WELCOME

STAR 'QR CODE'
DIGITAL CONTENT MATERIAL

DIFFERENTIAL EQUATIONS - INDEX

8.DIFFERENTIAL EQUATIONS

Sections	*No. of periods (12)*	*Weightage in IPE [2×4=8]*
1. Introductory Concepts	3	
2. Variables Separable	3	4 Marks
3. Homogeneous Differential Equations	2	4 Marks
4. Non-Homogeneous Differential Equations	2	4 Marks
5. Linear Differential Equations	2	4 Marks

Some physical situations are interpreted in a clear sense by means of rates of change of related quantities rather than the direct relations between the quantities themselves. That is why, differential equations arise naturally as models for several problems in Engineering, Physical sciences, Biology and Social studies.

Our discussions in this chapter 'differential equations' are mainly about the solutions of various kinds of differential equations.

Once again we remind that the symbol $\frac{dy}{dx}$ can be used in a sense of our need i.e, as the ratio of 2 differentials dy and dx as well as, a symbol for the derivative of y w.r.t x.

SYNOPSIS POINTS

1. The order of differential equation is the order of the highest order derivative in it.

2. The degree of a differential equation is the degree of the highest order derivative, when it is expressed as a polynomial equation in derivatives after eliminating the fractional powers, if any.

   Eg: The order of $6\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6x=0$ is 2 and its degree is 1

   The order of $\left(2-\left(\frac{d^2y}{dx^2}\right)^2\right)^3=\left(\frac{dy}{dx}\right)^2$ is 2 and its degree is 6

3. Forming a differential equation from a given relation: Differentiate the given relation as many times as the number of arbitrary constants in the given relation and eliminate the arbitrary constant (s) from the given relation by replacing the arbitrary constants in then original equation with the derivatives.

4. Solving a differential equation in the "Variables separable" form: Separate both the variables x and y along with dx and dy and integrate accordingly. Here, the constant of integration can be modified accordingly.

5. Procedure of solving homogeneous differential equation:

   1. Write the given differential equation in form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ ....(1)
   2. Take y=vx hence we get $\frac{dy}{dx}=v+x\frac{dv}{dx}$ .....(2)
   3. Find $x\frac{dv}{dx}$ by substituting the above in (1)
   4. Separate the variables in x and v, and resolve into partial fractions, if required.
   5. Integrate and simplify
   6. Replace v by y/x in the final step to get the solution interms of x and y

6. Non-homogeneous differential equations are of the form $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

   Here, various possible cases are according as (1) $b_1+a_2=0$ (2) $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ (3) $\frac{a_1}{a_2}=\frac{b_1}{b_2}$

7.1. The solution of the linear differential equation $\frac{dy}{dx}+P(x)y=Q(x)$ in y is $y(I.F)=\int Q(I.F)dx+c$

where I.F(Integrating factor) $= e^{\int Pdx}$. Thus the solution of $\frac{dy}{dx}+P(x)y=Q(x)$ is $y.e^{\int Pdx}=\int Q.e^{\int Pdx}dx+c$

7.2. The solution of the linear differential equation $\frac{dx}{dy}+P(y)x=Q(y)$ in x is $x.e^{\int Pdy}=\int Q.e^{\int Pdy}+c$

ADDITIONAL QUESTIONS WITH SOLUTIONS

1 **Solve** $\frac{dy}{dx}=\frac{6x+5y+7}{2x+18y-14}$

Sol: Given that $\frac{dy}{dx}=\frac{6x+5y+7}{2x+18y-14}$........(1)

Comparing $\frac{6x+5y+7}{2x+18y-14}$ with $\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ we get $\frac{a_1}{a_2}=\frac{6}{2}=3;\frac{b_1}{b_2}=\frac{5}{18}\Rightarrow\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

Put x = X+h, y = Y+k. Then $\frac{dy}{dx}=\frac{dY}{dX}$

$(1)\Rightarrow\frac{dY}{dX}=\frac{6(X+h)+5(Y+k)+7}{2(X+h)+18(Y+k)-14}=\frac{6X+5Y+(6h+5k+7)}{2X+18Y+(2h+18k-14)}$ ....(2)

Now we choose h, k such that 6h+5k+7=0, 2h+18k–14=0

Solving the above two equations we get h = –2, k =1

Hence (2) becomes $\frac{dY}{dX}=\frac{6X+5Y}{2X+18Y}$ ..(3). This is a homogeneous D.E in X & Y

Now, we take the substitution $Y=VX\Rightarrow\frac{dY}{dX}=V+X\frac{dv}{dx}$

$(3)\Rightarrow V+X\frac{dV}{dX}=\frac{6X+5VX}{2X+18VX}$

$\Rightarrow X\frac{dV}{dX}=\frac{6+5V}{2+18V}-V=\frac{6+5V-2V-18V^2}{2+18V}=\frac{6+3V-18V^2}{2+18V}=\frac{3(2+V-6V^2)}{18V+2}$

$\Rightarrow\frac{18V+2}{6V^2-V-2}dV=-3\frac{dX}{X}$

(On resolving $\frac{18V+2}{6V^2-V-2}$ into partial fractions, we get $\frac{18V+2}{(2V+1)(3V-2)}=\frac{2}{2V+1}+\frac{6}{3V-2}$)

$\Rightarrow\int\frac{18V+2}{(2V+1)(3V-2)}dV=\int\left[\frac{2}{2V+1}+\frac{6}{3V-2}\right]dV=-3\int\frac{dX}{X}$

$\Rightarrow 3\log X+2.\frac{1}{2}\log(2V+1)+6.\frac{1}{3}\log(3V-2)=c$

$\Rightarrow 3\log X+\log(2V+1)+2\log(3V-2)=\log c$

$\Rightarrow\log X^3+\log(2V+1)+\log(3V-2)^2=\log c\Rightarrow\log\left[X^3(2V+1)(3V-2)^2\right]=\log c$

$\Rightarrow X^3(2V+1)(3V-2)^2=c\Rightarrow X^3\left(2\frac{Y}{X}+1\right)\left(3\frac{Y}{X}-2\right)^2=c$

$\Rightarrow(2Y+X)(3Y-2X)^2=c\Rightarrow[2(y-1)+(x+2)][3(y-1)-2(x+2)]^2=c$

$\Rightarrow(x+2y)(3y-2x-7)^2=c$

2 **Solve (x–y)dy = (x+y+1)dx.**

Sol: Given that (x–y)dy = (x+y+1)dx $\Rightarrow \frac{dy}{dx} = \frac{x+y+1}{x-y}$........(1)

Comparing $\frac{x+y+1}{x-y}$ with $\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ we get $\frac{a_1}{a_2} = \frac{1}{1} = 1; \frac{b_1}{b_2} = \frac{1}{-1} = -1 \Rightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Put x = X+h, y = Y+k. Then $\frac{dy}{dx} = \frac{dY}{dX}$

$(1) \Rightarrow \frac{dY}{dX} = \frac{(X+h)+(Y+k)+1}{(X+h)-(Y+k)} = \frac{X+Y+(h+k+1)}{X-Y+(h-k)}$ ....(2)

Now we choose h, k such that h+k+1=0, h–k =0

Solving the above two equations, we get $h = -\frac{1}{2}, k = -\frac{1}{2}$

Hence (2) becomes $\frac{dY}{dX} = \frac{X+Y}{X-Y}$ ..(3). This is a homogeneous D.E in X & Y

Now, we take the substitution $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dv}{dx}$

$(3) \Rightarrow V + X\frac{dV}{dX} = \frac{X+VX}{X-VX} = \frac{1+V}{1-V}$

$\Rightarrow X\frac{dV}{dX} = \frac{1+V}{1-V} - V = \frac{1+V-V+V^2}{1-V} = \frac{1+V^2}{1-V} \Rightarrow \frac{1-V}{1+V^2}dV = \frac{dX}{X}$

$\Rightarrow \int\frac{dX}{X} = \int\frac{1-V}{1+V^2}dV = \int\frac{dV}{1+V^2} - \int\frac{V}{1+V^2}dV = \int\frac{dV}{1+V^2} - \frac{1}{2}\int\frac{2V}{1+V^2}dV$

$\Rightarrow \log|X| = Tan^{-1}V - \frac{1}{2}\log\left(1+V^2\right) + c \Rightarrow 2\log|X| + \log\left(1+V^2\right) + \log c = 2Tan^{-1}V$

$\Rightarrow \log|cX^2(1+V^2)| = 2Tan^{-1}V \Rightarrow \log cX^2\left(\frac{X^2+Y^2}{X^2}\right) = 2Tan^{-1}\left(\frac{Y}{X}\right)$

$\Rightarrow \log c\left[\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2\right] = 2Tan^{-1}\left(\frac{y+\frac{1}{2}}{x+\frac{1}{2}}\right)$

$\Rightarrow \log c\left[\frac{(2x+1)^2}{4} + \frac{(2y+1)^2}{4}\right] = 2Tan^{-1}\left(\frac{2y+1}{2x+1}\right)$

PQ **Solve (x–y–2)dx+(x–2y–3)dy=0** [Ans: $(x^2-2y^2-2x-4y-2) = c\left[\frac{x-y\sqrt{2}-\sqrt{2}-1}{x+y\sqrt{2}+\sqrt{2}-1}\right]^{\frac{1}{\sqrt{2}}}$]

3 **Solve $\frac{dy}{dx}=\frac{2x+9y-20}{6x+2y-10}$** EAM Q

Sol: Given that $\frac{dy}{dx}=\frac{2x+9y-20}{6x+2y-10}$ ....(1)

Comparing $\frac{2x+9y-20}{6x+2y-10}$ with $\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, we get $\frac{a_1}{a_2}=\frac{2}{6}=\frac{1}{3};\frac{b_1}{b_2}=\frac{9}{2}\Rightarrow\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

Put x = X+h, y = Y+k. Then $\frac{dy}{dx}=\frac{dY}{dX}$

$$(1)\Rightarrow\frac{dY}{dX}=\frac{2(X+h)+9(Y+k)-20}{6(X+h)+2(Y+k)-10}=\frac{2X+9Y+(2h+9k-20)}{6X+2Y+(6h+2k-10)}\text{ ....(2)}$$

Now we choose h, k such that 2h+9k–20=0, 6h+2k–10=0

Solving the above two equations we get $h=1,k=2$

Hence (2) becomes $\frac{dY}{dX}=\frac{2X+9Y}{6X+2Y}$ ..(3). This is a homogeneous D.E in X & Y

Now, we take the substitution $Y=VX\Rightarrow\frac{dY}{dX}=V+X\frac{dv}{dx}$

$$(3)\Rightarrow V+X\frac{dV}{dX}=\frac{2X+9VX}{6X+2VX}=\frac{2+9V}{6+2V}$$

$$\Rightarrow X\frac{dV}{dX}=\frac{2+9V}{6+2V}-V=\frac{2+9V-6V-2V^2}{6+2V}=\frac{2+3V-2V^2}{6+2V}\Rightarrow\frac{6+2V}{2+3V-2V^2}dV=\frac{dX}{X}$$

(On resolving $\frac{6+2V}{2+3V-2V^2}$ into partial fractions, we get $\frac{6+2V}{(2-V)(1+2V)}=\frac{2}{2-V}+\frac{2}{1+2V}$)

$$\Rightarrow\int\frac{dX}{X}=\int\left[\frac{2}{2-V}+\frac{2}{1+2V}\right]dV\Rightarrow\log X=-2\log(2-V)+\log(1+2V)+\log c$$

$$\Rightarrow\log X+\log(2-V)^2-\log(1+2V)=\log c\Rightarrow\log X\left[\frac{(2-V)^2}{(1+2V)}\right]=\log c$$

$$\Rightarrow X\left[\frac{(2-V)^2}{(1+2V)}\right]=c\Rightarrow X(2-V)^2=c(1+2V)\Rightarrow X\left(2-\frac{Y}{X}\right)^2=c\left(1+2\frac{Y}{X}\right)$$

$$\Rightarrow X\left(\frac{2X-Y}{X}\right)^2=c\left(\frac{X+2Y}{X}\right)\Rightarrow(2X-Y)^2=c(X+2Y)$$

$$\Rightarrow c[2(x-1)-(y-2)]^2=(x-1)+2(y-2)\ \Rightarrow x+2y-5=c(2x-y)^2$$

4 **Solve** $\frac{dy}{dx} - y = -2e^{-x}$.

Sol: Given D.E is $\frac{dy}{dx} - y = -2e^{-x} \Rightarrow \frac{dy}{dx} + y(-1) = -2e^{-x}$. This is a linear D.E in y.

It is in the form $\frac{dy}{dx} + yP(x) = Q(x)$ where $P(x) = -1$ and $Q(x) = -2e^{-x}$

Now, $I.F = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$

Hence, the solution is $y(I.F) = \int (I.F)Qdx$

$\Rightarrow y(e^{-x}) = \int e^{-x}(-2e^{-x})dx = -2\int e^{-2x}dx = -\cancel{2}\frac{e^{-2x}}{-\cancel{2}} + c$

$\Rightarrow y(e^{-x}) = e^{-2x} + c \Rightarrow y = \frac{e^{-2x}}{e^{-x}} + \frac{c}{e^{-x}}$ $\therefore$ The solution is $y = e^{-x} + c.e^{x}$

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5 **Solve sec x dy = (y + sin x) dx.**

Sol: Given D.E is sec x dy = (y + sin x) dx.

$\frac{dy}{dx} = \frac{y+\sin x}{\sec x} = \frac{y}{\sec x} + \frac{\sin x}{\sec x} = y\cos x + \sin x\cos x \Rightarrow \frac{dy}{dx} + y(-\cos x) = \sin x\cos x$

It is in the form $\frac{dy}{dx} + yP(x) = Q(x)$ where $P(x) = -\cos x$ and $Q(x) = \sin x\cos x$

Now, $I.F = e^{\int P(x)dx} = e^{\int -\cos x dx} = e^{-\int \cos x dx} = e^{-\sin x}$

Hence, the solution is $y(I.F) = \int (I.F)Qdx \Rightarrow ye^{-\sin x} = \int e^{-\sin x}(\sin x\cos x)\,dx$

Put $-\sin x = t \Rightarrow -\cos x dx = dt \Rightarrow \cos x dx = -dt$

$\therefore y\left(e^{-\sin x}\right) = \int e^{t}(-t)(-dt) = \int te^{t}dt = e^{t}(t-1) + c$

$\Rightarrow y.\, e^{-\sin x} = e^{-\sin x}(-\sin x - 1) + c \Rightarrow y = -\frac{\cancel{e^{-\sin x}}}{\cancel{e^{-\sin x}}}(\sin x + 1) + \frac{c}{e^{-\sin x}}$

$\Rightarrow y = -(\sin x + 1) + c.\, e^{\sin x}$.

6 **Solve** $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{1-x^2}$

Sol: Given D.E is $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{1-x^2}$

$$\Rightarrow \frac{dy}{dx}+\left(\frac{2x}{1-x^2}\right)y=\frac{x\sqrt{1-x^2}}{1-x^2}\Rightarrow \frac{dy}{dx}+y\left(\frac{2x}{1-x^2}\right)=\frac{x}{\sqrt{1-x^2}}$$

It is in the form $\frac{dy}{dx}+yP(x)=Q(x)$ where $P(x)=\frac{2x}{1-x^2}$ and $Q(x)=\frac{x}{\sqrt{1-x^2}}$

Here $P(x)=\frac{2x}{1-x^2}\Rightarrow \int P(x)dx=\int\frac{2x}{1-x^2}dx=-\int\frac{-2x}{1-x^2}dx=-\log|1-x^2|=\log(1-x^2)^{-1}=\log\frac{1}{1-x^2}$

Now, $I.F=e^{\int P(x)dx}=e^{\log\frac{1}{1-x^2}}=\frac{1}{1-x^2}$ $\qquad \therefore$ The solution is $y(I.F)=\int(I.F)Qdx$

$$\Rightarrow y\left(\frac{1}{1-x^2}\right)=\int\left(\frac{1}{1-x^2}\right)\frac{x}{\sqrt{1-x^2}}dx=\int\frac{xdx}{(1-x^2)^{3/2}}=\frac{-1}{2}\int\frac{-2xdx}{(1-x^2)^{3/2}}=-\frac{1}{2}\frac{(-2)}{\sqrt{1-x^2}}+c \qquad \left[\because \int\frac{dx}{x^{3/2}}=\frac{-2}{\sqrt{x}}\right]$$

$$\therefore \frac{y}{1-x^2}=\frac{1}{\sqrt{1-x^2}}+c\Rightarrow y=\frac{1-x^2}{\sqrt{1-x^2}}+c(1-x^2)\Rightarrow y=\sqrt{1-x^2}+c(1-x^2).$$

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7 **Solve** $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

Sol: Given D.E is $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

$$\Rightarrow \frac{dy}{dx}-y\left[\frac{x-2}{x(x-1)}\right]=\frac{x^3(2x-1)}{x(x-1)}\Rightarrow \frac{dy}{dx}+y\left[\frac{(2-x)}{x(x-1)}\right]=\frac{x^2(2x-1)}{(x-1)}$$

It is in the form $\frac{dy}{dx}+yP(x)=Q(x)$ where $P(x)=\frac{2-x}{x(x-1)}$ and $Q(x)=\frac{x^2(2x-1)}{(x-1)}$

Here $P(x)=\frac{(2-x)}{x(x-1)}\Rightarrow \int P(x)dx=\int\frac{2-x}{x(x-1)}dx=\int\left(\frac{-2}{x}+\frac{1}{x-1}\right)dx$ $\qquad$ On resolving into P.F

$$\Rightarrow -2\int\frac{dx}{x}+\int\frac{dx}{x-1}=-2\log x+\log(x-1)=\log\frac{(x-1)}{x^2}$$

Now, $I.F.=e^{\int P(x)dx}=e^{\log\frac{(x-1)}{x^2}}=\frac{x-1}{x^2}$ $\qquad \therefore$ The solution is $y(I.F)=\int(I.F)Q(x)dx$

$$\Rightarrow y\left(\frac{x-1}{x^2}\right)=\int\left(\frac{x-1}{x^2}\right)\frac{x^2(2x-1)}{(x-1)}dx=\int(2x-1)dx=x^2-x+c$$

$$\therefore \frac{y(x-1)}{x^2}=x^2-x+c\Rightarrow y(x-1)=x^2(x^2-x+c)$$

The Illustrative table of DIFFERENTIAL EQUATIONS

S.No.	Name of the D.E / form Solving procedure	Illustrations
1.	**1.1. Variables separable** $f(x,y)dx+g(x,y)dy=0$ (or) $\frac{dy}{dx}=f(x,y)$ (or) $f(x)dx=g(y)dy$ After the variables have been separated, integrate and add the constant of integration 'c' accordingly.	**1.** $\mathbf{ydx-xdy=0} \Rightarrow xdy=ydx \Rightarrow \frac{dy}{y}=\frac{dx}{x} \Rightarrow \int\frac{dy}{y}=\int\frac{dx}{x} \Rightarrow \log y=\log x+\log c \Rightarrow \log y=\log xc \Rightarrow \mathbf{y=cx}$ **2.** $\mathbf{(xy^2+x)dx+(yx^2+y)dy=0} \Rightarrow (x^2+1)ydy=-(y^2+1)xdx \Rightarrow \int\frac{ydy}{y^2+1}dy=-\int\frac{xdx}{x^2+1} \Rightarrow \mathbf{(x^2+1)(y^2+1)=c}$ **3.** $\mathbf{\tan y\sec^2 xdx+\tan x\sec^2 ydy=0} \Rightarrow \tan x\sec^2 ydy=-\tan y\sec^2 xdx \Rightarrow \int\frac{\sec^2 y}{\tan y}dy=-\int\frac{\sec^2 x}{\tan x}dx \Rightarrow \mathbf{(\tan x)(\tan y)=c}$ **4.** $\mathbf{\frac{dy}{dx}=xy+x+y+1} \Rightarrow \frac{dy}{dx}=(x+1)(y+1) \Rightarrow \frac{dy}{y+1}=(x+1)dx \Rightarrow \int\frac{dy}{y+1}=\int(x+1)dx \Rightarrow \mathbf{\log(y+1)=\frac{x^2}{2}+x+c}$ **5.** $\mathbf{(1-x^2)\frac{dy}{dx}+xy=5x} \Rightarrow (1-x^2)\frac{dy}{dx}=x(5-y) \Rightarrow \frac{dy}{5-y}=\frac{x}{1-x^2}dx \Rightarrow \int\frac{dy}{y-5}=\frac{1}{2}\int\frac{2x}{x^2-1}dx \Rightarrow \mathbf{y-5=c\sqrt{x^2-1}}$
2.	**1.2. D. E reducible to variables separable.** $\frac{dy}{dx}=f(ax+by)$ ......(1) 1. put x+by=z, differentiate w.r.t x and find $\frac{dy}{dx}$ 2. Put z & replace $\frac{dy}{dx}$ in (1) 3. Then (1) reduces to variables separable in x and z. 4. solve & replace z by ax+by to get the solution in terms of x,y.	**1.** $\mathbf{\frac{dy}{dx}=(x+y)^2}$....(1) put $x+y=z \Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1 \therefore (1) \Rightarrow \frac{dz}{dx}-1=z^2 \Rightarrow \frac{dz}{dx}=1+z^2$ $\Rightarrow \frac{dz}{1+z^2}=dx \Rightarrow \int\frac{dz}{1+z^2}=\int dx \Rightarrow \tan^{-1}z=x+c \Rightarrow \mathbf{\tan^{-1}(x+y)=x+c}$ **2.** $\mathbf{\frac{dy}{dx}=(4x+9y+1)^2}$....(1) put $4x+9y+1=z \Rightarrow 4+9\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{9}\left(\frac{dz}{dx}-4\right) \therefore (1) \Rightarrow \frac{1}{9}\left(\frac{dz}{dx}-4\right)=z^2$ $\Rightarrow \frac{dz}{dx}-4=9z^2 \Rightarrow \frac{dz}{4+9z^2}=dx \Rightarrow \int\frac{dz}{2^2+(3z)^2}=\int dx \Rightarrow \frac{1}{3}\cdot\frac{1}{2}.\mathrm{Tan}^{-1}\frac{3z}{2}=x+c \Rightarrow \mathbf{\frac{1}{6}Tan^{-1}\frac{3(4x+9y+1)}{2}=x+c}$ **3.** $\mathbf{(2x-4y+5)\frac{dy}{dx}+(x-2y+3)=0}$.....(1) put $x-2y=z \Rightarrow 1-2\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(1-\frac{dz}{dx}\right)$ $\therefore (1) \Rightarrow \frac{dy}{dx}=\frac{-(x-2y+3)}{2(x-2y)+5}$ $\Rightarrow \frac{1}{2}\left(1-\frac{dz}{dx}\right)=-\frac{(z+3)}{2z+5} \Rightarrow \frac{dz}{dx}=\frac{4z+11}{2z+5} \Rightarrow \frac{2z+5}{4z+11}dz=dx \Rightarrow \int\frac{4z+10}{4z+11}dz=\int 2dx \Rightarrow \int\left(1-\frac{1}{4z+11}\right)dz=\int 2dx \Rightarrow \mathbf{4x+8y+\log(4x-8y+11)+c=0}$ **4.** $\mathbf{\frac{dy}{dx}=\sin(x+y)+\cos(x+y)} \Rightarrow \frac{dz}{dx}=1+\sin z+\cos z \Rightarrow \int\frac{dz}{1+\sin z+\cos z}=\int dx \Rightarrow \mathbf{\log\left(1+\tan\frac{x+y}{2}\right)=x+c}$ **5.** $\mathbf{\frac{dy}{dx}-x\tan(y-x)=1}, y-x=z \Rightarrow \frac{dz}{dx}=x\tan z \Rightarrow \int\cot zdz=\int xdx \Rightarrow \log\sin z=\frac{x^2}{2}+\log c \Rightarrow \mathbf{\sin(y-x)=ce^{\frac{x^2}{2}}}$

3.	**2.1. Homogeneous D.E of first order and first degree.** $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$....(1) where f, g are H.F. of same degree 1. put $y=vx$ & $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (1) 2. find $x\frac{dv}{dx}$ 3. separate the variables x and v. 4. solve and replace v by $\frac{y}{x}$.	**1.** $2xy\frac{dy}{dx}=x^2+y^2 \Rightarrow \frac{dy}{dx}=\frac{x^2+y^2}{2xy}$, this is a homogenious D.E $\therefore$ put $y=vx \to \frac{dy}{dx}=v+x\frac{dv}{dx}$ $\therefore (1) \Rightarrow v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{2xvx}=\frac{1+v^2}{2v}$ $\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{2v}-v=\frac{1-v^2}{2v} \Rightarrow \frac{2v}{1-v^2}dv=\frac{dx}{x} \Rightarrow \int\frac{2v}{v^2-1}dv=-\int\frac{dx}{x} \Rightarrow \log(v^2-1)=-\log x+\log c \Rightarrow (v^2-1)=\frac{c}{x} \Rightarrow \mathbf{x^2-y^2=cx.}$ **2.** $y^2dx+(x^2-xy+y^2)dy=0 \Rightarrow \frac{dy}{dx}=\frac{-y^2}{x^2-xy+y^2}$, put $y=vx \Rightarrow \int\frac{v^2-v+1}{(1+v^2)v}dv=-\int\frac{dx}{x} \Rightarrow \mathbf{y=ce^{\tan^{-1}\frac{y}{x}}}$ **3.** $xdy-ydx-\sqrt{x^2+y^2}dx=0 \Rightarrow \frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}$, put $y=vx \Rightarrow \int\frac{dv}{\sqrt{1+v^2}}=\int\frac{dx}{x} \Rightarrow \log(v+\sqrt{1+v^2})=\log x+\log c \Rightarrow \mathbf{y+\sqrt{x^2+y^2}=cx^2}$ **4.** $(ydx+xdy)x\cos\frac{y}{x}=(xdy-ydx)y\sin\frac{y}{x} \Rightarrow \frac{dy}{dx}=\frac{xy\cos\frac{y}{x}+y^2\sin\frac{y}{x}}{xy\sin\frac{y}{x}-x^2\cos\frac{y}{x}}$, put $y=vx \Rightarrow \frac{v\sin v-\cos v}{v\cos v}dv=2\frac{dx}{x}$ $\Rightarrow \int\left(\tan v-\frac{1}{v}\right)dv=2\int\frac{dx}{x} \Rightarrow \log\sec v-\log v=2\log x+\log c \Rightarrow \sec v=vx^2c \Rightarrow \mathbf{\sec\frac{y}{x}=xyc}$
4.	**2.2. Non-Homogeneous D.E** $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ **Case(i):** $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}, b_1+a_2=0$, Regroup the terms to get d(xy) and Integrate **Case(ii):** $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, Put $a_1x+b_1y=z$ as in 1.2 **Case(iii):** $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}, b_1+a_2\neq0$ put $x=X+h$ and $y=Y+k$ $\Rightarrow \frac{dY}{dX}=\frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$ find h,k such that $a_1h+b_1k+c_1=0$, $a_2h+b_2k+c_2=0$ Solve $\frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$ by taking $Y=vX$ Replace v by $\frac{Y}{X}$ and X,Y by x−h, y−k	**1.** $\frac{dy}{dx}=\frac{2x-y+1}{x+2y-3}, \frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ & $b_1+a_2=0$; cross multiplyig the terms of the given D.E, $xdy+2ydy-3dy=2xdx-ydx+dx$ Now, regrouping the terms in the above equation we get $(xdy+ydx)+2ydy-3dy-2xdx-1dx=0 \Rightarrow$ $d(xy)+(2y-3)dy-(2x+1)dx=0 \Rightarrow \int d(xy)+\int(2y-3)dy-\int(2x+1)dx=\int 0 \Rightarrow \mathbf{xy+y^2-3y-x^2-x=c}$ — — — — — **1.** $(2x+2y+3)\frac{dy}{dx}=x+y+1 \Rightarrow \frac{dy}{dx}=\frac{x+y+1}{2x+2y+3}; \frac{a_1}{a_2}=\frac{b_1}{b_2}$ $\therefore$ put $x+y=z$ then $\frac{dz}{dx}=\frac{3z+4}{2z+3} \Rightarrow \int\frac{2z+3}{3z+4}dz=\int dx$ $\Rightarrow \int\left(\frac{2}{3}+\frac{1}{3(3z+4)}\right)dz=\int dx \Rightarrow \frac{2}{3}z+\frac{1}{3}.\frac{1}{3}\log(3z+4)=x+\frac{c}{9} \Rightarrow \mathbf{6(x+y)+\log(3x+xy+4)=9x+c}$ — — — — — **1.** $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$ here $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}, b_1+a_2\neq0, x=X+h; y=Y+k \Rightarrow \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$ and $h+2k-3=0, 2h+k-3=0 \Rightarrow (h,k)=(1,1)$ put $Y=vX \Rightarrow X\frac{dv}{dX}=\frac{1-v^2}{2+v} \Rightarrow \frac{2+v}{1-v^2}dv=\frac{dX}{X} \Rightarrow \int\left(\frac{2}{1-v^2}+\frac{v}{1-v^2}\right)dv=\int\frac{dX}{X} \Rightarrow X+Y=c(X-Y)^3 \Rightarrow \mathbf{x+y-2=c(x-y)^3}$ **2.** $\frac{dy}{dx}=\frac{x+y+1}{x-y} \Rightarrow \frac{dY}{dX}=\frac{X+Y}{X-Y}, (h,k)=\left(-\frac{1}{2},-\frac{1}{2}\right), Y=vX \Rightarrow \frac{1-v}{1+v^2}dv=\frac{dX}{X} \Rightarrow 2\tan^{-1}v=\log(X^2+Y^2)c$ $\Rightarrow \mathbf{2\tan^{-1}\left(\frac{2y+1}{2x+1}\right)=\log(c(x^2+y^2+x+y+\frac{1}{2}))}$

5.	**Linear Differential Equations:** (1) $\frac{dy}{dx}+P(x)y=Q(x)$ I.F = $e^{\int P dx}$ ***Sol*** : $y(I.F)=\int Q(I.F)dx+c$	**1.** $\frac{dy}{dx}+\frac{y}{x}=x^2, P=\frac{1}{x}, Q=x^2; I.F=e^{\int pdx}=e^{\int(1/x)dx}=e^{\log x}=x \therefore \text{sol.is } y(x)=\int x^2(x)dx+c \Rightarrow xy=\frac{x^4}{4}+c$ **2.** $(1+x^2)\frac{dy}{dx}+2xy=\cos x \Rightarrow \frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{\cos x}{1+x^2} \Rightarrow P=\frac{2x}{1+x^2}, Q=\frac{\cos x}{1+x^2}; I.F=e^{\int\frac{2x}{1+x^2}}=e^{\log(1+x^2)}=1+x^2 \Rightarrow y(1+x^2)=\int\frac{\cos x}{1+x^2}(1+x^2)dx=\sin x+c$ **3.** $\cos^2 x\frac{dy}{dx}+y=\tan x \Rightarrow \frac{dy}{dx}+\sec^2 xy=\tan x\sec^2 x, I.F=e^{\int\sec^2 xdx}=e^{\tan x} \therefore \text{sol.is } y(e^{\tan x})=\int e^{\tan x}\tan x\sec^2 xdx,$ put $\tan x=t \Rightarrow \sec^2 xdx=dt \Rightarrow$ then $\int e^t tdt=e^t(t-1)=e^{\tan x}(\tan x-1) \therefore ye^{\tan x}=e^{\tan x}(\tan x-1)+c$
	(2) $\frac{dx}{dy}+P(y)x=Q(y)$ $I.F=e^{\int P dy}$ ***Sol*** : $x(I.F)=\int Q(I.F)dy+c$	**1.** $\frac{dy}{dx}-\frac{1}{y}x=2y^2 \text{ or } (x+2y^3)\frac{dy}{dx}=y \Rightarrow y\frac{dx}{dy}=x+2y^3 \Rightarrow y\frac{dx}{dy}-x=2y^3 \Rightarrow \frac{dx}{dy}-\frac{x}{y}=2y^2$ which is in the form $\frac{dx}{dy}+Px=Q$ $\therefore I.F=e^{\int Pdx}=e^{\int-\frac{1}{y}dx}=e^{-\log y}=e^{\log y^{-1}}=\frac{1}{y} \therefore$ solution is $x\left(\frac{1}{y}\right)=\int 2y^2\left(\frac{1}{y}\right)dy \Rightarrow \frac{x}{y}=y^2+c.$ **2.** $(1+y^2)dx=(\tan^{-1}y-x)dy \Rightarrow (1+y^2)\frac{dx}{dy}=\tan^{-1}y-x \Rightarrow \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$ $I.F=e^{\int(1/1+y^2)dy}=e^{\tan^{-1}y}$ $\therefore xe^{\mathrm{Tan}^{-1}y}=\int\frac{\mathrm{Tan}^{-1}y}{1+y^2}e^{\mathrm{Tan}^{-1}y}dy$, put $\mathrm{Tan}^{-1}y=t \Rightarrow \frac{1}{1+y^2}dy=dt \Rightarrow \int e^t tdt=e^t(t-1)$ $\Rightarrow xe^{\mathrm{Tan}^{-1}y}=e^{\mathrm{Tan}^{-1}y}(\mathrm{Tan}^{-1}y-1)+c.$