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DIGITAL MATERIAL

INVERSE T' FUNCTIONS-INDEX

8.INVERSE TRIGONOMETRIC FUNCTIONS

1. INTRODUCTION PAGE

Sections	No. of periods (6)	Weightage in IPE [1x4=4]
1. InverseTrigonometric Functions	6	4 Marks

Any of the six trigonometric functions, is not a bijective function with its actual domain and codomain R. But, by restricting the domain and codomain properly, we can make them into bijective functions. Then, with the restricted domains and codomains, we can define their inverses, known as"inverse Trigonometric functions".

Eg: sinx:R→R *is neither one one nor onto. 'sinx' defined, with domain R is not one one because,* $\sin 30^0 = \frac{1}{2}$; $\sin 150^0 = \frac{1}{2}$; *(or, we can draw a line drawn parallel to the X-axis intersecting the sine curve at more than one point). Hence, if we restrict the domain to ts principal range i.e.,* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ *then sinx becomes a one one function.*

Also, sinx with codomain R is not an onto function. Because, the Range of sinx is [−1,1]. Hence, if we restrict the codomain to [−1,1] then sinx becomes an onto function. This restricted sine function of x is denoted by Sinx.Thus Sinx : $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \to [-1,1]$ *is a bijective function. Hence,* $Sin^{-1}x : [-1,1] \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ *is known as inverse sine function. Similarly, the other inverse trigonometric functions are defined with appropriate domains and codomains.*

The following 'Remarks' are to be kept in mind, while dealing with, the Inverse Trigonometric functions.

- *Write capital initials for inverse trigonometric functions.*
- *All the six inverse trigonometric functions are merely angles*
- *In the first chapter of trigonometry, we know that sin(an angle)=a real value. Now, we have to note that* Sin^{-1}*(a real value)=an angle.*
- $Sin^{-1}x$ *is also written as Arc sinx.*
- $Sin^{-1}x \neq \frac{1}{\sin x}$ *i.e.,* $Sin^{-1}x$ *and* $(\sin x)^{-1}$ *are different. Infact* $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$
- *In* $Sin^{-1}x$*, x is a real value belonging to [−1,1] and the (functional) value of* $Sin^{-1}x$ *is an angle in radian measure.*
- $Sin^{-1}x = \theta \to x$ *is a real value and* θ *is also a real value in radian measure.*
- $Sin^{-1}(\sin x) = x$ *and this x is the measure of angle in radians.*
- $\sin(Sin^{-1}x) = x$ *and this x is a real value.*
- *Note that* $\sin(Sin^{-1}x) \neq Sin^{-1}(\sin x)$.
  Infact, $Sin^{-1}(\sin\theta) = \theta$ *for* $-\pi/2 \leq \theta \leq \pi/2$ *and* $\sin(Sin^{-1}x) = x$ *for* $-1 \leq x \leq 1$

2. SINGLE PAGE SYNOPSIS

1. Domains and Ranges of Inverse Trigonometric Functions:

Function	Domain	Range
$\text{Sin}^{-1}x$	$[-1,1]$	$[-\pi/2,\pi/2]$
$\text{Cos}^{-1}x$	$[-1,1]$	$[0,\pi]$
$\text{Tan}^{-1}x$	R	$(-\pi/2,\pi/2)$
$\text{Cot}^{-1}x$	R	$(0,\pi)$
$\text{Sec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi/2)\cup(\pi/2,\pi]$
$\text{Csc}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[-\pi/2,0)\cup(0,\pi/2]$

2.2. $\text{Sin}^{-1}(\sin\theta)=\theta \text{ for } \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ | $\sin(\text{Sin}^{-1}x)=x \text{ for } x\in[-1,1]$

$\text{Cos}^{-1}(\cos\theta)=\theta \text{ for } \theta\in[0,\pi]$ | $\cos(\text{Cos}^{-1}x)=x \text{ for } x\in[-1,1]$

$\text{Tan}^{-1}(\tan\theta)=\theta \text{ for } \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ | $\tan(\text{Tan}^{-1}x)=x \text{ for } x\in R$

3. $\text{Sin}^{-1}(-x)=-\text{Sin}^{-1}x;\ \text{Cos}^{-1}(-x)=\pi-\text{Cos}^{-1}x;\ \text{Tan}^{-1}(-x)=-\text{Tan}^{-1}x;\ \text{Cot}^{-1}(-x)=\pi-\text{Cot}^{-1}x$

4. $\text{Sin}^{-1}x+\text{Cos}^{-1}x=\frac{\pi}{2};\qquad \text{Tan}^{-1}x+\text{Cot}^{-1}x=\frac{\pi}{2};\qquad \text{Sec}^{-1}x+\text{Csc}^{-1}x=\frac{\pi}{2}$

5. (i) $\text{Sin}^{-1}(x)=\text{Cosec}^{-1}\frac{1}{x}$ if $x\in[-1,0)\cup(0,1]$ (ii) $\text{Cos}^{-1}(x)=\text{Sec}^{-1}\frac{1}{x}$ if $x\in[-1,0)\cup(0,1]$

(iii) $\text{Cot}^{-1}x=\text{Tan}^{-1}\frac{1}{x}$ if $x>0$ (iv) $\text{Cot}^{-1}x=\pi+\text{Tan}^{-1}\frac{1}{x}$ if $x<0$

6.

$$\text{Tan}^{-1}x+\text{Tan}^{-1}y=\begin{cases}\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy<1\\ \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)+\pi, & \text{if } xy>1,\ x>0,\ y>0\\ \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)-\pi, & \text{if } xy>1,\ x<0,\ y<0\\ \frac{\pi}{2}, & \text{if } xy=1\end{cases}$$

7. $\text{Tan}^{-1}x-\text{Tan}^{-1}y=\text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$

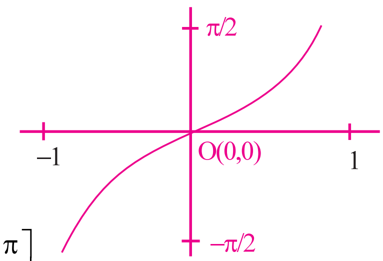
8. $2\text{Sin}^{-1}x=\text{Sin}^{-1}\left(2x\sqrt{1-x^2}\right)$ | $3\text{Sin}^{-1}x=\text{Sin}^{-1}(3x-4x^3)$

$2\text{Cos}^{-1}x=\text{Cos}^{-1}(2x^2-1)$ | $3\text{Cos}^{-1}x=\text{Cos}^{-1}(4x^3-3x)$

$2\text{Tan}^{-1}x=\text{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$ | $3\text{Tan}^{-1}x=\text{Tan}^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

3. PROOFS OF INVERSE T' FUNCTIONS

Def 1: The function $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$ defined by $f(x)=\sin x$ is a bijection. The inverse of f from $[-1,1]$ into $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is called inverse sine function or Arc sine function and it is denoted by $\mathrm{Sin}^{-1}x$ or Arc sinx.



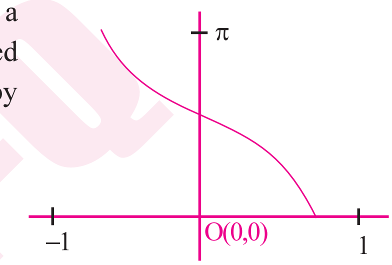
Note 1: $\mathrm{Sin}^{-1}x=\theta \Leftrightarrow x=\sin\theta,\ \forall\ x\in[-1,1]$

Note 2: If $x\in[-1,1]$ then $\sin(\mathrm{Sin}^{-1}x)=x$

Note 3: If $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ then $\mathrm{Sin}^{-1}(\sin\theta)=\theta$

Note 4: The domain of $\mathrm{Sin}^{-1}x$ is $[-1,1]$ and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Def 2: The function $f:[0,\pi]\to[-1,1]$ defined by $f(x)=\cos x$ is a bijection. The inverse of f from $[-1,1]$ into $[0,\pi]$ is called inverse cos function or Arc cos function and it is denoted by $\mathrm{Cos}^{-1}x$.

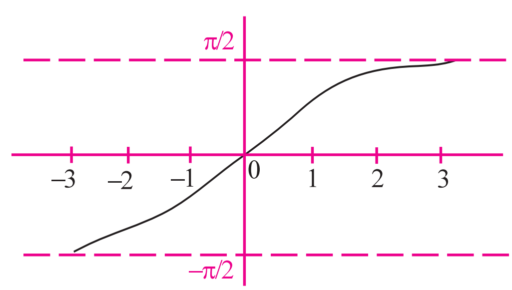


Note 1: $\mathrm{Cos}^{-1}x=\theta \Leftrightarrow x=\cos\theta,\ \forall\ x\in[-1,1]$

Note 2: If $x\in[-1,1]$ then $\cos(\mathrm{Cos}^{-1}x)=x$

Note 3: If $\theta\in[0,\pi]$ then $\mathrm{Cos}^{-1}(\cos\theta)=\theta$

Def 3: The function $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to R$ defined by $f(x)=\tan x$ is a bijection. The inverse of f from R into $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is called inverse tan function or Arc tan function and it is denoted by $\mathrm{Tan}^{-1}x$

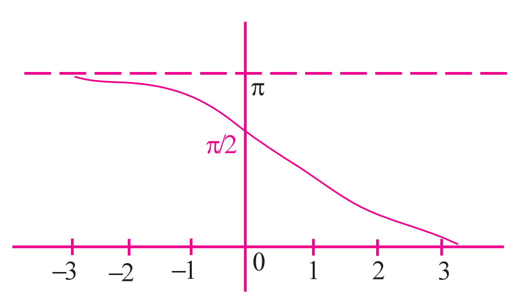


Note 1: $\mathrm{Tan}^{-1}x=\theta \Leftrightarrow x=\tan\theta,\ x\in R$

Note 2: If $x\in R$, then $\tan(\mathrm{Tan}^{-1}x)=x$

Note 3: If $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then $\mathrm{Tan}^{-1}(\tan\theta)=\theta$.

Def 4: The function $f:(0,\pi)\to R$ defined by $f(x)=\cot x$ is a bijection. The inverse of f from R into $(0,\pi)$ is called inverse cot function or Arc cot function and it is denoted by $\mathrm{Cot}^{-1}x$ or Arc cotx.



Note 1: $\mathrm{Cot}^{-1}x=\theta \Leftrightarrow x=\cot\theta,\ \forall\ x\in R$.

Note 2: If $x\in R$, then $\cot(\mathrm{Cot}^{-1}x)=x$

Note 3: If $\theta\in(0,\pi)$ then $\mathrm{Cot}^{-1}(\cot\theta)=\theta$.

Note 4: $\mathrm{Cot}^{-1}x=\theta\in(0,\pi/2) \Leftrightarrow x=\cot\theta\in(0,\infty)$

$\mathrm{Cot}^{-1}x=\theta\in(\pi/2,\pi) \Leftrightarrow x=\cot\theta\in(-\infty,0)$

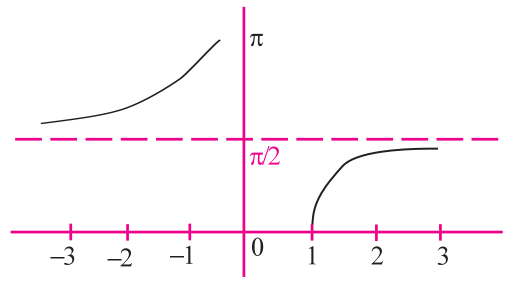
Def 5: The function $f:\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]\to(-\infty,-1]\cup[1,\infty)$ defined by $f(x)=\sec x$ is a bijection. The inverse of f from $(-\infty,-1]\cup[1,\infty)$ into $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ is called inverse sec function. It is denoted by $\mathrm{Sec}^{-1}x$ or Arc secx.

Note 1: $\mathrm{Sec}^{-1}x=\theta \Leftrightarrow x=\sec\theta,\ \forall\ x\in(-\infty,-1]\cup[1,\infty)$

Note 2: If $x\in(-\infty,-1]\cup[1,\infty)$ then $\sec(\mathrm{Sec}^{-1}x)=x$

Note 3: If $\theta=\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, then $\mathrm{Sec}^{-1}(\sec\theta)=\theta$.

Note 4: $\mathrm{Sec}^{-1}x=\theta\in[0,\pi/2) \Leftrightarrow x=\sec\theta\in[1,\infty)$
$\mathrm{Sec}^{-1}x=\theta\in(\pi/2,\pi] \Leftrightarrow x=\sec\theta\in(-\infty,-1]$



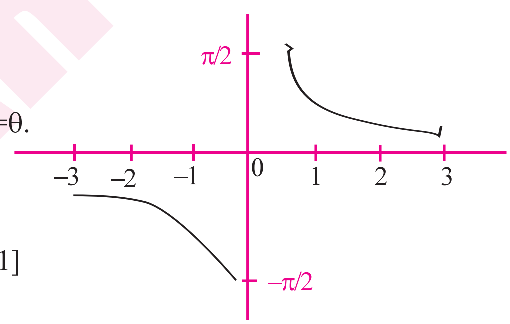
Def 6: The function $f:\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]\to(-\infty,-1]\cup[1,\infty)$ defined by $f(x)=\mathrm{cosec}x$ is a bijection. The inverse of f from $(-\infty,-1]\cup[1,\infty)$ into $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ is called inverse cosec function or Arc cosec function and it is denoted by $\mathrm{Cosec}^{-1}x$ or Arc cosecx.

Note 1: $\mathrm{Cosec}^{-1}x=\theta \Leftrightarrow x=\mathrm{Cosec}\theta,\ \forall\ x\in(-\infty,-1]\cup[1,\infty)$

Note 2: If $x\in(-\infty,-1]\cup[1,\infty)$ then $\mathrm{cosec}(\mathrm{cosec}^{-1}x)=x$

Note 3: If $\theta\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ then $\mathrm{Cosec}^{-1}(\mathrm{cosec}\theta)=\theta$.

Note 4: $\mathrm{Cosec}^{-1}x=\theta\in(0,\pi/2] \Leftrightarrow x=\mathrm{cosec}\theta\in[1,\infty)$;
$\mathrm{Cosec}^{-1}x=\theta\in[-\pi/2,0) \Leftrightarrow x=\mathrm{cosec}\theta\in(-\infty,-1]$



Domains and Ranges of the six Inverse Trigonometric Functions:

Function	Domain	Range
$\mathrm{Sin}^{-1}x$	$[-1,1]$	$[-\pi/2,\pi/2]$
$\mathrm{Cos}^{-1}x$	$[-1,1]$	$[0,\pi]$
$\mathrm{Tan}^{-1}x$	R	$(-\pi/2,\pi/2)$
$\mathrm{Cot}^{-1}x$	R	$(0,\pi)$
$\mathrm{Sec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi/2)\cup(\pi/2,\pi]$
$\mathrm{Csc}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[-\pi/2,0)\cup(0,\pi/2]$

Theorem 1: If $x \in [-1,1]$ then $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$.

Proof: Let $x \in [-1,1]$ then $-x \in [-1,1]$

Let $\text{Sin}^{-1}(-x) = \theta$, then $-x = \sin\theta$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Hence $\text{Sin}^{-1}x = -\theta = -\text{Sin}^{-1}(-x)$

$\therefore \text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$.

Theorem 2: If $x \in [-1,1]$ then $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}x$.

Proof: Let $x \in [-1,1]$ then $-x \in [-1,1]$

Let $\text{Cos}^{-1}(-x) = \theta$, then $-x = \cos\theta$ and $\theta \in [0,\pi]$

Hence $x = -\cos\theta = \cos(\pi - \theta)$ and $\pi - \theta \in [0,\pi]$

Hence $\text{Cos}^{-1}x = \pi - \theta = \pi - \text{Cos}^{-1}(-x)$

$\therefore \text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}x$.

Similarly, we can establish the following

If $x \in R$, then $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}x$

If $x \in R$, then $\text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1}x$

If $x \le -1$ or $x \ge 1$, then $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}x$

If $x \le -1$ or $x \ge 1$, then $\text{Cosec}^{-1}(-x) = -\text{Cosec}^{-1}x$.

Theorem 3: (i) $\text{Sin}^{-1}(x) = \text{Cosec}^{-1}\left(\frac{1}{x}\right)$ for $x \in [-1,0) \cup (0,1]$

(ii) $\text{Cos}^{-1}(x) = \text{Sec}^{-1}\left(\frac{1}{x}\right)$ for $x \in [-1,0) \cup (0,1]$

(iii) $\text{Tan}^{-1}x = \text{Cot}^{-1}\left(\frac{1}{x}\right)$ for $x \in (0,\infty)$ and $\text{Tan}^{-1}x = -\pi + \text{Cot}^{-1}\left(\frac{1}{x}\right)$ for $x \in (-\infty,0)$

Proof: (i) Let $\text{Sin}^{-1}x = \theta$ then $x = \sin\theta \Rightarrow \frac{1}{x} = \text{cosec}\,\theta \Rightarrow \text{Cosec}^{-1}\left(\frac{1}{x}\right) = 0 = \text{Sin}^{-1}x$

(ii) Let $\text{Cos}^{-1}x = \theta$ then $x = \cos\theta \Rightarrow \frac{1}{x} = \sec\theta \Rightarrow \text{Sec}^{-1}\left(\frac{1}{x}\right) = \theta = \text{Cos}^{-1}x$

(iii) Suppose $x \in (0,\infty)$

Let $\text{Tan}^{-1}x = \theta$, then $x = \tan\theta \Rightarrow \frac{1}{x} = \cot\theta \Rightarrow \text{Cot}^{-1}\left(\frac{1}{x}\right) = \theta = \text{Tan}^{-1}x$

Suppose $x \in (-\infty,0)$. Let $x = -y$, then $\text{Tan}^{-1}x = \text{Tan}^{-1}(-y) = -\text{Tan}^{-1}y$

$$= -\text{Cot}^{-1}\left(\frac{1}{y}\right) = \pi - \text{Cot}^{-1}\left(\frac{1}{y}\right) - \pi = -\pi + \text{Cot}^{-1}\left(-\frac{1}{y}\right) = -\pi + \text{Cot}^{-1}\frac{1}{x}$$

Rem : (i) $\text{Cot}^{-1}x = \text{Tan}^{-1}\frac{1}{x}$ if $x>0$ (ii) $\text{Cot}^{-1}x = \pi + \text{Tan}^{-1}\frac{1}{x}$ if $x<0$

Theorem 3: If $-1 \le x \le 1$, then $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Proof: If x=0, then $\cos^{-1} + \sin^{-1}x = \cos^{-1}(0) + \sin^{-1}(0) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

Let $0<x\le 1$, then $0 \le \cos^{-1}x < \frac{\pi}{2}, 0 < \sin^{-1}x \le \frac{\pi}{2}$

Let $\sin^{-1}x = \theta$, then $0 < \theta \le \frac{\pi}{2}$ and $\sin\theta = x$

Hence, $0 \le \left(\frac{\pi}{2} - \theta\right) < \frac{\pi}{2}$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = x$

$\therefore \cos^{-1}x = \frac{\pi}{2} - \theta \Rightarrow \cos^{-1}x + \theta = \frac{\pi}{2} \Rightarrow \cos^{-1}x + \sin^{-1} = \frac{\pi}{2}$

Let $-1 \le x < 0$ then $\frac{\pi}{2} < \cos^{-1}x \le \pi, \frac{-\pi}{2} \le \sin^{-1}x < 0$

Let $\sin^{-1}x = \theta$, then $x = \sin\theta$ and $\frac{-\pi}{2} \le \theta < 0$. Hence $\frac{\pi}{2} \ge -\theta > 0$

Hence $\frac{\pi}{2} < \left(\frac{\pi}{2} - \theta\right) \le \pi$

We have $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = x$

$\therefore \cos^{-1}x = \frac{\pi}{2} - \theta$

$\therefore \cos^{-1}x + \theta = \frac{\pi}{2} \Rightarrow \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Theorem 4: $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for all $x \in R$

Proof: Let $x \in R$ and $\tan^{-1}x = \alpha$, then $x = \tan\alpha$ and $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$

We have $\cot\left(\frac{\pi}{2} - \alpha\right) = \tan\alpha = x$

Case (i) : $x \ge 0$

In this case $0 \le \alpha < \frac{\pi}{2}$, hence $0 < \left(\frac{\pi}{2} - \alpha\right) \le \frac{\pi}{2}$

Now, from equation (1) we have $\cot^{-1}x = \frac{\pi}{2} - \alpha$

Hence, $\alpha + \cot^{-1}x = \frac{\pi}{2} \Rightarrow \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

Case (ii) : $x < 0$

In this case $-\frac{\pi}{2} < \alpha < 0$, hence $\frac{\pi}{2} < \left(\frac{\pi}{2} - \alpha\right) \le \pi$

Now, from equation (1) we have $\text{Cot}^{-1}x = \frac{\pi}{2} - \alpha$

Hence, $\text{Tan}^{-1}x + \text{Cot}^{-1} = \frac{\pi}{2}$

Note 1: If $x \le -1$ or $x \ge 1$, then $\text{Sec}^{-1}x + \text{Cosec}^{-1}x = \pi/2$.

Note 2: If $x \in (-\infty, -1] \cup [1, \infty)$ then $\text{Sec}^{-1}x + \text{Cosec}^{-1}x = \pi/2$.

Theorem 5: If $x \ge 0$, $y \ge 0$ and $x^2 + y^2 \le 1$, then $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \text{Sin}^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

Proof: Suppose that $x \ge 0$, $y \ge 0$ and $x^2 + y^2 \le 1$.

Let $\text{Sin}^{-1}x = \alpha$ and $\text{Sin}^{-1}y = \beta$. Then $\sin\alpha = x$, $\sin\beta = y$ and both α & β belong to $\left[0, \frac{\pi}{2}\right]$

Hence $\cos\alpha = \sqrt{1-x^2}$, $\cos\beta = \sqrt{1-y^2}$ and $0 \le (\alpha + \beta) \le \pi$

We have $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \sqrt{(1-x^2)(1-y^2)} - xy \ge 0$ since $x^2 + y^2 \le 1$

Hence, $\alpha + \beta \notin \left(\frac{\pi}{2}, \pi\right) \Rightarrow 0 \le (\alpha+\beta) \le \frac{\pi}{2}$

Now, $\sin(a+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Hence, $\alpha + \beta = \text{Sin}^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

Thus, $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \text{Sin}^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

Note: (i) $\text{Sin}^{-1}x - \text{Sin}^{-1}y = \text{Sin}^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ if $x, y \in [0,1]$

(ii) $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \text{Cos}^{-1}\left[\sqrt{(1-x^2)(1-y^2)} - xy\right]$ if $x, y \in [0,1]$

(iii) $\text{Sin}^{-1}x - \text{Sin}^{-1}y = \text{Cos}^{-1}\left[\sqrt{(1-x^2)(1-y^2)} + xy\right]$ if $0 \le y \le x \le 1$

(iv) $\text{Cos}^{-1}x + \text{Cos}^{-1}y = \text{Cos}^{-1}\left[xy - \sqrt{(1-x^2)(1-y^2)}\right]$ if $x, y \in [0,1]$

(v) $\text{Cos}^{-1}x - \text{Cos}^{-1}y = \text{Cos}^{-1}\left[xy + \sqrt{(1-x^2)(1-y^2)}\right]$ if $0 \le x \le y \le 1$

(vi) $\text{Cos}^{-1}x + \text{Cos}^{-1}y = \text{Sin}^{-1}\left[y\sqrt{(1-x^2)} + x\sqrt{(1-y^2)}\right]$ if $x, y \in [0,1]$ and $x^2 + y^2 \ge 1$

(vii) $\text{Cos}^{-1} - \text{Cos}^{-1}y = \text{Sin}^{-1}\left[y\sqrt{1-x^2} - x\sqrt{1-y^2}\right]$ if $x, y \in [0,1]$

Theorem 6: For x>0, y>0, if (i) xy<1 then $\text{Tan}^{-1}x+\text{Tan}^{-1}y=\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) xy>1 then $\text{Tan}^{-1}x+\text{Tan}^{-1}y=\pi+\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof: Let $\text{Tan}^{-1}x=\alpha,\ \text{Tan}^{-1}y=\beta \Rightarrow x=\tan\alpha,\ y=\tan\beta$

Given that x>0, y>0 $\Rightarrow \tan\alpha>0,\ \tan\beta>0 \Rightarrow \alpha,\beta\in\left(0,\frac{\pi}{2}\right)\Rightarrow 0<\alpha+\beta<\pi$

$\Rightarrow 0<\alpha+\beta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\alpha+\beta<\pi$ .....(A) $(\because xy\neq 1,\ \tan\alpha\tan\beta\neq 1\Rightarrow \alpha+\beta\neq\pi/2)$

Also, $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{x+y}{1-xy}$ ....(B)

As $x>0,\ y>0,\ xy\neq 1$, then 2 cases arise viz., xy<1 and xy>1

Case (i): $xy<1$

Now $\frac{x+y}{1-xy}>0\Rightarrow \tan(\alpha+\beta)>0$ .....(C)

$(A),(C)\Rightarrow 0<\alpha+\beta<\frac{\pi}{2}\Rightarrow \alpha+\beta$ is in the defined domain of tan function.

$\therefore (B)\Rightarrow \alpha+\beta=\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)\Rightarrow \text{Tan}^{-1}x+\text{Tan}^{-1}y=\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

Case (ii): $xy>1$

Now, $\frac{x+y}{1-xy}>0\Rightarrow \tan(\alpha+\beta)<0$ ......(D)

$(A),(D)\Rightarrow \frac{\pi}{2}<\alpha+\beta<\pi\Rightarrow \alpha+\beta$ is not in the defined domain of tan function

$\Rightarrow\left(\frac{\pi}{2}-\pi\right)<(\alpha+\beta)-\pi<(\pi-\pi)\Rightarrow -\frac{\pi}{2}<(\alpha+\beta-\pi)<0$

$\therefore \tan(\alpha+\beta-\pi)=-\tan(\pi-(\alpha+\beta))=\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{x+y}{1-xy}$

$\Rightarrow \alpha+\beta-\pi=\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)\Rightarrow \alpha+\beta=\pi+\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

Theorem 7: If $x<0,\ y<0$ and $xy>1$ then $\text{Tan}^{-1}x+\text{Tan}^{-1}y=-\pi+\text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof: Given that x<0, y<0 −x>0, −y>0 ⇒ (−x)(−y)>0 ⇒ xy>0

Let $\text{Tan}^{-1}(-x)=\alpha \Rightarrow \tan\alpha=x$, $\text{Tan}^{-1}(-y)=\beta \Rightarrow \tan\beta=-y$

$-x>0,-y>0\Rightarrow 0<\alpha<\frac{\pi}{2},0<\beta<\frac{\pi}{2}\Rightarrow 0<\alpha+\beta<\pi$ .....(A)

$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{-x-y}{1-(-x)(-y)}=-\left(\frac{x+y}{1-xy}\right)<0\ \ (\because xy>1\Rightarrow 1-xy<0)$

$$\therefore (A) \Rightarrow \frac{\pi}{2} < \alpha+\beta < \pi \Rightarrow \frac{\pi}{2}-\pi < \alpha+\beta-\pi < 0 \Rightarrow -\frac{\pi}{2} < \alpha+\beta-\pi < 0$$

$$\therefore \tan(\alpha+\beta-\pi) = -\tan(\pi-(\alpha+\beta)) = \tan(\alpha+\beta) = -\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \alpha+\beta-\pi = \mathrm{Tan}^{-1}\left(-\left(\frac{x+y}{1-xy}\right)\right) = -\mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \alpha+\beta = \pi - \mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right) \Rightarrow \mathrm{Tan}^{-1}(-x)+\mathrm{Tan}^{-1}(-y) = \pi - \mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow -\mathrm{Tan}^{-1}x - \mathrm{Tan}^{-1}y = \pi - \mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right) \Rightarrow \mathrm{Tan}^{-1}x + \mathrm{Tan}^{-1}y = -\pi + \mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$$

Theorem 8: If x>0, y>0 then $\mathrm{Tan}^{-1}x - \mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$

Proof: Let $\mathrm{Tan}^{-1}x=\alpha$, $\mathrm{Tan}^{-1}y=\beta \Rightarrow \tan\alpha=x, \tan\beta=y$

$$x>0, y>0 \Rightarrow \tan\alpha>0, \tan\beta>0 \Rightarrow 0<\alpha<\frac{\pi}{2}, 0<\beta<\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < a-\beta < \frac{\pi}{2}$$

Now, $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \frac{x-y}{1+\tan\alpha\tan\beta} = \frac{x-y}{1+xy}$

If x>0 & y>0 then 2 cases arise viz., x>y or x<y

If x>y then $\frac{x-y}{1+xy} > 0 \Rightarrow \tan(\alpha-\beta)>0 \ \therefore 0<(\alpha-\beta)<\frac{\pi}{2}$

$$\Rightarrow \tan(\alpha-\beta) = \frac{x-y}{1+xy} \Rightarrow \alpha-\beta = \mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right) \Rightarrow \mathrm{Tan}^{-1}x - \mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$$

If x<y then $\frac{x-y}{1+xy} < 0 \Rightarrow \tan(\alpha-\beta)>0 \quad \therefore -\frac{\pi}{2} < (\alpha-\beta) < 0$

$$\therefore \tan(\alpha-\beta) = \frac{x-y}{1+xy} \Rightarrow \alpha-\beta = \mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right) \Rightarrow \mathrm{Tan}^{-1}x - \mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$$

Corollary: If $x<0, y<0$ then $\mathrm{Tan}^{-1}x - \mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$

Corollary: (i) $2\mathrm{Sin}^{-1}x = \mathrm{Sin}^{-1}2x\sqrt{1-x^2}$ if $x \le \frac{1}{\sqrt{2}} = \pi - \mathrm{Sin}^{-1}2x\sqrt{1-x^2}$ if $x > \frac{1}{\sqrt{2}}$

(ii) $2\mathrm{Cos}^{-1} = \mathrm{Cos}^{-1}(2x^2-1)$ if $x \ge \frac{1}{\sqrt{2}} = \pi - \mathrm{Cos}^{-1}(1-2x^2)$ if $x < \frac{1}{\sqrt{2}}$

Corollary: $2\mathrm{Tan}^{-1}x = \mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$ if $|x|<1 = \pi + \mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$ if $|x|>1$

Theorem 9: $2\text{Tan}^{-1}x = \text{Sin}^{-1}\dfrac{2x}{1+x^2}, \forall\, x \in R$

$$= \text{Cos}^{-1}\frac{1-x^2}{1+x^2}, \text{ if } x \geq 0 = -\text{Cos}^{-1}\frac{1-x^2}{1+x^2}, \text{ if } x < 0$$

Theorem 10: (i) $3\text{Sin}^{-1}x = \text{Sin}^{-1}(3x-4x^2)$ for $0 \leq x \leq \dfrac{1}{2}$

(ii) $3\text{Cos}^{-1}x = \text{Cos}^{-1}(4x^3-3x)$ for $\dfrac{\sqrt{3}}{2} \leq x \leq 1$

(iii) $3\text{Tan}^{-1}x = \text{Tan}^{-1}\left(\dfrac{3x-x^2}{1-3x^2}\right)$ for $0 \leq x \leq \dfrac{1}{\sqrt{3}}$

Proof: (i) Let $\text{Sin}^{-1}x=\theta$, then $x=\sin\theta$.

$$0 \leq x \leq \frac{1}{2} \Rightarrow 0 \leq \sin\theta \leq \frac{1}{2} \Rightarrow 0 \leq \theta \leq \frac{\pi}{6} \Rightarrow 0 \leq 3\theta \leq \frac{\pi}{2}$$

$$\sin 3\theta = \sin 3\theta = 3\sin\theta - 4\sin^3\theta = 3\sin\theta - 4\sin^3\theta = 3x - 4x^3$$

$$0 \leq 3\theta \leq \frac{\pi}{2}, \sin 3\theta = 3x - 4x^3 \Rightarrow 3\theta = \text{Sin}^{-1}(3x-4x^3)$$

$$\Rightarrow 3\text{Sin}^{-1}x = \text{Sin}^{-1}(3x-4x^3)$$

(ii) Let $\text{Cos}^{-1}x=\theta$, then $x=\cos\theta$.

$$\frac{\sqrt{3}}{2} \leq x \leq 1 \Rightarrow \frac{\sqrt{3}}{2} \leq \cos\theta \leq 1 \Rightarrow \frac{\pi}{6} \geq \theta \geq 0 \Rightarrow 0 \leq \theta \leq \frac{\pi}{6} \Rightarrow 0 \leq 3\theta \leq \frac{\pi}{2}$$

$$\cos 3\theta = \cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$$

$$0 \leq 3\theta \leq \frac{\pi}{2}, \cos 3\theta = 4x^3 - 3x \leq 3\theta = \text{Cos}^{-1}(4x^3-3x) \Rightarrow 3\text{Cos}^{-1}x = \text{Cos}^{-1}(4x^3-3x)$$

(iii) Let $\text{Cos}^{-1}x=\theta$, then $x=\cos\theta$

$$0 \leq x \leq \frac{1}{\sqrt{3}} \Rightarrow 0 \leq \tan\theta \leq \frac{1}{\sqrt{3}} \Rightarrow 0 \leq \tan\theta \leq \frac{1}{\sqrt{3}} \Rightarrow 0 \leq \theta \leq \frac{\pi}{6} \Rightarrow 0 \leq 3\theta \leq \frac{\pi}{2}$$

$$\tan 3\theta = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \frac{3x-x^3}{1-3x^2}$$

$$0 \leq 3\theta \leq \frac{\pi}{2}, \tan 3\theta = \frac{3x-x^3}{1-3x^2} \Rightarrow 3\theta = \text{Tan}^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) \Rightarrow 3\text{Tan}^{-1}x\left(\frac{3x-x^3}{1-3x^2}\right)$$

Theorem 11: (i) $\text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1-x^2}$ for $0 \le x \le 1$ (ii) $\text{Sin}^{-1}x = -\text{Cos}^{-1}\sqrt{1-x^2}$ for $-1 \le x \le 0$

Proof: Let $\text{Sin}^{-1}x=\theta$, then $x=\sin\theta$

(i) Suppose $0\le x\le 1$. Now $0\le\sin\theta\le 1 \Rightarrow 0\le\theta\le\pi/2$

$$0 \le \theta \le \frac{\pi}{2} \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-x^2} \Rightarrow \theta = \text{Cos}^{-1}\sqrt{1-x^2}$$

$$\Rightarrow \text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1-x^2}$$

(ii) Suppose $-1\le x<0$. Now $-1 \le \sin\theta \Rightarrow -\frac{\pi}{2} \le \theta \le 0 \le \frac{\pi}{2} \ge -\theta > 0$

$$0 < -\theta \le \frac{\pi}{2} \le \cos(-\theta) = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2} \Rightarrow -\theta = \text{Cos}^{-1}\sqrt{1-x^2}$$

$$\Rightarrow \text{Sin}^{-1}x = -\text{Cos}^{-1}\sqrt{1-x^2}$$

Corollary: (i) $\text{Cos}^{-1}x = \text{Sin}^{-1}\sqrt{1-x^2}$ for $0\le x\le 1$ (ii) $\text{Cos}^{-1}x = \pi - \text{Sin}^{-1}\sqrt{1-x^2}$ for $-1\le x\le 0$

Theorem 12: (i) $\text{Sin}^{-1}x = \text{Tan}^{-1}\frac{x}{\sqrt{1-x^2}} = \text{Sec}^{-1}\frac{1}{\sqrt{1-x^2}}$ for $0\le x\le 1$

(ii) $\text{Cos}^{-1}x = \text{Tan}^{-1}\frac{\sqrt{1-x^2}}{x} = \text{Cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$ for $0<x\le 1$

(iii) $\text{Tan}^{-1}x = \text{Sin}^{-1}\frac{x}{\sqrt{1+x^2}} = \text{Cos}^{-1}\frac{1}{\sqrt{1+x^2}}$ for $x\ge 0$

Proof: (i) Let $\text{Sin}^{-1}x=\theta$, then $x=\sin\theta$

$0\le x\le 1 \Rightarrow 0\le\sin\theta\le 1 \Rightarrow 0\le\theta\le\pi/2$

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x}{\sqrt{1-x^2}}, \sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}$$

$$0 \le \theta \le \frac{\pi}{2}, \tan\theta = \frac{x}{\sqrt{1-x^2}}, \sec\theta = \frac{1}{\sqrt{1-x^2}} \Rightarrow \theta = \text{Tan}^{-1}\frac{x}{\sqrt{1-x^2}} = \text{Sec}^{-1}\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \text{Sin}^{-1}x = \text{Tan}^{-1}\frac{x}{\sqrt{1-x^2}} = \text{Sec}^{-1}\frac{1}{\sqrt{1-x^2}}$$

(ii) Let $\text{Cos}^{-1}x=\theta$, then $x=\cos\theta$

$0<x\le 1 \Rightarrow 0<\cos\theta\le 1 \Rightarrow 0\le\theta\le\pi/2$

$$\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}, \text{Cosec}\,\theta = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{1-x^2}}$$

$$0 \le \theta \le \frac{\pi}{2}, \tan\theta = \frac{\sqrt{1-x^2}}{x}, \text{cosec}\,\theta = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \theta = \text{Tan}^{-1}\frac{\sqrt{1-x^2}}{x} = \text{Cosec}^{-1}\frac{1}{\sqrt{1-x^2}} \Rightarrow \text{Cosec}^{-1}x = \text{Tan}^{-1}\frac{\sqrt{1-x^2}}{x} = \text{Cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$$

ADDITIONAL QUESTIONS ON INVERSE T' FUNCTIONS

1. If $\mathbf{Cos^{-1}\left(\frac{p}{a}\right)+Cos^{-1}\left(\frac{q}{a}\right)=\alpha}$, prove that $\mathbf{\frac{p^2}{a^2}-\frac{2pq}{a^2}\cos\alpha+\frac{q^2}{b^2}=\sin^2\alpha}$.

Sol: Let, $\cos^{-1}\left(\frac{q}{a}\right)=A,\ \cos^{-1}\left(\frac{q}{b}\right)=B$

Then, $\cos A=\left(\frac{p}{a}\right), \cos B=\left(\frac{q}{b}\right)$

$$\cos^{-1}\left(\frac{p}{a}\right)+\cos^{-1}\left(\frac{q}{b}\right)=\alpha \quad \Rightarrow A+B=\alpha \quad \Rightarrow \cos(A+B)=\cos\alpha$$

$\Rightarrow \cos A\,.\cos B - \sin A.\sin B=\cos\alpha$

$\Rightarrow \cos A.\cos B-\sqrt{1-\cos^2 A}.\sqrt{1-\cos^2 B}=\cos\alpha$

$$\Rightarrow \left(\frac{p}{a}\right).\left(\frac{q}{b}\right)-\sqrt{1-\left(\frac{p^2}{a^2}\right)}.\sqrt{1-\left(\frac{q^2}{b^2}\right)}=\cos\alpha$$

$$\Rightarrow \left(\frac{pq}{ab}\right)-\cos\alpha=\sqrt{1-\left(\frac{p^2}{a^2}\right)}.\sqrt{1-\left(\frac{q^2}{b^2}\right)}$$

$$\Rightarrow \left(\frac{pq}{ab}-\cos\alpha\right)^2=\left[1-\frac{p^2}{a^2}\right]\left[1-\frac{q^2}{b^2}\right]$$

$$\Rightarrow \frac{p^2q^2}{a^2b^2}+\cos^2\alpha-\frac{2pq}{ab}.\cos\alpha=1-\frac{q^2}{b^2}-\frac{p^2}{a^2}+\frac{p^2q^2}{a^2b^2}$$

$$\Rightarrow \frac{p^2}{a^2}-\frac{2pq}{ab}\cos\alpha+\frac{q^2}{b^2}=1-\cos^2\alpha \quad \Rightarrow \frac{p^2}{a^2}-\frac{2pq}{ab}\cos\alpha+\frac{q^2}{b^2}=\sin^2\alpha$$

2. Solve $Sin^{-1}x + Sin^{-1}2x = \frac{\pi}{3}$

Sol: Let $Sin^{-1}x = A \Rightarrow \sin A = x \Rightarrow \cos A = \sqrt{1-x^2}$

$Sin^{-1}2x = B \Rightarrow \sin B = 2x \Rightarrow \cos B = \sqrt{1-(2x)^2} = \sqrt{1-4x^2}$

Now, $A + B = \frac{\pi}{3} \Rightarrow \cos(A+B) = \cos\frac{\pi}{3} = \frac{1}{2} \Rightarrow \cos A \cos B - \sin A \sin B = \frac{1}{2}$

$\Rightarrow \sqrt{1-x^2}\sqrt{1-4x^2} - x(2x) = \frac{1}{2} \Rightarrow \sqrt{1-x^2}\sqrt{1-4x^2} = 2x^2 + \frac{1}{2}$

On squaring both sides, we get $(1-x^2)(1-4x^2) = \left(2x^2 + \frac{1}{2}\right)^2$

$\Rightarrow 1 - 4x^2 - x^2 + 4x^4 = (2x^2)^2 + 2(2x^2).\frac{1}{2} + \left(\frac{1}{2}\right)^2$

$\Rightarrow 1 - 5x^2 + 4x^4 = 4x^4 + 2x^2 + \frac{1}{4} \Rightarrow 7x^2 = \frac{3}{4} \Rightarrow x^2 = \frac{3}{28} \Rightarrow x = \pm\frac{\sqrt{3}}{2\sqrt{7}}$

But $x = -\frac{\sqrt{3}}{2\sqrt{7}}$ is not valid.

If so, $Sin^{-1}x$, $Sin^{-1}2x$ in L.H.S become negative, but RHS is positive.

$\therefore$ the only solution is $x = \frac{\sqrt{3}}{2\sqrt{7}}$

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3. Show that $\sec^2(Tan^{-1}2) + \operatorname{cosec}^2(Cot^{-1}2) = 10$

Sol : L.H.S = $[1+\tan^2(Tan^{-1}2)] + [1+\cot^2(Cot^{-1}2)] = 1+4+1+4 = 10$.

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4. Prove that $2Sin^{-1}\left(\frac{3}{5}\right) - Cos^{-1}\frac{5}{13} = Cos^{-1}\left(\frac{323}{325}\right)$

Sol : Let, $\sin^{-1}\left(\frac{3}{5}\right) = \theta \Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \cos\theta = \frac{4}{5}$ $\qquad \Rightarrow 2\sin^{-1}\frac{3}{5} = 2\theta$

Also, $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25} \Rightarrow 2\theta = \sin^{-1}\frac{24}{25}$

Hence, the given problem reduces to $\sin^{-1}\frac{24}{25} - \cos^{-1}\frac{5}{13} = \cos^{-1}\frac{323}{325}$

Let, $\sin^{-1}\frac{24}{25} = \alpha \Rightarrow \sin\alpha = \frac{24}{25} \Rightarrow \cos\alpha = \frac{7}{25}$

$\cos^{-1}\frac{5}{13} = \beta \Rightarrow \cos\beta = \frac{5}{13} \Rightarrow \sin\beta = \frac{12}{13}$

Now $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{7}{25}\times\frac{5}{13} + \frac{24}{25}\times\frac{12}{13} = \frac{35+288}{325} = \frac{323}{325}$

$\Rightarrow \alpha - \beta = Cos^{-1}\frac{323}{325} . \Rightarrow Sin^{-1}\frac{24}{25} - Cos^{-1}\frac{5}{13} = Cos^{-1}\frac{323}{325} \quad \Rightarrow 2Sin^{-1}\frac{3}{5} - Cos^{-1}\frac{5}{13} = Cos^{-1}\frac{323}{325}$.

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5. **If $\alpha = \text{Tan}^{-1}\left(\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ then prove that $x^2 = \sin 2\alpha$.**

A. Put, $x^2 = \sin 2\theta$

$$\therefore \alpha = \text{Tan}^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right) = \text{Tan}^{-1}\left(\frac{\sqrt{1+\sin 2\theta}-\sqrt{1-\sin 2\theta}}{\sqrt{1+\sin 2\theta}+\sqrt{1-\sin 2\theta}}\right)$$

$$= \text{Tan}^{-1}\left[\frac{\sqrt{(\cos\theta+\sin\theta)^2}-\sqrt{(\cos\theta-\sin\theta)^2}}{\sqrt{(\cos\theta+\sin\theta)^2}+\sqrt{(\cos\theta-\sin\theta)^2}}\right]$$

$$= \text{Tan}^{-1}\left[\frac{(\cos\theta+\sin\theta)-(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)+(\cos\theta-\sin\theta)}\right]$$

$$= \text{Tan}^{-1}\left[\frac{2\sin\theta}{2\cos\theta}\right] = \text{Tan}^{-1}(\tan\theta) = 0 \ \Rightarrow \sin 2\alpha = x^2$$

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6. **Solve $\text{Sin}^{-1}\dfrac{3x}{5} + \text{Sin}^{-1}\dfrac{4x}{5} = \text{Sin}^{-1}x$**

A. $\sin^{-1}\dfrac{3x}{5} + \sin^{-1}\dfrac{4x}{5} = \sin^{-1}x$

$$\Rightarrow x = \sin\left(\sin^{1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5}\right) = \frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}$$

$$x = 0 \ \text{ or } \ 25 = 3\sqrt{25-16x^2} + 4\sqrt{25-9x^2}$$

$$x \neq 0 \Rightarrow 4\sqrt{25-9x^2} = 25 - 3\sqrt{25-16x^2}$$

On squaring both sides, we get, $16(25-9x^2) = 625 - 150\sqrt{25-16x^2} + 9(25-16x^2)$

$$\Rightarrow 400 - 144x^2 = 625 - 150\sqrt{25-16x^2} + 225 - 144x^2$$

$$\Rightarrow 150\sqrt{25-16x^2} = 450$$

$$\Rightarrow \sqrt{25-16x^2} = 3 \Rightarrow 25 - 16x^2 = 9 \Rightarrow 16x^2 = 16 \Rightarrow x = \pm 1$$

Thusd $x = -1, 0, 1$ we can verify that all these values of x satisify the given equation.