# WELCOME

## STAR 'QR CODE'

## DIGITAL MATERIAL

## TRIGONOMETRIC EQUATIONS -INDEX

# 7.TRIGONOMETRIC EQUATIONS

| Sections | No. of periods (6) | Weightage in IPE [1x4=4] |
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## 1. INTRODUCTION PAGE

*Any equation consisting of trigonometric function of a variable angle is called a trigonometric equation. Unlike the algebraic equations, the trigonometric equations (when no restrictions are imposed on the variable angle) consist of infinite solutions. This peculiarity is obtained due to the periodic property of trigonometric functions. Eg:* $\sin\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o, 540^o, 720^o$..... *as well as* $\theta = -180^o, -360^o, -540^o$..... *Here, each one is an integral multiple of* $180^o$. *Hence, all of these solutions can be expressed simply, as* $\theta = n(180^o), \forall n \in Z$ *or* $\theta = \{n\pi, n \in Z\}$ *and this type of solution is known as general solution. Thus, the set of all values of the variable angle satisfying the given trigonometric equation is called the general solution (G.S) of the trigonometric equation.*

*The numerically least angle, say* $\alpha$ *of the given variable angle* $\theta$ *satisfying the given equation, generally in the interval* $[0, 2\pi]$ *is called the Principal Solution (P.S) of* $\theta$. *These principal solutions of various trigonometric functions are restricted to suitable invervals.*

*The principal solutions of the angles of the 6 trigonometric functions are as follows:*

| Trigonometric function | Principal Solution | Remarks |
|---|---|---|
| $\sin\theta$ | $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | $\sin\theta > 0 \Rightarrow \alpha \in (0, \pi/2]$<br>$\sin\theta < 0 \Rightarrow \alpha \in [-\pi/2, 0)$ |
| $\cos\theta$ | $\alpha \in [0, \pi]$ | $\cos\theta > 0 \Rightarrow \alpha \in [0, \pi/2)$<br>$\cos\theta < 0 \Rightarrow \alpha \in (\pi/2, \pi]$ |
| $\tan\theta$ | $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ | $\tan\theta > 0 \Rightarrow \alpha \in (0, \pi/2)$<br>$\tan\theta < 0 \Rightarrow \alpha \in (-\pi/2, 0)$ |
| $\csc\theta$ | $\alpha \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ | |
| $\sec\theta$ | $\alpha \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ | |
| $\cot\theta$ | $\alpha \in (0, \pi)$ | |

# 2. PROOFS OF T'C EQUATIONS OF SOME BASIC CONCEPTS

# 7.TRIGONOMETRIC EQUATIONS

**Def :** The set of all values of $\theta$ which satisfy a trigonometric equation $f(\theta)=0$, is called general solution or solution set of $f(\theta)=0$.

**Def 1:** If $\sin\theta=k$, $k\in[-1,1]$ then the unique value $\alpha$ of $\theta$ in $[-\pi/2, \pi/2]$ is called the Principal value(PV) of $\theta$ or Principal solution (P.S) of $\sin\theta=k$.

**Eg 1:** Principal solution of $\sin\theta=1/2$ is $\pi/6$

**Eg 2:** Principal solution of $\sin\theta=-1/\sqrt{2}$ is $-\pi/4$

**Def 2:** If $\cos\theta=k$, $k\in[-1,1]$ then the unique value $\alpha$ of $\theta$ in $[0,\pi]$, is called the Principal solution of $\theta$

**Eg 1:** Principal solution of $\cos\theta=1/\sqrt{2}$ is $\pi/4$

**Eg 2:** Principal solution of $\cos\theta=-1/2$ is $2\pi/3$

**Def 3:** If $\tan\theta=k$, $k\in R$ then the unique value $\alpha$ of $\theta$ in $(-\pi/2, \pi/2)$ is called principal solution of $\theta$.

**Eg 1:** Principal solution of $\tan\theta=\sqrt{3}$ is $\pi/3$

**Eg 2:** Principal solution of $\tan\theta=-1$ is $-\pi/4$

**Def 4:** If $\text{cosec}\theta=k$, $k\in(-\infty,-1]\cup[1,\infty)$ then the unique value $\alpha$ of $\theta$ in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$, is called Principal solution of $\theta$.

**Eg:** The principal solution of $\csc\theta=2$ is $\pi/6$

**Def 5:** If $\sec\theta=k$, $k\in(-\infty,-1]\cup[1,\infty)$ then the unique value $\alpha$ of $\theta$ in $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ is called the Principal solution of $\theta$.

**Eg:** The principal solution of $\sec\theta=2$ is $\pi/4$

**Def 6:** If $\cot\theta=k$, $k\in R$ then the unique value of $\theta$ in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ is called the Principal solution of $\theta$.

**Eg :** The principal solution of $\cot\theta = 1/\sqrt{3}$ is $\pi/3$.

**1. Theorem: The general solution of the equation $\sin\theta = 0$ is $\theta = n\pi$, $n \in Z$**

**Proof:** For $\theta\in[0,2\pi)$, $\sin\theta=0 \Leftrightarrow \theta\in\{0,\pi\}$

Further for any $\theta\in R$ and $k=\left[\frac{\theta}{2\pi}\right]$, $(\theta-2k\pi)\in[0,2\pi)$ and $\sin\theta=\sin(\theta-2k\pi)$ so that

$\sin\theta=0\Leftrightarrow(\theta-2k\pi)\in\{0,\pi\}\Leftrightarrow \theta=2k\pi$ or $(2k+1)\pi$.

Hence $\sin\theta=0 \Leftrightarrow \theta=n\pi$ for some $n\in Z$.

Hence the solution set for $\sin\theta=0$ is $\{n\pi: n\in Z\}$.

Usually we express it as $\theta=n\pi$, $n\in Z$ is general solution of the equation $\sin\theta=0$.

**Note 1**: $\cos\theta = 0 \Leftrightarrow \sin\left(\theta - \frac{\pi}{2}\right) = 0 \Leftrightarrow \left(\theta - \frac{\pi}{2}\right) \in \{n\pi : n \in Z\}$

$$\Leftrightarrow \theta \in \{(2n+1)\frac{\pi}{2} : n \in Z\}$$

**Note 2**: $\tan\theta = 0 \Leftrightarrow \sin\theta = 0 \Leftrightarrow \theta \in \{n\pi : n \in Z\}$

**2. Theorem: The general solution of the equation sinθ=k, |k|≤1, is $\theta = n\pi + (-1)^n\alpha$, n∈Z, α is the principal solution**:

**Proof**: For |k|≤1, ∃ unique $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that sinα=k i.e., α is the principal solution of sinθ=k

then sinθ=sinα ⇔ sinθ−sinα=0

$$\Leftrightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0 \Leftrightarrow \cos\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

If $\cos\left(\frac{\theta+\alpha}{2}\right) = 0$ then $\frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}$ for some n∈Z

$$\frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2} \Leftrightarrow \theta + \alpha = (2n+1)\pi$$

$$\Leftrightarrow \theta = (2n+1)\pi - \alpha = (2n+1)\pi + (-1)^{2n+1}\alpha \quad .....(i)$$

If $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$ then $\frac{\theta-\alpha}{2} = n\pi$, for some n∈Z

$$\Leftrightarrow \theta - \alpha = 2n\pi \Leftrightarrow \theta = 2n\pi + \alpha = 2n\pi + (-1)^{2n}\alpha \quad .....(ii)$$

Hence from (i) & (ii) general solution of the equation sinθ=k is $\theta = n\pi + (-1)^n\alpha$, n∈Z (or) the solution set $\theta = \{n\pi + (-1)^n\alpha \mid n \in Z\}$, where α is principal solution of sinθ=k.

**Note**: For k∈(−∞,−1]∪[1,∞), general solution of the equation cosecθ=k is $\theta = n\pi + (-1)^n\alpha$ , n∈Z where α is the Principal solution of the equation $\sin\theta = \frac{1}{k}$

**3.Theorem: The general solution of the equation cosθ=k, |k|≤1 is θ=2nπ±α, n∈Z, α is the principal solution**:

**Proof**: For |k|≤1, ∃ unique α∈[0,π] such that cosα=k i.e., α is the principal solution of cosθ=k then cosθ=cosα ⇔ cosθ−cosα=0

$$\Leftrightarrow -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0 \Leftrightarrow \sin\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\sin\left(\frac{\theta+\alpha}{2}\right) = 0 \Leftrightarrow \frac{\theta+\alpha}{2} = \in \{n\pi : n \in Z\} \Leftrightarrow \theta \in \{2n\pi - \alpha : n \in Z\}$$

$$\sin\left(\frac{\theta-\alpha}{2}\right) = 0 \Leftrightarrow \frac{\theta-\alpha}{2} \in \{n\pi : n \in Z\} \Leftrightarrow \theta \in \{2n\pi + \alpha : n \in Z\}$$

Hence general solution of the equation cosθ=k is 2nπ±α, n∈Z (or)
the solution set θ={2nπ±α | n∈Z}, where α is the principal solutionof cosθ=k

**Note**: For k∈(−∞,−1)∪(1,∞), general solution of the equation secθ=k is θ=2nπ+α, n∈Z where α is the principal solution of the equation cosθ=1/k.

**4. Theorem: The general solution of the equation tanθ=k, k∈R is θ=nπ+α, n∈Z, α is the principal solution:**

**Proof**: For k∈R, ∃ unique $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that tanα=k i.e., α is the principal solution of tanθ=k then tanθ=tanα

$$\Leftrightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha} \Leftrightarrow \sin\theta - \cos\alpha = \cos\theta\sin\alpha \Leftrightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\Leftrightarrow \sin(\theta - \alpha) = 0 \Leftrightarrow \theta - \alpha = n\pi \text{ or } \theta = n\pi + \alpha, \text{ for some } n \in Z.$$

For any $n \in Z$, we have $\tan(n\pi + \alpha) = \tan\alpha = k$

Hence general solution of the equation tanθ=k is θ=nπ+α, n∈Z (or)

the solution set θ={nπ+α | n∈Z}, where a is α solution of tanθ=k

**Note**: For k∈R, general solution of the equation cotθ=k is θ=nπ+α, n∈Z, where α is the principal solution of the equation tanθ=1/k.

**5. Theorem: The general solution of $\sin^2\theta=\sin^2\alpha$ is θ=nπ±α, nπ±α, n∈Z**

$$\sin^2\theta = \sin^2\alpha \Leftrightarrow \frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$$

⇔cos2θ=cos2α ⇔ 2θ=2nπ±2α or θ=nπ±α for some n∈Z

Hence general solution of the equation $\sin^2\theta=\sin^2\alpha$ is θ=nπ±α, n∈Z.

**Note**: Similarly, for $\cos^2\theta=\cos^2\alpha$ and $\tan^2\alpha=\tan^2\alpha$, the general solution is θ=nπ±α, n∈Z.

**Rem**: The general solution seems to differ for certain trigonometric equations, but after proper verification the angles coincide with the solution.

# ADDITIONAL QUESTIONS ON T' EQUATIONS

**1. Solve 4sinx.sin2x.sin4x=sin3x**

**Sol:** L.H.S = 4sinx.sin2x.sin4x = (2sinx)(2sin4xsin2x)

=(2sinx)(cos(4x–2x)–cos(4x+2x))=(2sinx)(cos2x–cos6x)

=2cos2x.sinx–2cos6x.sinx = sin(2x+x)–sin(2x–x)–2cos6xsinx.

=sin3x–sinx–2cos6x.sinx.

Now, the given equation is sin3x–sinx–2cos6xsinx=sin3x

$\Rightarrow$ sinx+2cos6xsinx=0 $\Rightarrow$ sinx(1+2cos6x)=0 $\Rightarrow$ sinx=0 (or) cos6x=$\frac{-1}{2}$

Now, sinx=0 $\Rightarrow$ x=nπ, n∈Z

Also, $\cos 6x = -\frac{1}{2} = \cos\frac{2\pi}{3}$. Here P.V is $\alpha = \frac{2\pi}{3}$

$\therefore 6x = 2n\pi \pm \frac{2\pi}{3}, n \in Z \Rightarrow n = \frac{2n\pi}{6} \pm \frac{2\pi}{6(3)} = \frac{n\pi}{3} \pm \frac{\pi}{9}, n \in Z$

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**2. If 0<θ<π, then solve cosθ.cos2θ.cos3θ=1/4**

**Sol:** Given that $\cos\theta\cos 2\theta\cos 3\theta = \frac{1}{4} \Rightarrow 2\cos 3\theta\cos\theta\cos 2\theta = \frac{1}{2}$

$\Rightarrow (\cos 4\theta + \cos 2\theta)\cos 2\theta = \frac{1}{2} \Rightarrow 2\cos 4\theta\cos 2\theta + 2\cos^2 2\theta = 1$

$\Rightarrow 2\cos 4\theta\cos 2\theta + 1 + \cos 4\theta = 1 \Rightarrow \cos 4\theta(2\cos 2\theta + 1) = 0$

$\Rightarrow \cos 4\theta = 0 \text{ or } \cos 2\theta = -\frac{1}{2} \Rightarrow \cos 4\theta = 0 \text{ (or) } \cos 2\theta = \cos\frac{2\pi}{3}$

$\Rightarrow 4\theta = (2n+1)\frac{\pi}{2}, \text{(or) } 2\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow \theta = (2n+1)\frac{\pi}{8}, \text{(or) } \theta = n\pi \pm \frac{\pi}{3}, n \in Z$

If 0<θ<π, then $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore$ the required solution set $= \left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{\pi}{3}, \frac{2\pi}{3}\right\}$

**3.** **Solve** $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}, 0 \le x \le 2\pi$

**Sol;** Given that, $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}$

We know that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\Rightarrow 2\cos\frac{5x}{2}\cos\frac{x}{2} = 2\sin x\cos\frac{x}{2} \Rightarrow 2\cos\frac{x}{2}\left(\cos\frac{5x}{2} - \sin x\right) = 0$

$\Rightarrow \cos\frac{x}{2} = 0 \text{ (or) } \cos\frac{5x}{2} - \sin x = 0$

Now, $\cos\frac{x}{2} = 0, \ \frac{x}{2} = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\pi; \ n \in Z$

$\because$ If $\cos\theta = 0$ then, G.S of $\theta = (2n+1)\frac{\pi}{2}; \ n \in Z$

And $\cos\frac{5x}{2} - \sin x = 0 \Rightarrow \cos\frac{5x}{2} - \cos\left(\frac{\pi}{2} - x\right) = 0 \Rightarrow \cos\frac{5x}{2} = \cos\left(\frac{\pi}{2} - x\right)$

$\Rightarrow \frac{5x}{2} = 2n\pi \pm \left(\frac{\pi}{2} - x\right) \Rightarrow x = \frac{4n\pi}{7} + \frac{\pi}{7} \text{ (or) } x = (4n-1)\frac{\pi}{3}$

If $\cos\theta = \cos\alpha$ then, G.S of $\theta = 2n\pi \pm \alpha; \ n \in Z$

Hence, the values of x in $0 \le x < 2\pi$ are $\frac{\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{13\pi}{7}$

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**4.** **If x is acute and $\sin(x+10^o)=\cos(3x-68^o)$ then find x.**

**Sol:** $\cos(3x - 68^0) = \sin(x + 10^0) \Rightarrow \cos(3x - 68^0) = \cos[90^0 - (x + 10^0)]$

$\Rightarrow \cos(3x - 68^0) = \cos(80^0 - x) \Rightarrow 3x - 68^0 = 2n\pi \pm (80^0 - x), n \in Z)$

$\Rightarrow 3x - 68^0 = 2n\pi + (80^0 - x)$ or $3x - 68^0 = 2n\pi - 80^0 + x$

$\Rightarrow 4x = 2n\pi + 148^0$ or $2x = 2n\pi - 12^0$

$\Rightarrow x = \frac{n\pi}{2} + 37^0 \text{ (or) } x = n\pi - 6^0 \Rightarrow x = 37^0$