# 6. BINOMIAL THEOREM

| Sections | No. of periods ( 16 ) | Weightage in IPE [1x2+1x4+1x7 =13] |
|---|---|---|
| **1. Binomial theorem for positive integral index and Binomial Coefficients** | 10 | 2 or 4 or 7 |
| **2. Binomial theorem for rational index** | 4 | 7 |
| **3. Approximations using Binomial theorem** | 2 | 4 |

Binomial is an algebraic expression with 2 terms. The Binomial theorem gives formulae for the expansion of a given binomial for any given index.
We recall the following basic concepts of Binomial theorem that are learnt in lower classes.

Observe the following expansions of the binomial (x+y).

$(x + y)^0 = 1$ 1

$(x + y)^1 = x + y$ 1 1

$(x + y)^2 = x^2 + 2xy + y^2$ 1 2 1

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$. 1 3 3 1

Observations:(i) The number of terms in the expansion is one more than the exponent
(ii) The exponents of x decrease gradually while the exponents of y increase graduallay and the sum of the exponents of x and y in each term is equal to the exponent of the binomial.
(iii) The coefficients of various terms are provided by the pascal triangle.

If the index of the binomial is high, then applying the pascal triangle pattern to determine the coefficients becomes a Cumbersome process. Hence we have the following general formulae to deal with such situations.

The binomial theorem for any positive integral index n is given by

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + .... + {}^nC_r x^{n-r} y^r + .... + {}^nC_n y^n$$

Here, ${}^nC_0, {}^nC_1, ....., {}^nC_r, ....., {}^nC_n$ are called the Binomial coefficients.

These binomial coefficients can be evaluated using the formula ${}^nC_r = \frac{n!}{r!(n-r)!}$

In section 1 we discuss about finding the required term, term independent of x, middle term (s), numerically greatest term(s), coefficients of desired powers of x etc., of a given binomial for a given index. We also discuss about trinomial expansion to some extent.

Establishing various kinds of relations among the binomial coefficients in the standard binomial expansion, is another important aspect of this section.
Section-2, deals with the binomial theorem for rational index (given by Issac Newton) with certain restrictions on the terms of the binomial .Determining the sums of certain types of infinite series are also discussed.

Approximations of certain patterns using Binomial theorem for rational index, are discussed in section-3.

# SYNOPSIS POINTS

**1.1. Binomial theorem**: $(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + .... + {}^nC_rx^{n-r}y^r + .... + {}^nC_ny^n$, $n \in N$

**1.2.** The **general term** of the binomial expansion is $T_{r+1} = {}^nC_rx^{n-r}y^r$.

**2. Standard Binomial expansion**:

$(1+x)^n = 1+nx+{}^nC_2x^2+....+{}^nC_rx^r+....+x^n = C_0+C_1x+C_2x^2+....+C_rx^r+......+C_nx^n$

**3.** In the expansion of $(1+x)^n$ (i) the coefficient of $x^r$ is ${}^nC_r$=the coefficient of $(r+1)^{th}$ term
(ii) the coefficient of $r^{th}$ term is ${}^nC_{r-1}$

**4. Numerically greatest term(s) in the expansion of $(1+\alpha)^n$**: Find $\dfrac{(n+1)|\alpha|}{|\alpha|+1}$

If this value is a mixed fraction with integral part r then the numerically greatest term is $T_{r+1}$
If the value is an integer, say r then $T_r$ and $T_{r+1}$ are the two numerically greatest terms.

**5.** (i) $C_0+C_1+C_2+.....+C_n = \sum_{r=0}^{n} C_r = 2^n$ (ii) $C_0+C_2+C_4+... = C_1+C_3+C_5+.... = 2^{n-1}$

**6.** $a.C_0+(a+d).C_1+(a+2d).C_2+...+(a+nd).C_n = (2a+nd).2^{n-1}$.

**7.** $C_0.C_r+C_1.C_{r+1}+C_2.C_{r+2}+....+C_{n-r}.C_n = {}^{2n}C_{(n+r)}$

**8.** $(1+x)^n = 1+nx+\dfrac{n(n-1)}{2!}x^2+\dfrac{n(n-1)(n-2)}{3!}x^3+........\infty,\ x \in R, |x|<1 \text{ and } n \in Q$

**9.** If x is a real number and $|x|<1$, then

(i) $(1+x)^{-1} = 1-x+x^2-x^3+....+(-1)^r x^r+......$

(ii) $(1+x)^{-2} = 1-2x+3x^2-4x^3+....+(-1)^r.(r+1)x^r+......$

(iii) $(1+x)^{-3} = 1-3x+6x^2-10x^3+....+(-1)^r.\dfrac{(r+1)(r+2)}{2!}x^r+......$

**10.** $(1+x)^{\frac{p}{q}} = 1+\dfrac{p}{1}.\left(\dfrac{x}{q}\right)+\dfrac{p(p-q)}{1.2.}\left(\dfrac{x}{q}\right)^2+....+\dfrac{p(p-q)...(p-(r-1)q)}{r!}.\left(\dfrac{x}{q}\right)^r+.....$

$(1-x)^{\frac{p}{q}} = 1-\dfrac{p}{1}\left(\dfrac{x}{q}\right)+\dfrac{p(p-q)}{1.2}\left(\dfrac{x}{q}\right)^2-.......+(-1)^r.\dfrac{p(p-q)(p+(r-1)q)}{r!}\left(\dfrac{x}{q}\right)^r+....$

$(1+x)^{\frac{-p}{q}} = 1-\dfrac{p}{1}\left(\dfrac{x}{q}\right)+\dfrac{p(p+q)}{1.2}\left(\dfrac{x}{q}\right)^2-.....+(-1)^r.\dfrac{p(p+q)....(p+(r-1)q)}{r!}.\left(\dfrac{x}{q}\right)^r+.......$

$(1-x)^{\frac{-p}{q}} = 1+\dfrac{p}{1}\left(\dfrac{x}{q}\right)+\dfrac{p(p+q)}{1.2}\left(\dfrac{x}{q}\right)^2+.....+\dfrac{p(p+q).....(p+(r-1)q)}{r!}.\left(\dfrac{x}{q}\right)^r+......$

# ADDITIONAL QUESTIONS WITH SOLUTIONS

## BINOMIAL THEOREM-1

**1. Find the number of terms in the expansion of** $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$

**Sol:** Number of terms in the expansion of $(x+a)^n$ is $(n+1)$

$\therefore$ number of terms in $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$ is $9+1=10$.

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**2. Expand** $(4x+5y)^7$

**Sol:** $(4x+5y)^7=\sum_{r=0}^{7}{}^7C_r(4x)^{7-r}(5y)^r={}^7C_0(4x)^7(5y)^0+{}^7C_1(4x)^6(5y)^1+{}^7C_2(4x)^5(5y)^2+{}^7C_3(4x)^4(5y)^3$

$+{}^7C_4(4x)^3(5y)^4+{}^7C_5(4x)^2(5y)^5+{}^7C_6(4x)^1(5y)^6+{}^7C_7(4x)^0(5y)^7$

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**3. Write down and simplify $7^{th}$ term in $(3x-4y)^{10}$**

**Sol:** General term in $(3x-4y)^{10}$ is $T_{r+1}=(-1)^r\,{}^{10}C_r(3x)^{10-4}(4y)^r$

By putting r=6, we get

$T_{6+1}=T_7=(-1)^6\,{}^{10}C_6(3x)^4(4y)^6={}^{10}C_4(3)^4(4)^6x^4y^6$

$$=\frac{10\times9\times8\times7}{1\times2\times3\times4}3^4.(4)^6x^4y^6=280\,(12)^5x^4y^6$$

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**4. Write down and simplify $10^{th}$ term in** $\left(\frac{3p}{4}-5q\right)^{14}$

**Sol:** General term in $\left(\frac{3p}{4'}-5q\right)^{14}$ is $T_{r+1}={}^{14}C_r\left(\frac{3p}{4}\right)^{14-r}(-5q)^r=(-1)^r\,{}^{14}C_r\left(\frac{3p}{4}\right)^{14-r}(5q)^r$

By putting r=9, we get

$T_{9+1}=T_{10}=(-1)^9\,{}^{14}C_9\left(\frac{3p}{4}\right)^5(5q)^9$

$$=-{}^{14}C_5\left(\frac{3}{4}\right)^5(5)^9p^5.q^9=-\frac{14\times13\times12\times11\times10}{1\times2\times3\times4\times5}\frac{3^5}{4^5}.5^9p^5.q^9=\frac{-(2002)3^5.5^9}{4^5}.p^5.q^9$$

**5. Write down and simplify $r^{th}$ term in** $\left(\frac{3a}{5}+\frac{5b}{7}\right)^8, (1 \le r \le 9)$.

**Sol:** The general term in $\left(\frac{3a}{5}+\frac{5b}{7}\right)^8$ is $T_{r+1} = {}^8C_r\left(\frac{3a}{5}\right)^{8-r}\left(\frac{5b}{7}\right)^r$

By putting r–1 in place of r, we get

$$T_{(r-1)+1} = T_r = {}^8C_{(r-1)}\left(\frac{3a}{5}\right)^{8-r+1}\left(\frac{5b}{7}\right)^{r-1} = {}^8C_{(r-1)}\left(\frac{3a}{5}\right)^{9-r}\left(\frac{5b}{7}\right)^{r-1} ;1 \le r \le 9$$

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**6. Find the coefficient of** $x^2$ in $\left(7x^3-\frac{2}{x^2}\right)^9$

**Sol:** The general term in $\left(7x^3-\frac{2}{x^2}\right)^9$ is $T_{r+1} = (-1)^r\,{}^9C_r(7x^3)^{9-r}\left(\frac{2}{x^2}\right)^r$

$$= (-1)^r\,{}^9C_r(7)^{9-r}(2)^r x^{27-3r}x^{-2r} \quad = (-1)^r\,{}^9C_r(7)^{9-r}(2)^r.x^{27-5r} \quad ..........(1)$$

For coefficient of $x^2$, put $27-5r = 2 \Rightarrow 5r = 25 \Rightarrow r = 5$

Put r = 5 in eq. (1), we get

$$T_{5+1} = (-1)^5\,{}^9C_5(7)^4(2)^5x^{27-25} = -{}^9C_4 7^4.2^5.x^2 = \left(\frac{9\times8\times7\times6}{1\times2\times3\times4}\right)7^4.2^5.x^2$$

$\therefore$ Coefficient of $x^2 = -126\times7^4\times2^5$

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**7. Find the coefficient of** $x^{-7}$ in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$

**Sol:** The general term in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$ is $T_{r+1} = (-1)^r.\,{}^7C_r\left(\frac{2x^2}{3}\right)^{7-r}\left(\frac{5}{4x^5}\right)^r$

$$= (-1)^r.\,{}^7C_r\left(\frac{2}{3}\right)^{7-r}\left(\frac{5}{4}\right)^r x^{14-2r}.x^{-5r}$$

$$\therefore T_{r+1} = (-1)^r.\,{}^7C_r\left(\frac{2}{3}\right)^{7-r}\left(\frac{5}{4}\right)^r x^{14-7r} \quad .........(1)$$

For coefficient of $x^{-7}$, put $14-7r = -7 \Rightarrow 7r = 21 \Rightarrow r = 3$

Put r =3 in eq. (1), we get

$$T_{3+1} = (-1)^3\,{}^7C_3\left(\frac{2}{3}\right)^4\left(\frac{5}{4}\right)^3x^{14-21} = \frac{-7\times6\times5}{1\times2\times3}\left(\frac{2}{3}\right)^4\left(\frac{5}{4}\right)^3x^{-7}$$

$\therefore$ Coefficient of $x^{-7} = -35\times\frac{1}{3^4}.\frac{5^3}{2^2} = \frac{-4375}{324}$

**8. Find the term independent of x in the expansion of** $\left(\frac{2x^2}{5}+\frac{15}{4x}\right)^9$

**Sol:** The general term in $\left(\frac{2x^2}{5}+\frac{15}{4x}\right)^9$ is $T_{r+1}={}^9C_r\left(\frac{2x^2}{5}\right)^{9-r}\left(\frac{15}{4x}\right)^r={}^9C_r\left(\frac{2}{5}\right)^{9-r}\left(\frac{15}{4}\right)^r x^{18-2r}.x^{-r}$

$$T_{r+1}={}^9C_r\left(\frac{2}{5}\right)^{9-r}\left(\frac{15}{4}\right)^r.x^{18-3r}\;\ldots\ldots\ldots\ldots(1)$$

To get the independent of x, put $18-3r=0 \Rightarrow r=6$

Put r=6 in eq. (1), we get

$$T_{6+1}={}^9C_6\left(\frac{2}{5}\right)^3\left(\frac{15}{4}\right)^6x^0={}^9C_3.\frac{2^3}{5^3}.\frac{3^6\times5^6}{4^6}=\frac{9\times8\times7}{1\times2\times3}.\frac{3^6.5^3}{2^9}=\frac{3^7\times5^3\times7}{2^7}$$

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**9. Find the numerically greatest term(s) in the expansion of $(3x+5y)^{12}$ when x=1/2, y=4/3**

**Sol:** Write $(3x+5y)^{12}=\left[3x\left(1+\frac{5y}{3x}\right)\right]^{12}=3^{12}x^{12}\left(1+\frac{5y}{3x}\right)^{12}$

On comparing $\left(1+\frac{5}{2}\frac{y}{x}\right)^{12}$ with $(1+x)$, we get $n=17, x=\frac{5}{3}.\frac{y}{x}=\frac{5}{3}.\frac{\left(\frac{4}{3}\right)}{\frac{1}{2}}=\frac{5}{3}.\frac{8}{3}=\frac{40}{9}$

Now $\frac{(n+1)|x|}{1+|x|}=\frac{(12+1)\left(\frac{40}{9}\right)}{1+\frac{40}{9}}=\frac{13\times40}{49}=\frac{520}{49}=10\frac{30}{49}$

which is not an integer $\quad \therefore m=\left[10\frac{30}{49}\right]=10$

N.G. term in $\left(1+\frac{5y}{3x}\right)^{12}$ is $T_{m+1}=T_{11}={}^{12}C_{10}\left(\frac{5}{3}\frac{y}{x}\right)^{10}={}^{12}C_{10}\left(\frac{5}{3}\times\frac{\left(\frac{4}{3}\right)}{\left(\frac{1}{2}\right)}\right)^{10}$

$$={}^{12}C_{10}\times\left(\frac{5}{3}\times\frac{8}{3}\right)^{10}={}^{12}C_{10}\left(\frac{40}{9}\right)^{10}$$

$\therefore$ N.G. term in $(3x+5y)^{12}$ is $=3^{12}.\left(\frac{1}{2}\right)^{12}{}^{12}C_{10}\left(\frac{40}{9}\right)^{10}={}^{12}C_{10}\left(\frac{3}{2}\right)^2\left(\frac{20}{3}\right)^{10}$

**10. Find the numerically greatest term (s) in the expansion of $(3+7x)^n$ when $x=\frac{4}{5}, n=15$**

**Sol:** Write $(3+7x)^n=\left[3\left(1+\frac{7}{3}x\right)\right]^n=3^n\left(1+\frac{7}{3}x\right)^n$

Now we first find N.G. term in $\left(1+\frac{7}{3}x\right)^n$

On comparing with $(1+x)^n$, we get $X=\frac{7}{3}x=\frac{7}{3}\times\frac{4}{5}=\frac{28}{15}$

n = 15

Now $\frac{(n+1)|x|}{1+|x|}=\frac{(15+1)\left(\frac{28}{15}\right)}{1+\frac{28}{15}}=\frac{16\times 28}{43}=\frac{448}{43}=10\frac{18}{43}$

Its integral part (m) = 10

$\therefore T_{m+1}=T_{11}$ is the N.G. term

$T_{11}$ in $\left(1+\frac{7}{3}x\right)^{15}={}^{15}C_{10}\left(\frac{7}{3}x\right)^{10}={}^{15}C_{10}\left(\frac{7}{3}\times\frac{4}{5}\right)^{10}={}^{15}C_{10}\left(\frac{28}{15}\right)^{10}$

$\therefore$ N.G. term in $(3+7x)^n$ is $=3^{15}.{}^{15}C_{10}\left(\frac{28}{15}\right)^{10}={}^{15}C_{10}.3^5\left(\frac{28}{5}\right)^{10}$

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**11. If the coefficient of $x^{11}$ and $x^{12}$ in the binomial expansion of $\left(2+\frac{8x}{3}\right)^n$ are equal, find n.**

**Sol:** The general term in $\left(2+\frac{8x}{3}\right)^n$ is $T_{r+1}={}^nC_r(2)^{n-r}\left(\frac{8x}{3}\right)^r$

we have r = 11, r = 12

$T_{12}={}^nC_{11}(2)^{n-11}\left(\frac{8x}{3}\right)^{11}$; $T_{13}={}^nC_{12}(2)^{n-12}\left(\frac{8x}{3}\right)^{12}$

$\because$ the coefficient of $x^{11}$ and $x^{12}$ are equal

$\Rightarrow {}^nC_{11}(2)^{n-11}\left(\frac{8}{3}\right)^{11}={}^nC_{12}(2)^{n-12}\left(\frac{8}{3}\right)^{12}\Rightarrow\frac{{}^nC_{12}}{{}^nC_{11}}=(2).\frac{3}{8}$

$\Rightarrow\frac{n-11}{12}=\frac{3}{4}\Rightarrow n-11=9\Rightarrow n=20\left(\because\frac{{}^nC_r}{{}^nC_{r-1}}=\frac{n-(r-1)}{r}\right)$

**12. Expand** $(3+x-x^2)^4$

Sol: $(3+x-x^2)^4 = [(3+x)-x^2]^4$

$= {}^4C_0(3+x)^4 - {}^4C_1(3+x)^3x^2 + {}^4C_2(3+x)^2(x^2)^2$

$-{}^4C_3(3+x)^1(x^2)^3 + {}^4C_4(x^2)^4$

$=(3+x)^4 - 4(3+x)^3x^2 + 6(3+x)^2x^4 - 4(3+x)x^6 + (1)(x^8)$

$=[{}^4C_0(3)^4 + {}^4C_1(3)^3x + {}^4C_2(3)^2x^2 + {}^4C_3(3)x^3 + {}^4C_4x^4]$

$-4[{}^3C_0(3)^3 + {}^3C_1(3^2)x + {}^3C_2(3)x^2 + {}^3C_3x^3]x^2$

$+6[{}^2C_6(3)^2 + {}^2C_1(3)x + {}^2C_2x^2]x^4 - 4(3+x)x^6 + x^8$

$=(81+108x+54x^2+12x^3+x^4) - 4x^2(27+27x+9x^2+x^3) + 6x^4(9+6x+x^2) - 4x^6(3+x) + x^8$

$=81+108x+(54-108)x^2+(12-108)x^3+(1-36+54)x^4+(-4+36)x^5+(6-12)x^6+(-4)x^7+x^8$

$=81+108x-54x^2-96x^3+19x^4+32x^5-6x^6-4x^7+x^8$

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**13. Find the sum of the coefficients of $x^{32}$ and $x^{-18}$ in the expansion of** $\left(2x^3-\frac{3}{x^2}\right)^{14}$

**Sol:** The general term in $\left(2x^3-\frac{3}{x^2}\right)^{14}$ is $T_{r+1} = {}^{14}C_r(2x^3)^{14-r}\left(-\frac{3}{x^2}\right)^r = (-1)^r\,{}^{14}C_r(2)^{14-r}.(3)^r.x^{42-3r}.x^{-2r}$

$= (-1)^r.{}^{14}C_r\,2^{14-r}(3)^r.x^{42-5r}$ ..........(1)

For coefficient of $x^{32}$, Put $42-5r=32$ $\Rightarrow 5r=10 \Rightarrow r=2$

Put r = 2 in equation (1), we get

$T_{2+1} = T_3 = (-1)^2\,{}^{14}C_2(2)^{12}(3)^2.x^{42-10} = {}^{14}C_2(2)^{12}(3)^2.x^{32}$

$\therefore$ Coefficient of $x^{32}$ is ${}^{14}C_2(2)^{12}(3)^2$............(2)

For coefficient of $x^{-18}$.

Put $42-5r=-18$ $\Rightarrow 5r=60 \Rightarrow r=12$

Put r = 12 in equation (1), we get

$T_{12+1} = T_{13} = (-1)^{12}\,{}^{14}C_{12}(2)^2(3)^{12}x^{42-60} = {}^{14}C_{12}(2)^2(3)^{12}.x^{-18}$

$\therefore$ Coefficient of $x^{18}$ is ${}^{14}C_{12}(2)^2 3^{12}$

Hence sum of the coefficients of $x^{32}$ and $x^{-18}$ is ${}^{14}C_2(2)^{12}(3)^2 + {}^{14}C_{12}(2)^2 3^{12}$

## BINOMIAL THEOREM-2

**14. Find $3C_0+6C_1+12C_2+.......+3.2^n.C_n$**

**Sol:** $3C_0+6C_1+12C_2+.......+3.2^n.C_n$

$= 3[C_0+2.C_1+4.C_2+.....+2^nC_n]$

$= 3[C_0+2.C_1+2^2.C_2+.....+2^nC_n]$

$[\because (1+x)^n = C_0+C_1x+C_2x^2+......+C_nx^n$ Put $x = 2$ $(1+2)n = C_0+2.C_1+2^2.C_2+.....+2^n.C_n]$

$= 3(3^n) = 3^{n+1}$

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**15. Prove that $2.C_0+5.C_1+8.C_2+......+(3n+2)C_n=(3n+4).2^{n-1}$**

**Sol:** Let $S = 2.C_0+5.C_1+8.C_2+.........+(3n-1)C_{n-1}+(3n+2)C_n$

$\because C_n = C_0.C_{n-1} = C_1......$

$S = (3n+2)\ C_0+(3n-1)C_1+(3n-4)C_2+.....+.....+5C_{n-1}+2C_n$

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**16. Prove that $C_0-4.\ C_1+7.\ C_2-10.\ C_3+......=0$, if n is an even positive integer.**

**Sol:** 1, 4, 7, 10 ................. are in A.P.

$T_{n+1} = a + nd = 1 + n(3) = 3n + 1$

$\therefore C_0 - 4.C_1 + 7.C_2 - 10.C_3 + ........ (n+1)$ terms

$= C_0 - 4.\ C_1 + 7.C_2 - 10.C_3 + ........ + (-1)^n\ (3n+1)C_n = \sum_{r=0}^{n} (-1)^r (3r+1)C_r$

$\sum_{r=0}^{n} \{(-1)^r (3r)C_r + (-1)^r C_r\} = 3.\sum_{r=0}^{n} (-1)^r r.C_1 + \sum_{r=0}^{n} (-1)^r.C_r \quad = 3(0) + 0 = 0$

$\therefore C_0 - 4C_1 + 7.C_2 - 10.C_3 + ....... = 0$

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**17. Show that $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+\frac{C_7}{8}+......=\frac{2^n-1}{n+1}$**

**Sol:** $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+\frac{C_7}{8}+..... \quad = \frac{{}^nC_1}{2}+\frac{{}^nC_3}{4}+\frac{{}^nC_5}{6}+\frac{{}^nC_7}{8}+......$

$= \frac{n}{2} + \frac{n(n-1)(n-2)}{4\times 3!} + \frac{n(n-1)(n-2)(n-3)(n-4)}{6+5!} + .......$

$$=\frac{1}{n+1}\left[\frac{(n+1)n}{2!}+\frac{(n+1)n(n-1)(n-2)}{4!}+....\frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{6!}+......\right]$$

$$=\frac{1}{n+1}\left[{}^{(n+1)}C_2+{}^{(n+1)}C_4+{}^{(n+1)}C_6+.....\right]$$

$$=\frac{1}{n+1}\left[{}^{(n+1)}C_0+{}^{(n+1)}C_2+{}^{(n+1)}C_4+......-{}^{(n+1)}C_0\right]=\frac{1}{n+1}[2^n-1]=\frac{2^n-1}{n+1}$$

$$\therefore \frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+.....=\frac{2^n-1}{n+1}$$

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**18. Find the sum of** $\frac{{}^{15}C_1}{{}^{15}C_0}+2\frac{{}^{15}C_2}{{}^{15}C_1}+3\frac{{}^{15}C_3}{{}^{15}C_2}+.......+15\frac{{}^{15}C_{15}}{{}^{15}C_{14}}$

**Sol:** $\frac{{}^{15}C_1}{{}^{15}C_0}+2\frac{{}^{15}C_2}{{}^{15}C_1}+3\frac{{}^{15}C_3}{{}^{15}C_2}+.......+15\frac{{}^{15}C_{15}}{{}^{15}C_{14}}$

$$=\left(\frac{15-0}{1}\right)+2\left(\frac{15-1}{2}\right)+3\frac{(15-2)}{3}+......+15\left(\frac{1}{15}\right)=15+14+13+.......+1$$

$$=\frac{(15)(15+1)}{1\times 2}=120$$

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**19. Find the sum of** $2^2.C_0+3^2.C_1+4^2.C_2+.......+(n+2)^2C_n$

**Sol:** $2^2.C_0+3^2.C_1+4^2.C_2+.........+(n+2)^2C_n$

$$=\sum_{r=0}^{n}(r+2)^2C_n \quad =\sum_{r=0}^{n}(r^2+4r+4)C_r \quad =\sum_{r=0}^{n}r^2C_r+4\sum_{r=0}^{n}r.C_r+4\sum_{r=0}^{n}C_r$$

$$=\sum_{r=0}^{n}r(r-1)C_r+\sum_{r=0}^{n}r.C_r+4\sum_{r=0}^{n}r.C_r+4\sum_{r=0}^{n}C_r=\sum_{r=2}^{n}r(r-1)C_r+5\sum_{r=1}^{n}r.C_r+4\sum_{r=0}^{n}C_r$$

$$=n(n-1)2^{n-2}+5n.2^{n-1}+4.2^n=(n^2+9n+4.4)2^{n-2}=(n^2+9n+16)2^{n-2}$$

**20. Prove that** $C_0^2 - C_1^2 + C_2^2 - C_3^2 \ldots\ldots\ldots + (-1)^n C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}\, {}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$

**Sol: Easy method:**

We know $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + \ldots\ldots + C_nx^n$ ---- (1)

$(x-1)^n = C_0x^n - C_1x^{n-1} + C_2x^{n-2} - \ldots\ldots + (-1)^n C_n$ ---- (2)

Multiplying (1) and (2) ,we get

$[C_0x^n - C_1x^{n-1} + C_2x^{n-2} - \ldots\ldots + (-1)^n C_n][C_0 + C_1x + C_2x^2 + \ldots + C_nx^n]$

$= (x-1)^n (1+x)^n = (x^2-1)^n$ .....(3)

Coefficient of $x^n$ in the product L.H.S of (3) is $C_0^2 - C_1^2 + C_2^2 - C_3^2 \ldots\ldots\ldots + (-1)^n C_n^2$

**To get R.H.S term, we verify the cases for n=1,2,3,4..**

**When n=1, we have** $(x^2-1)^n = (x^2-1)^1 = (x^2-1)$

**When n=2, we have** $(x^2-1)^n = (x^2-1)^2 = (x^4-2x^2 + 1)$

**When n=3, we have** $(x^2-1)^n = (x^2-1)^3 = (x^6-3x^4 + 3x^2 - 1)$

**When n=4, we have** $(x^2-1)^n = (x^2-1)^4 = (x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$

Hence, we can generalise the following conclusions:

Coefficient of $x^n$ in the R.H.S of (3) is 0 if n odd

Coefficient of $x^n$ in the R.H.S of (3) is $(-1)^{(n/2)} \cdot {}^nC_{(n/2)}$ if n is even

$C_0^2 - C_1^2 + C_2^2 - C_3^2 \ldots\ldots\ldots + (-1)^n C_n^2 = 0$ if n is odd

$= (-1)^{n/2}\, {}^nC_{(n/2)}$ if n is even.

## BINOMIAL THEOREM-3

**21. Find the set of values of x for which $(5+x)^{3/2}$ is valid**

**Sol:** $(5+x)^{3/2} = \left[5\left(1+\frac{x}{5}\right)\right]^{3/2} = 5^{3/2}\left(1+\frac{x}{5}\right)^{3/2}$

$\therefore$ $(5+x)^{3/2}$ is valid when $\left|\frac{x}{5}\right| < 1 \Rightarrow |x| < 5 \Rightarrow x \in (-5,5)$

---

**22. Find the set of values of x for which $(7+3x)^{-5}$ is valid**

**Sol:** $(7+3x)^{-5} = \left[7\left(1+\frac{3}{7}x\right)\right]^{-5} = 7^{-5}\left(1+\frac{3}{7}x\right)^{-5}$

$(7+3x)^{-5}$ is valid when $\left|\frac{3x}{7}\right| < 1 \Rightarrow |x| < \frac{7}{3} \Rightarrow x \in \left(\frac{-7}{3}, \frac{7}{3}\right)$

---

**23. Find the set of values of x for which $(4 - x/3)^{-1/2}$ is valid**

**Sol:** $\left(4-\frac{x}{3}\right)^{-1/2} = \left[4\left(1-\frac{x}{12}\right)\right]^{-1/2}$

$\left(4-\frac{x}{3}\right)^{-1/2}$ is valid when $\left|\frac{-x}{12}\right| < 1 \Rightarrow |x| < 12 \Rightarrow x \in (-12,12)$

---

**24. Wrie down the first 3 terms in the expansion of $(8-5x)^{2/3}$**

**Sol:** $\left[8\left(1-\frac{5}{8}x\right)^{2/3}\right] = (2^3)^{2/3}\left[1-\frac{5}{8}x\right]^{2/3} = 4\left[\left(1-\frac{5x}{8}\right)^{2/3}\right]$

$(1-x)^{p/q} = 1 - p\left(\frac{x}{q}\right)q + \frac{(p)(p-q)}{1.2}\left(\frac{x}{q}\right)^2 - \ldots..$

Here $X = \frac{5x}{8}, p=2, q=3, \frac{X}{q} = \frac{\frac{5x}{8}}{3} = \frac{5x}{24} \therefore (8-5x)^{2/3}$

$= 4\left[1 - 2\left(\frac{5x}{24}\right) + \frac{(2)(2-3)}{1.2}\left(\frac{5x}{24}\right)^2 - \ldots.\right] = 4\left[1 - \frac{5x}{12} - \left(\frac{5x}{24}\right)^2 + \ldots..\right]$

$\therefore$ The first 3 terms of $(8-5x)^{2/3}$ are $4, \frac{-5x}{3}, \frac{-25}{144}x^2$

**25. Wrie down the first 3 terms in the expansion of $(1 + 4x)^{-4}$**

**Sol:** We have $(1+x)^{-n} = 1 - nx + \frac{(n)(n+1)}{1.2}x^2 + .....$ Here n = 4, x = 4x

$$\therefore (1+4x)^{-4} = 1 - 4(4x) + \frac{(4)(5)}{1.2}.(4x)^2 + ........ = 1 = 16x + 160x^2 - .......$$

$\therefore$ The first 3 terms of $(1 + 4x)^{-4}$ are 1, $-16x$, $160x^2$.

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**26. Wrie down the first 3 terms in the expansion of $(2 - 7x)^{-3/4}$**

**Sol:** Write $(2-7x)^{-3/4} = \left[2\left(1-\frac{7}{2}x\right)\right]^{-3/4} = (2)^{-3/4}\left[2\left(1-\frac{7}{2}x\right)^{-\frac{3}{4}}\right]$

Now $(1-X)^{-p/q} = 1 + p\left(\frac{x}{q}\right) + \frac{(p)(p+q)}{1.2}\left(\frac{x}{q}\right)^2 + .....$

Here $p = 3, q = 4, X = \frac{7}{2}x, \frac{x}{q} = \frac{\frac{7}{2}x}{4} = \frac{7}{8}x \therefore 2^{-3/4}\left(1-\frac{7}{2}x\right)^{-3/4}$

$$= 2^{-3/4}\left[1 + (3)\left(\frac{7}{8}x\right) + \frac{(3)(3+4)}{1.2}\left(\frac{7}{8}x^2\right) + .......\right] \qquad = 2^{-3/4}\left[1 + \frac{21}{8}x + \frac{1029}{128}x^2 .......\right]$$

$\therefore$ First 3 terms of $(2 - 7x)^{-3/4}$ are $= 2^{-3/4}, 2^{-3/4}\left(\frac{21}{8}x\right), 2^{-3/4}\left(\frac{1029}{128}\right)x^2$

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**27. Write down the general term in the expansion $\left(1+\frac{4x}{5}\right)^{\frac{5}{2}}$**

**Sol:** General term of $(1 - x)^{p/q}$ is $T_{r+1} = \frac{(p)(p-q)......[p-(r-1)}{(r)!}\left(\frac{X}{q}\right)^r$

Here p = 5, q = 2, $X = \frac{4x}{5}, \frac{X}{q} = \frac{\frac{4x}{5}}{2} = \frac{2x}{5}$ $T_{r+1}\left(1+\frac{4x}{5}\right)^{5/2}$ is

$$= \frac{(5)(5-2)(5-2\times2)(5-2\times3).......[5-(r-1)2]}{r!}\left(\frac{2x}{5}\right)^r$$

$$= (-1)^{r-3}\frac{2.3.1.1.3.5.......(2r-7)}{r!}\left(\frac{2x}{5}\right)^r$$

–

**28. Write down the general term in the expansion $\left(3-\frac{5x}{4}\right)^{-1/2}$**

**Sol:** $\left(3-\frac{5x}{4}\right)^{-1/2}=\left[3\left(\left(3-\frac{5x}{4}\right)^{-1/2}\right)\right]=(3)^{1/3}\left[1-\frac{5x}{12}\right]^{-1/2}$

General term of $(1-x)^{-p/q}$ is $T_{r+1}=\frac{(p)(p+q)(p+2q)......[p+(r-1)q]}{(r)!}\left(\frac{X}{q}\right)^r$

Here p = 1, q = 2

$x=\frac{5x}{12},\frac{x}{q}=\frac{\left(\frac{5x}{12}\right)}{2}=\frac{5x}{24}$

$\therefore T_{r+1}$ of $\left(3-\frac{5x}{4}\right)^{-1/2}=3^{-1/2}\left[\frac{(1)(1+2)(1+2.2)......[1+(r-1)2]}{r!}\right]\left(\frac{5x}{24}\right)^r$

$=\frac{1}{\sqrt{3}}\left[\frac{1.3.5.....(2r-1)}{r!}\right]\left(\frac{5x}{24}\right)^r$

---

**29. If |x| is so small that $x^2$ and higher powers of x may be neglected, then find an approximate value of $\frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}$**

**Sol:** $\frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}=\left(1+\frac{3x}{2}\right)^{-4}\left[8\left(1+\frac{9}{8}x\right)\right]^{1/3}(1+2x)^{-1}$

$\left(1+\frac{3x}{2}\right)^{-4}.8^{1/3}\left(1+\frac{9}{8}x\right)^{1/3}(1+2x)^{-1}\simeq 2\left[1-\frac{4}{1}\left(\frac{3x}{2}\right)\right]\left[1+\frac{4}{3}\left(\frac{9x}{8}\right)\right][1+(-2)(2x)]$

$\because x^2$ and higher powers of x are neglected

$\simeq 2(1-6x)\left(1+\frac{3x}{8}\right)(1-4x)\simeq 2\left(1-6x+\frac{3x}{8}\right)(1-4x)$ $\because x^2$ and higher powers of x are neglected

$\simeq 2\left(1-\frac{45}{8}x\right)(1-4x)\simeq 2\left(1-4x-\frac{45}{8}x\right)$ $\because x^2$ and higher powers of x are neglected

$=2\left(1-\frac{77}{8}x\right)$

**30. Find an approximate value of $\sqrt[5]{32.16}$ corrected to 4 decimal places.**

**Sol:** $\sqrt[5]{32.16} = (32+0.16)^{1/5} = (32)^{1/5}.\left(1+\frac{0.16}{32}\right)^{1/5} = 2[1+0.005]^{1/5}$

$$= 2\left[1+\frac{1}{5}(0.005)+\frac{\frac{1}{5}\left(-\frac{4}{5}\right)}{2!}(0.005)^2+....\right]$$

$$= 2\left[1+0.001-\frac{2}{25}(0.000025)+....\right] \simeq 2(1.000998) \simeq 2.001996 \qquad \therefore \sqrt[5]{32.16} \simeq 2.0019$$

---

**31. Find an approximate value of $\sqrt[7]{199}$ corrected to 4 decimal places.**

**Sol:** $\sqrt{199} = (196+3)^{1/2} = (199)^{1/2}\left(1+\frac{3}{196}\right)^{1/2} = 14[1+0.0153]^{1/2}$

$$= 14\left[1+\frac{1}{2}(0.0153)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}(0.0153)^2+......\right] = 14\left[1+0.00765-\frac{1}{8}(0.00023409)+....\right]$$

$$\simeq 14(1.00765) \simeq 14.1071 \ \therefore \sqrt{199} \simeq 14.1071$$

---

**32. Find an approximate value of $\sqrt[3]{1002}-\sqrt[3]{998}$ corrected to 4 decimal places.**

**Sol:** $\sqrt[3]{1000}\sqrt[3]{1+\frac{2}{1000}} \quad \sqrt[3]{1000}\sqrt[3]{1-\frac{2}{1000}} = 10\ [(1+0.002)^{1/3} - (1-0.002)^{1/3}$

$$= \left(\left(1+\frac{1}{3}(0.002)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}(0.002)^2+....\right)-\left(1-\frac{1}{3}(0.002)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}(0.002)^2+....\right)\right)$$

$$\simeq 10\left[\frac{2}{3}(0.002)\right] \simeq 10\left[\frac{0.004}{3}\right] \simeq \frac{0.04}{3} \simeq 0.0133$$

**33. Find an approximate value of $(1.02)^{3/2}-(0.98)^{3/2}$ corrected to 4 decimal places.**

**Sol:** $[1+0.02]^{3/2}-[1-0.02]^{3/2}=\left[1+\frac{3}{2}(0.02)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}(0.02)^2+......\right]$

$$=\left[1-\frac{3}{2}(0.02)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}(0.02)^2+......\right]=2\left[\frac{3}{2}(0.02)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-1\right)}{3!}(0.02)^3+......\right]$$

$$=2\left[0.03+\frac{1}{6}.\frac{3}{2}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(0.000008)\right]=2[0.03-0.000001]=0.05999$$