

Sections	No. of periods (4)	Weightage in IPE [1x2=2]
1. Hyperbolic Functions	4	2 Marks

We recall that, $\sin\theta, \cos\theta$ are known as circular functions because the parametric equations $x=\sin\theta$, $y=\cos\theta$ of these two trigonometric functions correspond to the coordinates of a point on the unit circle $x^2+y^2=1$. Similarly, the functions concerned to present topic are called Hyperbolic functions, because the parametric equations of these functions viz., $x=\sinh x$, $y=\cosh x$ correspond to the coordinate of a point on the Hyperbola $x^2-y^2=1$.

These hyperbolic functions are used to solve some problems related to Engineering. For example, the tension at any point in a cable suspended by its ends and hanging under its own weight such as electric transmission lines may be computed with hyperbolic functions. Also these hyperbolic functions are useful in solving some differential equations in Calculus.

Basically, there are six Hyperbolic functions each of which, is expressed in terms of exponential functions of x viz., e^x, e^{-x}.

In the higher classes, it will be known that, this e^x is expressible in the sum of infinite series as $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

The six basic hyperbolic functions in terms of e^x and e^{-x} are defined as follows:

$\frac{e^x - e^{-x}}{2}$ *is known as sine hyperbolic function and it is denoted by sinhx. This sinhx is a real valued function with domain R. Thus,* $\sinh x = \frac{e^x - e^{-x}}{2}, \forall x \in R$. $\frac{e^x + e^{-x}}{2}$ *is known as cosine hyperbolic function and it is denoted by coshx. This coshx is a real valued function with domain R. Thus,* $\cosh x = \frac{e^x + e^{-x}}{2}, \forall x \in R$

$\frac{e^x - e^{-x}}{e^x + e^{-x}}$ *is known as tangent hyperbolic function and it is denoted by tanhx. This tanhx is a real valued function with domain R. Thus,* $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \forall x \in R$

The reciprocal functions of the above hyperobic functions are as follows:

$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}, x \neq 0$; $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$; $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}, x \neq 0$

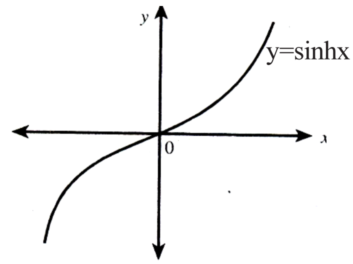
2. Definitions, Graphs & Proofs of Hyperbolic &Inverse Hyperbolic Functions

For any real x, the exponential function e^x is defined as $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Def 1: The hyperbolic sine function, denoted by sinhx

is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$

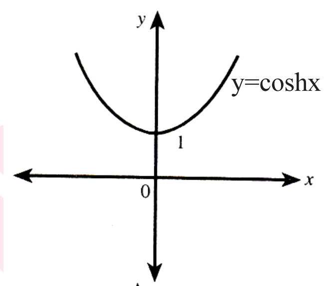
The domain of sinhx is R and Range is R



Def 2: The hyperbolic cosine function, denoted by coshx

is defined as $\cosh x = \frac{e^x + e^{-x}}{2}$

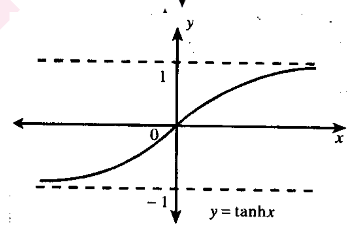
The domain of coshx is R and Range is $[1,\infty)$



Def 3: The hyperbolic tangent function, denoted by tanhx

is defined as $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

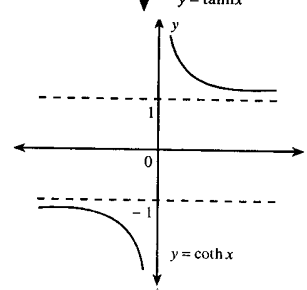
The domain of tanhx is R and Range is $(-1,1)$



Def 4: The hyperbolic cotangent function, denoted by cothx

is defined as $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

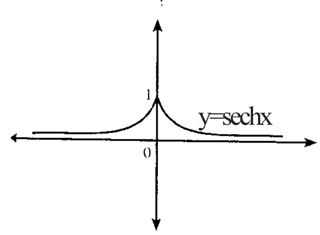
The domain of cothx is $R-\{0\}$ & Range is $(-\infty,1)\cup(1,\infty)$



Def 5: The hyperbolic secant function, denoted by sechx

is defined as $\text{sech}x = \frac{2}{e^x + e^{-x}}$

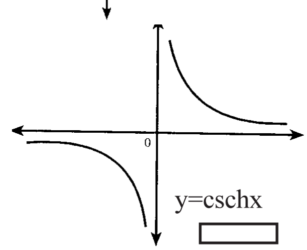
The domain of sechx is R & Range is $(0,1]$



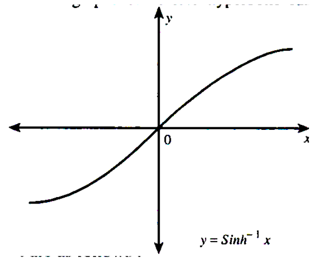
Def 6: The hyperbolic cosecant function, denoted by cosechx

is defined as $\text{cosech}x = \frac{2}{e^x - e^{-x}}$

The domain of cosechx is $R-\{0\}$ & Range is $R-\{0\}$



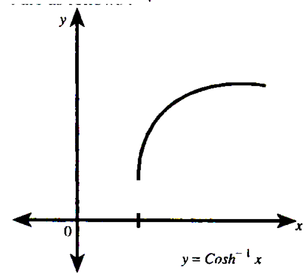
8.2.INVERSE HYPERBOLIC FUNCTIONS



Def1: The function $f:R \to R$ defined by $f(x)=\sinh x$ is a bijection. The inverse of f from R into R is called inverse hyperbolic sine function and it is denoted by $Sinh^{-1}x$.

Note 1 : $Sinh^{-1}x=t \Leftrightarrow x=\sinh t, \ \forall \ x \in R$.

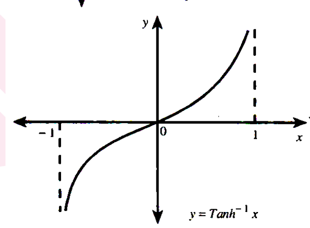
Note 2 : $\sinh(Sinh^{-1}x)=x$, $Sinh^{-1}(\sinh t)=t$.



Def 2: The function $f:[0,\infty) \to [1,\infty)$ defined by $f(x)=\cosh x$ is a bijection. The inverse of f from $[1,\infty)$ into $[0,\infty)$ is called inverse hyperbolic cosine function and it is denoted by $Cosh^{-1}x$.

Note 1: $Cosh^{-1}x=t \Leftrightarrow x=\cosh t, \ \forall \ x \in [1,\infty)$

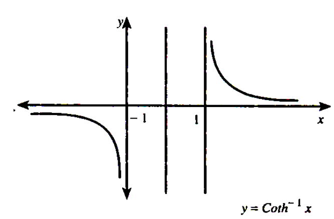
Note 2: $\cosh(Cosh^{-1}x)=x$, $Cosh^{-1}(\cosh t)=t$, where $x \in [1,\infty)$, $t \in [0,\infty)$



Def 3: The function $f:R \to (-1,1)$ defined by $f(x)=\tanh x$ is a bijection. The inverse of f from $(-1,1)$ into R is called inverse hyperbolic tangent function and it is denoted by $Tanh^{-1}x$.

Note 1: $Tan^{-1}x=t \Leftrightarrow x=Tanh t \ \forall \ x \in R$

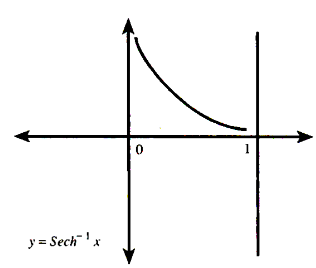
Note 2: $\tanh(Tanh^{-1}x)=x$, $Tanh^{-1}(\tanh t)=t$ for $x \in R$, $t \in (-1,1)$



Def 4: The function $f:R-\{0\} \to (-\infty,-1) \cup (1,\infty)$ defined by $f(x)=\coth x$ is a bijection.The inverse of f from $(-\infty,-1) \cup (1,\infty)$ into $R-\{0\}$ is called inverse hyperbolic cotangent function. and it is denoted by $Coth^{-1}x$.

Note 1: $Coth^{-1}x=t \Leftrightarrow x=\coth t, \forall x \in R-\{0\}$

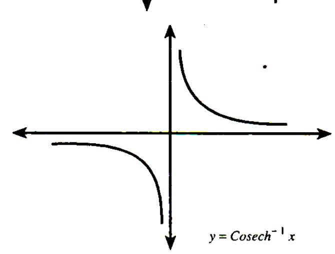
Note 2: $\coth(Coth^{-1}x)=x$, $Coth^{-1}(\coth t)=t$ for $x \in R$-$\{0\}$, $t \in (-\infty,-1) \cup (1,\infty)$



Def 5: The function $f:[0,\infty) \to (0,1]$ defined by $f(x)=\operatorname{sech} x$ is a bijection. The inverse of f from $(0,1]$ into $[0,\infty)$ is called inverse hyperbolic secant function and it is denoted by $Sech^{-1}x$.

Note 1: $Sech^{-1}x=t \Leftrightarrow x=\operatorname{sech} t, \ \forall x \in (0,1]$

Note: $\operatorname{sech}(Sech^{-1}x)=x$, $Sech^{-1}(\operatorname{sech} t)=t$ for $x \in (0,1]$, $t \in [0,\infty)$.



Def 6: The function $f : R-\{0\} \to R-\{0\}$ defined by $f(x)=\operatorname{cosech} x$ is a bijection. The inverse of f from $R-\{0\}$ into $R-\{0\}$ is called inverse hyperbolic cosecant function and it is denoted by $Cosech^{-1}x$.

Note 1: $Cosec^{-1}x=t \Leftrightarrow x=\operatorname{cosech} t, \ \forall x \in R$-$\{0\}$.

Note 2: $\operatorname{cosech}(Cosech^{-1}x)=x$, $Cosech^{-1}(\operatorname{cosech} t)=t$ for $x \in R-\{0\}$, $t \in R-\{0\}$

The domains and ranges of the inverse hyperbolic functions are as follows:

	Function	Domain	Range
1.	$\text{Sinh}^{-1}x$	R	R
2.	$\text{Cosh}^{-1}x$	$[1,\infty)$	$[0,\infty)$
3.	$\text{Tanh}^{-1}x$	$(-1,1)$	R
4.	$\text{Coth}^{-1}x$	$(-\infty,-1)\cup(1,\infty)$	$R-\{0\}$
5.	$\text{Sech}^{-1}x$	$(0,1]$	$[0,\infty)$
6.	$\text{Cosec}^{-1}x$	$R-\{0\}$	$R-\{0\}$

Theorem 1: $\text{Sinh}^{-1}x = \log_e(x+\sqrt{x^2+1})$ for $x\in R$

Proof: Let $\text{Sinh}^{-1}x = t$, then $\sinh t = x \Rightarrow \sinh t = \frac{e^t - e^{-t}}{2} \Rightarrow 2x = e^t - \frac{1}{e^t}$

$$\Rightarrow (e^t)^2 - 2xe^t - 1 = 0 \Rightarrow e^t = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $e^t > 0$ we have $e^t = x + \sqrt{x^2+1} \Rightarrow t = \log_e(x+\sqrt{x^2+1})$

$$\Rightarrow \text{Sinh}^{-1}x = \log_e(x+\sqrt{x^2+1})$$

Theorem 2: $\text{Cosh}^{-1}x = \log_e(x+\sqrt{x^2-1})$ for $x \geq 1$

Proof: Let $\text{Cosh}^{-1}x = t$, then $\cosh t = x$ and $t > 0 \Rightarrow x = \cosh t = \frac{e^t + e^{-t}}{2}$

$$\Rightarrow (e^t)^2 + 2xe^t + 1 = 0 \Rightarrow e^t = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$$\Rightarrow t = \log_e(x \pm \sqrt{x^2-1}) = \log_e(x+\sqrt{x^2-1})^{\pm 1} = \pm\log_e(x+\sqrt{x^2-1})$$

Since $e > 0$ we have $t = \log_e(x+\sqrt{x^2-1})$

Theorem 3: $\text{Tanh}^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$ for $x \in (-1,1)$

Proof: Let $\text{Tanh}^{-1}x = t$, then $x = \tanh t$

$$\Rightarrow \frac{1}{x} = \frac{e^t + e^{-t}}{e^t - e^{-t}} \Rightarrow \frac{1+x}{1-x} = \frac{e^t + e^{-t} + e^t - e^{-t}}{e^t + e^{-t} - e^t + e^{-t}} = \frac{2e^t}{2e^{-t}} = e^{2t}$$

$$\Rightarrow 2t = \log_e\left(\frac{1+x}{1-x}\right) \text{ for } x \in (-1,1) \Rightarrow t = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \text{Tanh}^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \text{ for } x \in (-1,1)$$

Theorem 4: $\operatorname{Coth}^{-1}x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right)$ for $|x|>1$

Proof: Let $\operatorname{Coth}^{-1}x = t$, then $x = \coth t$

$$\Rightarrow x = \coth t = \frac{e^t+e^{-t}}{e^t-e^{-t}} \Rightarrow \frac{x+1}{x-1} = \frac{e^t+e^{-t}+e^t-e^{-t}}{e^t+e^{-t}-e^t+e^{-t}} = \frac{2e^t}{2e^{-t}} = e^{2t}$$

$$\Rightarrow 2t = \log_e\left(\frac{x+1}{x-1}\right) \text{ for } |x|>1 \Rightarrow t = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right) \Rightarrow \operatorname{Cot}^{-1}x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right) \text{ for } |x|>1$$

Theorem 5: $\operatorname{Sech}^{-1}x = \log_e\left[\frac{1+\sqrt{1-x^2}}{x}\right]$ for $x \in (0,1]$

Proof: Let $\operatorname{Sech}^{-1}x = t$, then $\operatorname{sech} t = x$ and $t \ge 0$

$$\Rightarrow x = \operatorname{sech} t = \frac{2}{e^t+e^{-t}} \Rightarrow e^t+e^{-t} = \frac{2}{x} \Rightarrow (e^t)^2 - \frac{2}{x}e^t + 1 = 0$$

$$\Rightarrow e^t = \frac{\frac{2}{x} \pm \sqrt{\frac{4}{x^2}-4}}{2} = \frac{1\pm\sqrt{1-x^2}}{x} \text{ for } x \in (0,1]$$

$$\Rightarrow t = \log_e\left(\frac{1\pm\sqrt{1-x^2}}{x}\right) = \log_e\left(\frac{1\pm\sqrt{1-x^2}}{x}\right)^{\pm 1} = \pm\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

Since $t \ge 0$ we have

$$t = \log_e\left[\frac{1+\sqrt{1-x^2}}{x}\right] \Rightarrow \operatorname{Sech}^{-1}x = \log_e\left[\frac{1+\sqrt{1-x^2}}{x}\right] \text{ for } x \in (0,1]$$

Theorem 6: $\operatorname{Cosech}^{-1}x = \log_e\left[\frac{1+\sqrt{1+x^2}}{x}\right]$ if $x>0$

$$= \log_e\left[\frac{1-\sqrt{1+x^2}}{x}\right] \text{ if } x<0$$

Proof: Let $\operatorname{Cosech}^{-1}x = t$, then $x = \operatorname{cosech} t$

$$\Rightarrow x = \operatorname{cosech} t = \frac{2}{e^t-e^{-t}} \Rightarrow e^t - e^{-t} = \frac{2}{x} \Rightarrow (e^t)^2 - \frac{2}{x}e^t - 1 = 0$$

$$\Rightarrow e^t = \frac{\frac{2}{x} \pm \sqrt{\frac{4}{x^2}+4}}{2} = \frac{1\pm\sqrt{1+x^2}}{x} \Rightarrow e^t = \frac{1+\sqrt{1+x^2}}{x} \text{ if } x>0 \text{ and } e^t = \frac{1-\sqrt{1+x^2}}{x} \text{ if } x>0$$

$$\Rightarrow t = \log_e\left[\frac{1+\sqrt{1+x^2}}{x}\right] \text{ if } x>0 \text{ and } t = \log_e\left[\frac{1+\sqrt{1+x^2}}{x}\right] \text{ if } x<0$$

$$\Rightarrow \operatorname{Cosech}^{-1}x = \log_e\left[\frac{1+\sqrt{1+x^2}}{x}\right] \text{ if } x>0 \qquad = \log_e\left[\frac{1-\sqrt{1+x^2}}{x}\right] \text{ if } x<0$$