WELCOME

STAR 'QR CODE'
DIGITAL CONTENT MATERIAL

STAR-Q

DEFINITE INTEGRALS - INDEX

7.DEFINITE INTEGRALS

SECTIONS	No. of periods (12)	Weightage in IPE $1 \times 2 + 1 \times 7 = 9$
1. Definite Integrals using the methods of integration	3	2 or 7
2. Properties of Definite Integrals	5	2 or 7
3. Reduction formulae	3	2 or 7
4*.Limit as a sum	1	

Definite integral is the most fundamental concept of integral calculus. There are several interpretations to 'definite integral'. Geometrically, the definite integral of a continuous function f(x) over a closed interval [a,b] gives the area enclosed by the curve f(x), the x-axis and the lines x=a, x=b.

If V(t) is the velocity of a particle in time t moving along a straight line then the expression $\int_a^b V(t)dt$ *denotes the distance travelled by the particle between the times a and b.*

Rectification (finding the length of an arc of a curve between two given points), evaluation of surface areas and volumes of solids of revolution, finding centre of gravity of arcs of certain curves etc., can be done using the concept of Definite Integral.

The problem of evaluating the definite integral was originally solved by using the concept of the Limit as a sum. *But that process is very tedious and lengthy one. Later, the second fundamental theorem of Integral Calculus paved an easy avenue to rectify such difficulties.*

In section-1 , fundamental theorem (II) *of integral calculus was stated and it is used throughout this chapter to evaluate the definite integral .*

In section-2, some properties of definite integrals were stated and proved. Using those, the evaluation of some definite integrals, can be done easily.

In section-3, the process of successive integration is dealt with under the name Reduction Formulae.

Though 'Limit as a sum' is not included in the prescribed syllabus, it is dealt with in section-4, in the light of various competitive exams.*

SYNOPSIS POINTS

1. If F(x) is an integral of f(x) defined on [a,b] then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

2. $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(\theta)d\theta$

3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for $a < c < b$

5. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$

6. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

7. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

8.1. $\int_0^a f(x)dx = \begin{cases} 2\int_0^{a/2} f(x)dx, & \text{if } f(a-x) = f(x) \\ 0, & \text{if } f(a-x) = -f(x) \end{cases}$

8.2. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

9. If f(x) is a periodic function with period a then $\int_0^{na} f(x)dx = = n\int_0^a f(x)dx$

10. For even 'n' $\int_0^{\pi/2} \sin^n x dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right).....\frac{1}{2}.\frac{\pi}{2} = \int_0^{\pi/2} \cos^n x dx$

For odd 'n' $\int_0^{\pi/2} \sin^n x dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right).....\frac{2}{3}.1 = \int_0^{\pi/2} \cos^n x dx$

11. $\int_0^{\pi/2} \sin^m x \cos^n x dx = \begin{cases} \frac{((m-1)(m-3)....(4)(2))((n-1)(n-3)....(2 \text{ or } 1))}{(m+n)(m+n-2)....(2 \text{ or } 1)}, & \text{if } m \text{ is odd \& } n \text{ is even or odd} \\ \frac{((m-1)(m-3)....(3)(1))((n-1)(n-3)....(4)(2))}{(m+n)(m+n-2)....(3)(1)}, & \text{if } m \text{ is even \& } n \text{ is odd} \\ \frac{((m-1)(m-3)....(3)(1))((n-1)(n-3)....(3)(1))}{(m+n)(m+n-2)....(4)(2)}.\frac{\pi}{2}, & \text{if } m \text{ is even \& } n \text{ is even (i.e., both m,n are even)} \end{cases}$

STAR-Q

ADDITIONAL QUESTIONS WITH SOLUTIONS

PROBLEMS ON 'LIMIT AS A SUM'

1 **Evaluate** $\underset{n\to\infty}{Lt}\left[\frac{1}{n+1}+\frac{1}{n+2}+..+\frac{1}{6n}\right]$

Sol: $\underset{n\to\infty}{Lt}\left[\frac{1}{n+1}+\frac{1}{n+2}+..+\frac{1}{6n}\right]$

$$=\underset{n\to\infty}{Lt}\sum_{r=1}^{5n}\frac{1}{n+r}=\underset{n\to\infty}{Lt}\sum_{r=1}^{5n}\frac{1}{n\left(1+\frac{r}{n}\right)}=\underset{n\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+\frac{r}{n}}$$

$$=\int_0^5\frac{1}{1+x}dx=\left[\log_e(1+x)\right]_0^5=\log_e 6$$

---

2 **Find** $\underset{n\to\infty}{\lim}\left(\frac{n!}{n^n}\right)^{1/n}$

Sol: Let $k=\underset{n\to\infty}{\lim}\left[\frac{n!}{n^n}\right]^{1/n}\Rightarrow\log k=\log\left[\underset{n\to\infty}{\lim}\left(\frac{n!}{n^n}\right)^{1/n}\right]=\underset{n\to\infty}{\lim}\log\left(\frac{n!}{n^n}\right)^{1/n}=\underset{n\to\infty}{\lim}\frac{1}{n}\log\left[\frac{1.2....n}{n.n....n}\right]$

$$=\underset{n\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)=\int_0^1\log x\,dx$$

Now applying the "By Parts Rule" we have

$$\log k=\left[x\log x\right]_0^1-\int_0^1 x.\frac{1}{x}dx=0-\left[-x\right]_0^1=-1\Rightarrow\log_e k=-1\Rightarrow k=e^{-1}=1/e$$

STAR-Q

3 **Evaluate** $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

Sol: Let $k=\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

$$\Rightarrow \log k = \underset{n\to\infty}{\mathrm{Lt}}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\ldots+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\frac{1}{n}\sum_{r=1}^{n}\log\left[1+\left(\frac{r}{n}\right)^2\right]=\int_0^1\log(1+x^2)dx=[x\log(1+x^2)]_0^1-\int_0^1\frac{2x^2}{1+x^2}xdx$$

$$=\log 2-2\int_0^1\left[1-\frac{1}{1+x^2}\right]dx=\log 2-2\left[x-\mathrm{Tan}^{-1}x\right]_0^1=\log 2-\left[2-\frac{\pi}{2}\right]$$

$$=\log 2+\frac{\pi}{2}-2=\log_e 2+\frac{\pi}{2}\log_e e-2\log_e e=\log[2e^{(\pi/2)-2}]\Rightarrow k=2e^{(\pi/2)-2}=2e^{(\pi-4)/2}$$

STAR-Q

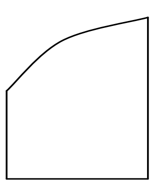
AREAS

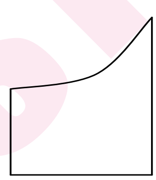
SECTIONS	No. of periods (5 to 7)	Weightage in IPE [1×7=7]
1. Areas	5 to 7	7

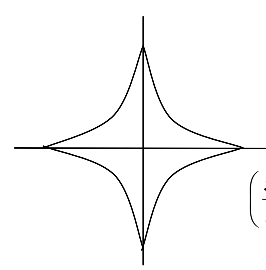
The foundation stones of integral calculus were laid in the **way out** *of solving the problem of areas of plane regions.*

A geometrical application of the concept of definite integral is the evaluation of area of a bounded curve within the given limits.

We are very familiar with the evaluation of areas of some regular geometrical plane figures like triangle, square, rectangle, circle etc., The areas of such figures can be found by using concerned formulas. But a practical situation may not always restrict regularity of curves. For example, the shapes of some crop fields are irregular because of the flow of water currents in irregular shaped canals. How to compute such areas? or How to find the areas of the figures in following shapes?







$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$

If the curves are expressed as $y=f(x)$*, then with the limits, say* $x=a$, $x=b$ *and* $y=0$*, the area of the bounded region is given by* $\int_a^b f(x)dx$

The process of finding the area of any bounded region of a curve is known as **quadrature.**

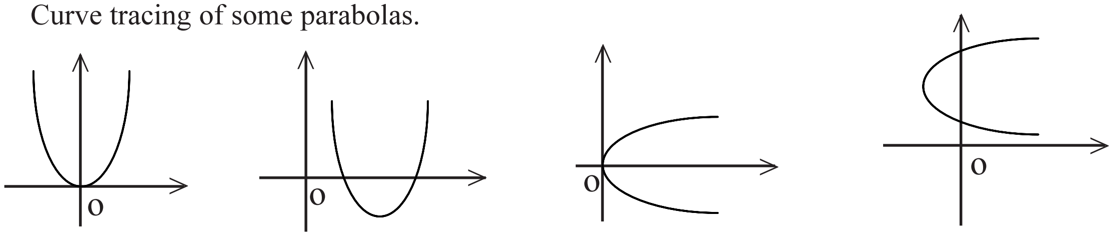
An extension of such problems is the evaluation of volumes of some geometrical figures.

Curve tracing *is an important prerequisite to deal with the problem of areas. Hence the techniques of* **curve tracing** *were discussed with adequate information in the beginning of the chapter.*

STAR-Q

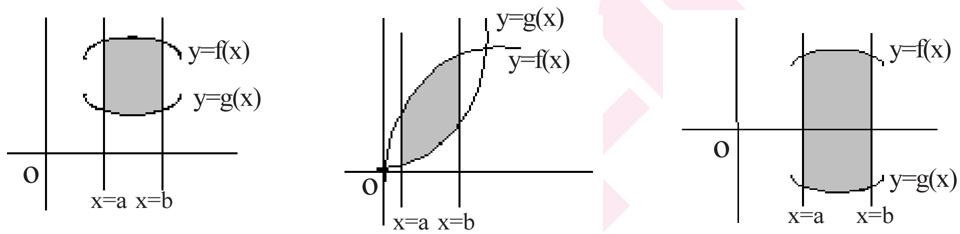
SYNOPSIS POINTS

1. Curve tracing of some parabolas.

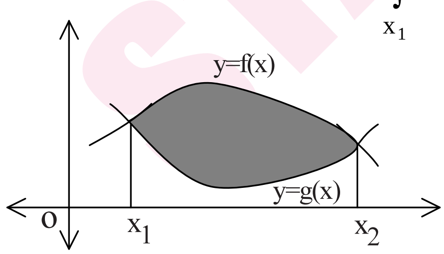


$y=x^2$ $\qquad$ $y=ax^2+bx+c$ $\qquad$ $x=y^2$ $\qquad$ $x=ay^2+by+c$

2.1. The area bounded by the upper boundary curve y=f(x) and the lower boundary curve y=g(x), between the lines x=a and x=b, is given by $A = \int_a^b (f(x) - g(x))dx$

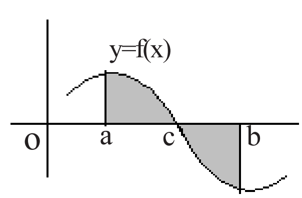


2.2. If the curves y=f(x) and y=g(x) intersect in two points (x_1,y_1) & (x_2,y_2) then the area enclosed between these two curves is given by $A = \int_{x_1}^{x_2} (f(x) - g(x))dx$



2.3. If f(x) is a continuous function on [a,b] and f(x)≥0 over [a,c] and f(x)≤0 over [c,b] for a<c<b then the area of the region bounded by y=f(x), X-axis and the lines x=a and x=b is

$$A = \int_a^b f(x)dx = \int_a^c f(x)dx - \int_c^b f(x)dx$$



☞ To get the point (s) of intersection of the given curve with the X-axis, put y=0 in the given relation and solve for x. Similarly, to get the point (s) of intersection with the Y-axis; put x=0 in the given relation and solve for y.