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DIGITAL MATERIAL

PRODUCT OF VECTORS -INDEX

PRODUCT OF TWO VECTORS

1. INTRODUCTION PAGE

Sections	No. of periods (13 to 15)	Weightage in IPE [2, 4 or 7 = 6 or 13]
1. Scalar product; Angle between vectors, Orthogonal projections, vector equation of plane	9	2 or 4 or 7 Marks
2. Vector product; vector areas	6	2 or 4 or 7 Marks

It is very familiar to science students that, the work (w) done by a body depends on (i) magnitude of force ($|\overline{F}|$ or $\overline{F}$) (ii) Magnitude of Displacement ($|\overline{S}|$ or $\overline{S}$) (iii) Cosine of angle between $\overline{F}$ and $\overline{S}$ ($\cos\theta$). Thus, a scalar quantity W is derived from two vector quantities, viz., $\overline{F}, \overline{S}$.

In Vector Algebra, a generalised treatment is given to such situations. The scalar Product of two vectors is denoted by $\overline{a}.\overline{b}$ or $(\overline{a}, \overline{b})$ or $S\overline{a}\overline{b}$ and it is also called as dot product or inner product.

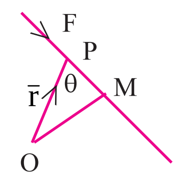
The first section starts with the definition of scalar product of two vectors. Determining the angle between two vectors is an immediate consequence of the definition of dot product. Orthogonal projection of vectors is a Mathematical application of the scalar product. Some important things to be remembered in this topic are (i) $\overline{a}, \overline{b}$ are perpendicular $\Rightarrow \overline{a}.\overline{b} = 0$ (ii) $\overline{i}.\overline{j} = 0$, $\overline{i}.\overline{i} = 1$ (iii) $\overline{a}.\overline{a} = |\overline{a}|^2 = a^2$. Some geometric and trigonometric problems can be solved easily using vector methods. The vector equation of the plane in the normal form is expressed using the dot product as $\overline{r}.\hat{n} = p$ and its cartesian form is $lx + my + nz = p$. Also, we deal with the vector equation of the sphere with centre C(c) and radius "a" i.e., $|\overline{r}|^2 - 2\overline{r}.\overline{c} + |\overline{c}|^2 = a^2$.

The science students are aware that certain vector quantities like Moment of force, $(\overline{T} = \overline{r} \times \overline{F})$ angular momentum $(\overline{l} = \overline{r} \times \overline{p})$ are expressed as products of magnitudes of some vector quantities. In Vector Algebra, a generalised treatment is given to such situations in the name of Vector Product or Cross product or Skew product, which is denoted by $\overline{a} \times \overline{b}$ or $[\overline{a}, \overline{b}]$ or $V\overline{a}\overline{b}$ for the vectors $\overline{a}, \overline{b}$. Here, we are forced to raise our thumb into space in order to follow the vector $\overline{a} \times \overline{b}$. An important interpretation of the cross product is that $|\overline{a} \times \overline{b}|$ which gives the area of the parallelogram with $\overline{a}, \overline{b}$ as adjacent sides and it is followed by the concept of vector area.

2. PROOFS ON RESULTS ON VECTOR PRODUCT

1. **Vector Product**:
Let $\bar{a}, \bar{b}$ be two non-zero and non-collinear vectors and θ be the angle between them then the vector product of $\bar{a}, \bar{b}$ is $\bar{a}x\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n}$ where $\hat{n}$ is the unit Pseudo vector perpendicular to the plane containing $\bar{a}$ and $\bar{b}$ and its direction is determined according to the Right Handed Screw Rule.
Thus, $\bar{a}x\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n}$

Eg: Let O be a fixed point and P be a point on the line of action of a force $\bar{F}$ and $\overline{OP} = \bar{r}$ be the P.V of P w.r.to O, θ be the angle between $\bar{r}$ & $\bar{F}$, $\hat{n}$ be the unit vector perpendicular to the plane containing $\bar{r}$ & $\bar{F}$ then the moment of force $\bar{m}$ (Torque or vector moment) about O is the cross product of $\bar{r}$ and $\bar{F}$ i.e, moment of force of F about O is $\bar{m} = \bar{r}x\bar{F} = |\bar{r}||\bar{F}|\sin\theta.\hat{n}$.



Rem: If either $\bar{a} = \bar{0}$ (or) $\bar{b} = \bar{0}$ (or) $\bar{a}, \bar{b}$ are collinear vectors then $\bar{a}x\bar{b} = \bar{0}$

Note 1: The cross product of two vectors is a vector quantity which is perpendicular to plane of $\bar{a}, \bar{b}$ such that $\bar{a}, \bar{b}$ and $\bar{a}x\bar{b}$ form a right handed system of vectors. Here, $\bar{a}x\bar{b}$ is perpendicular to both $\bar{a}$ & $\bar{b}$. Hence $\bar{a}x\bar{b}$ is parallel to $\hat{n}$. Also, $\bar{a}, \bar{b}, \hat{n}$ form a right handed system of vectors.

Note 2: $\bar{a}x\bar{b} \neq \bar{b}x\bar{a}$ (the cross product is not commutative). Infact $\bar{a}x\bar{b} = -\bar{b}x\bar{a}$ (anti commutative law) but $|\bar{a}x\bar{b}| = |\bar{b}x\bar{a}|$

Note 3: $\bar{a}x\bar{b} = \bar{0} \Leftrightarrow \bar{a}, \bar{b}$ are collinear or $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$

Note 4: $\bar{a}x\bar{a} = \bar{0}$

Note 5: $(\bar{a}x\bar{b})x\hat{n} = 0$ ($\because$ $\bar{a}x\bar{b}$ and $\hat{n}$ are parallel)

Note 6: $(\bar{a}x\bar{b}).\bar{a} = 0, (\bar{a}x\bar{b}).\bar{b} = 0$, ($\because$ $\bar{a}x\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$)

Note 7: $\bar{a}$ is perpendicular to $\bar{b} \Leftrightarrow \bar{a}x\bar{b} = |\bar{a}||\bar{b}|\hat{n} \Leftrightarrow |\bar{a}x\bar{b}| = |\bar{a}||\bar{b}|$

Note 8: $(l\bar{a})x(m\bar{b}) = lm(\bar{a}x\bar{b})$ for the scalars $l, m \in R$.

***Note 9**: $|\bar{a}x\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$. ($\because$ $|\sin\theta| = \sin\theta$ for $0 \le \theta \le 180^0$ and $|\hat{n}| = 1$)

Note 10: $|\bar{a}x\bar{b}| \le |\bar{a}||\bar{b}|$ ($\because$ $|\sin\theta| \le 1$)

Note 11: $\bar{a}x(\bar{b}+\bar{c}) = \bar{a}x\bar{b} + \bar{a}x\bar{c}$ (Proof is given in the next chapter. Vide Theorem: 9)

2. **Unit vectors perpendicular to the plane containing a,b**:
From the definition of cross product, $\bar{a}x\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n}$ also $|\bar{a}x\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$.

$$\therefore \hat{n} = \frac{\bar{a}x\bar{b}}{|\bar{a}||\bar{b}|\sin\theta} = \frac{\bar{a}x\bar{b}}{|\bar{a}x\bar{b}|}$$

Note 1: The unit vectors perpendicular to the plane containing $\bar{a}$ and $\bar{b} = \pm\dfrac{\bar{a}x\bar{b}}{|\bar{a}x\bar{b}|}$

Note 2: The vector of magnitude k perpendicular to plane of $\bar{a}, \bar{b}$ = $k\left(\dfrac{\bar{a}x\bar{b}}{|\bar{a}x\bar{b}|}\right)$

3. If θ is the angle between $\bar{a},\bar{b}$ then $\sin\theta = \dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}$ ($\because$ from **Note 9** : $|\bar{a}x\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$)

4. Cross product on the orthonormal traid ($\bar{i},\bar{j},\bar{k}$)

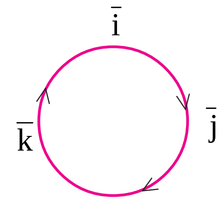
(i) $\bar{i}x\bar{i} = \bar{0}$ ($\because$ $\bar{i},\bar{i}$ are collinear)

(ii) $\bar{i}x\bar{j} = |\bar{i}||\bar{j}|\sin 90^{o}.\bar{k} = 1.1.1.\bar{k} = \bar{k}$

Hence, $\bar{i}x\bar{i} = \bar{0}, \bar{j}x\bar{j} = \bar{0}, \bar{k}x\bar{k} = \bar{0}$

$\bar{i}x\bar{j} = \bar{k}, \bar{j}x\bar{k} = \bar{i}, \bar{k}x\bar{i} = \bar{j}$

and $\bar{j}x\bar{i} = -\bar{k}, \bar{k}x\bar{j} = -\bar{i}, \bar{i}x\bar{k} = -\bar{j}$



5. Analytic expression for cross product:

5.1. If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}, \bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ **then** $\bar{a}x\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Proof: $\bar{a}x\bar{b} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})x(b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$

$= (a_1\bar{i})x(b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) + (a_2\bar{j})x(b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) + (a_3\bar{k})x(b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$

$= (a_1\bar{i})x(b_1\bar{i}) + (a_1\bar{i})x(b_2\bar{j}) + (a_1\bar{i})x(b_3\bar{k}) + (a_2\bar{j})x(b_1\bar{i}) + (a_2\bar{j})x(b_2\bar{j}) + (a_2\bar{j})x(b_3\bar{k})$

$+ (a_3\bar{k})x(b_1\bar{i}) + (a_3\bar{k})x(b_2\bar{j}) + (a_3\bar{k})x(b_3\bar{k})$

$= a_1b_1(\bar{i}x\bar{i}) + a_1b_2(\bar{i}x\bar{j}) + a_1b_3(\bar{i}x\bar{k}) + a_2b_1(\bar{j}x\bar{i}) + a_2b_2(\bar{j}x\bar{j})$

$+ a_2b_3(\bar{j}x\bar{k}) + a_3b_1(\bar{k}x\bar{i}) + a_3b_2(\bar{k}x\bar{j}) + a_3b_3(\bar{k}x\bar{k})$

$= a_1b_1(\bar{0}) + a_1b_2(\bar{k}) + a_1b_3(-\bar{j}) + a_2b_1(-\bar{k}) + a_2b_2(\bar{0}) + a_3b_1(\bar{j}) + a_2b_3(\bar{i}) + a_3b_2(-\bar{i}) + a_3b_3(\bar{0})$

$= \bar{i}(a_2b_3 - a_3b_2) - \bar{j}(a_1b_3 - a_3b_1) + \bar{k}(a_1b_2 - a_2b_1) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

5.2. If $\bar{r}$ **is any vector then** $\bar{i}\times(\bar{r}\times\bar{i}) + \bar{j}\times(\bar{r}\times\bar{j}) + \bar{k}\times(\bar{r}\times\bar{k}) = 2\bar{r}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\bar{i}\times(\bar{r}\times\bar{i}) = \bar{i}\times((x\bar{i} + y\bar{j} + z\bar{k})\times\bar{i})$

$= \bar{i}\times(x(\bar{i}\times\bar{i}) + y(\bar{j}\times\bar{i}) + z(\bar{k}\times\bar{i})) = \bar{i}\times(x(\bar{0}) + y(-\bar{k}) + z(\bar{j}))$

$= \bar{i}\times(-y\bar{k} + z\bar{j})$

$= (-y)(\bar{i}\times\bar{k}) + z(\bar{i}\times\bar{j}) = (-y)(-\bar{j}) + z(\bar{k}) = y\bar{j} + z\bar{k}$

similarly, the other symmetric expressions give $\bar{j}\times(\bar{r}\times\bar{j}) = z\bar{k} + x\bar{i}$ and $\bar{k}\times(\bar{r}\times\bar{k}) = x\bar{i} + y\bar{j}$

$\therefore$ $\bar{i}\times(\bar{r}\times\bar{i}) + \bar{j}\times(\bar{r}\times\bar{j}) + \bar{k}\times(\bar{r}\times\bar{k}) = (y\bar{j} + z\bar{k}) + (z\bar{k} + x\bar{i}) + (x\bar{i} + y\bar{j}) = 2(x\bar{i} + y\bar{j} + z\bar{k}) = 2\bar{r}$.

6. Inter relations between cross product and dot product:

6.1. If $\bar{a},\bar{b}$ **are any two vectors then** $|\bar{a}x\bar{b}|^2 + (\bar{a}.\bar{b})^2 = |\bar{a}|^2|\bar{b}|^2$ **(or)** $|\bar{a}x\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - (\bar{a}.\bar{b})^2 = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} \\ \bar{a}.\bar{b} & \bar{b}.\bar{b} \end{vmatrix}$

Proof: Let θ be the angle between $\bar{a}$ & $\bar{b}$

then $\bar{a}x\bar{b} = |\bar{a}||\bar{b}|\sin\theta\hat{n} \Rightarrow \bar{a}x\bar{b} = |\bar{a}||\bar{b}|\sin\theta$ ($\because$ $|\hat{n}| = 1$)

Also $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|\cos\theta$.

$\therefore$ $|\bar{a}x\bar{b}|^2 + (\bar{a}.\bar{b})^2 = |\bar{a}|^2|\bar{b}|^2\sin^2\theta + |\bar{a}|^2|\bar{b}|^2\cos^2\theta = |\bar{a}|^2|\bar{b}|^2(\sin^2\theta + \cos^2\theta) = |\bar{a}|^2|\bar{b}|^2$

Hence $|\bar{a}x\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - (\bar{a}.\bar{b})^2 = (\bar{a}.\bar{a})(\bar{b}.\bar{b}) - (\bar{a}.\bar{b})(\bar{a}.\bar{b}) = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} \\ \bar{a}.\bar{b} & \bar{b}.\bar{b} \end{vmatrix}$

6.2. If $|\bar{a}x\bar{b}|=|\bar{a}.\bar{b}|$ then $\langle \bar{a}.\bar{b} \rangle=45^o$

Proof: $|\bar{a}x\bar{b}|=|\bar{a}.\bar{b}|$

$\Rightarrow |\bar{a}||\bar{b}|\sin\theta = |\bar{a}||\bar{b}||\cos\theta| \Rightarrow \sin\theta=|\cos\theta| \Rightarrow \theta=45^o$.

7. If $\bar{a},\bar{b},\bar{c}$ are the sides of a triangle then $\bar{a}x\bar{b}=\bar{b}x\bar{c}=\bar{c}x\bar{a}$.

Proof: $\bar{a},\bar{b},\bar{c}$ are the sides of ΔABC

$\Rightarrow \bar{a}+\bar{b}+\bar{c}=\bar{0}$

$\Rightarrow \bar{a}\times(\bar{a}+\bar{b}+\bar{c})=\bar{a}\times\bar{0}$

$\Rightarrow \bar{a}\times\bar{a}+\bar{a}\times\bar{b}+\bar{a}\times\bar{c}=\bar{0}\Rightarrow\bar{0}+\bar{a}\times\bar{b}+\bar{a}\times\bar{c}=\bar{0}$

$\Rightarrow \bar{a}\times\bar{b}+\bar{a}\times\bar{c}=\bar{0}\Rightarrow\bar{a}\times\bar{b}=-(\bar{a}\times\bar{c})\Rightarrow\bar{a}\times\bar{b}=\bar{c}\times\bar{a}$ .....(1)

Again $\bar{a}+\bar{b}+\bar{c}=\bar{0}\Rightarrow\bar{b}\times(\bar{a}+\bar{b}+\bar{c})=\bar{b}\times\bar{0}$

$\Rightarrow \bar{b}\times\bar{a}+\bar{b}\times\bar{b}+\bar{b}\times\bar{c}=\bar{0}$

$\Rightarrow \bar{b}\times\bar{a}+\bar{0}+\bar{b}\times\bar{c}=\bar{0}\Rightarrow\bar{b}\times\bar{c}=-\bar{b}\times\bar{a}$

$\Rightarrow \bar{b}\times\bar{c}=\bar{a}\times\bar{b}$ ....(2)

from (1) & (2) $\bar{a}\times\bar{b}=\bar{b}\times\bar{c}=\bar{c}\times\bar{a}$

8. Geometrical Intepretation of Cross product (Areas, Vector Areas):

Vector Area: Let A be the area of the region enclosed by a plane curve C having anti clockwise orientation and if $\hat{n}$ is the unit vector perpendiuclar to the plane of curve in the right handed system then A$\hat{n}$ is called the vector area of the plane region bounded by the curve C.

Note: The magnitude of vector area of a plane curve is equal to the area of the curve and its direction is perpendicular to the plane of the curve.

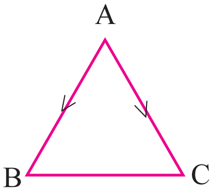
Theorem 1.1: The vector Area of ΔABC is $\frac{1}{2}(\overline{AB}\times\overline{AC})$

Proof: Let $\hat{n}$ be the unit vector perpendicular to the plane of A,B,C such that $\overline{AB},\overline{AC}$ and $\hat{n}$ form a right handed system of vectors.

Now, vector area of $\Delta ABC=(\text{Area of }\Delta ABC)\hat{n}=\frac{1}{2}(AB.AC)\sin A(\hat{n})$

$=\frac{1}{2}|\overline{AB}||\overline{AC}|\sin A(\hat{n})$

$=\frac{1}{2}(\overline{AB}\times\overline{AC})$



Corollory1 : The Area of ΔABC is $\Delta=\frac{1}{2}|\overline{AB}x\overline{AC}|$

Also, the area of ΔABC is $\Delta=\frac{1}{2}|\overline{AB}\times\overline{AC}|=\frac{1}{2}|\overline{BC}\times\overline{BA}|=\frac{1}{2}|\overline{CA}\times\overline{CB}|$

Corollory2: If A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are the vertices of ΔABC then the area of the triangle is

$$\Delta=\frac{1}{2}|\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a}|$$

Condition for collinearity of 3 points A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) :

Corollary 3: The points A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are collinear $\Leftrightarrow$ $\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}=\bar{0}$ i.e., $\overline{AB}\times\overline{AC}=\bar{0}$

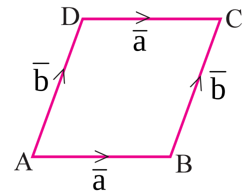
Proof: If A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are collinear then vector area of ΔABC is $\bar{0}$ and vice versa.

Hence, $\overline{AB}\times\overline{AC}=\bar{0}$ or $\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}=\bar{0}$

Theorem 2: If ABCD is a parallelogram such that $\overline{AB}=\overline{a},\overline{BC}=\overline{b}$ then its vector area is $\overline{a}x\overline{b}$.

Proof: Vector area of the parallelogram ABCD

$$= 2(\text{Vector Area of } \Delta ABD) = 2\left(\frac{1}{2}(\overline{AB}x\overline{AD})\right) = \overline{AB}x\overline{AD} = \overline{a}x\overline{b}$$



Geometrical interpretation of cross product:

Corollary: Area of the parallelogram with adjacent sides represented by $\overline{a},\overline{b}$ is $|\overline{a}x\overline{b}|$

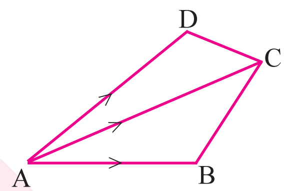
Theorem 3:If AC,BD are the diagonals of a quadrilateral ABCD then its vector Area is $\frac{1}{2}(\overline{AC}x\overline{BD})$

Proof: Vector area of the quadrilateral ABCD=Vector area of ΔABC+vector area of ΔACD.

$$=\frac{1}{2}(\overline{AB}\times\overline{AC})+\frac{1}{2}(\overline{AC}\times\overline{AD})$$

$$=\frac{1}{2}(\overline{AC}\times(-\overline{AB})+\overline{AC}\times\overline{AD})=\frac{1}{2}\left((\overline{AC}\times\overline{BA})+(\overline{AC}\times\overline{AD})\right)$$

$$=\frac{1}{2}(\overline{AC}\times(\overline{BA}+\overline{AD}))=\frac{1}{2}(\overline{AC}\times\overline{BD})$$



Corollary: The area of the quadrilateral ABCD with diagonals AC, BD is $\frac{1}{2}|\overline{AC}x\overline{BD}|$

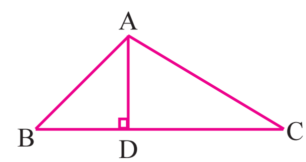
Note: If $\overline{d}_1,\overline{d}_2$ are the diagonals of a quadrilateral (or parallelogram) then its area is $\frac{1}{2}|\overline{d}_1\times\overline{d}_2|$

8. The length of the perpendicular from a point to a line in space:

Consider ΔABC and AD be the length of the perpendicular from A on to BC

Now, Area of ΔABC $=\frac{1}{2}(\text{base})(\text{height})=\frac{1}{2}|\overline{BC}||\overline{AD}|$

$$\Rightarrow|\overline{AD}|=\frac{2(\text{Area of }\Delta ABC)}{|\overline{BC}|}=\frac{2.\frac{1}{2}|\overline{BC}\times\overline{BA}|}{|\overline{BC}|}=\frac{|\overline{BC}\times\overline{BA}|}{|\overline{BC}|}$$

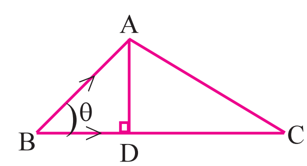


Ex: If B=(2,1,–2), C=(0,–5,1) then find the perpendicular distance from A(1,4,–2) to BC.

Sol: Let $\overline{OA}=\overline{i}+4\overline{j}-2\overline{k},\overline{OB}=2\overline{i}+\overline{j}-2\overline{k},\overline{OC}=-5\overline{j}+\overline{k}$

Now, $\overline{BA}=\overline{OA}-\overline{OB}=(\overline{i}+4\overline{j}-2\overline{k})-(2\overline{i}+\overline{j}-2\overline{k})-(-\overline{i}+3\overline{j})$

$\overline{BC}=\overline{OC}-\overline{OB}=(-5\overline{j}+\overline{k})-(2\overline{i}+\overline{j}-2\overline{k})=-2\overline{i}-6\overline{j}+3\overline{k}$



From ΔABD, $\sin\theta=\frac{|\overline{AD}|}{|\overline{BA}|}$ ......(1)

By the definition of cross product, $\sin\theta=\frac{|\overline{BA}\times\overline{BC}|}{|\overline{BA}||\overline{BC}|}$ .....(2)

From (1) & (2) $\frac{|\overline{AD}|}{|\overline{BA}|}=\frac{|\overline{BA}\times\overline{BC}|}{|\overline{BA}||\overline{BC}|}\Rightarrow|\overline{AD}|=\frac{|\overline{BA}\times\overline{BC}|}{|\overline{BC}|}$ ....(3)

Now, $\overline{BA}\times\overline{BC}=\begin{vmatrix}\overline{i}&\overline{j}&\overline{k}\\-1&3&0\\-2&-6&3\end{vmatrix}=\overline{i}(9-0)-\overline{j}(-3-0)+\overline{k}(6+6)=3(3\overline{i}+\overline{j}+4\overline{k})$

$\Rightarrow|\overline{BA}\times\overline{BC}|=3\sqrt{3^2+1^2+4^2}=3\sqrt{26}$

Also, $|\overline{BC}|=\sqrt{(-2)^2+(-6)^2+3^2}=\sqrt{49}=7$ $\therefore|\overline{AD}|=\frac{3\sqrt{26}}{7}$ [from (3)]

3.PRODUCT OF THREE, FOUR VECTORS

3. INTRODUCTION PAGE

Sections	No. of periods (8 to 9)	Weightage in IPE [1x4 or 1x7]
1. Scalar Triple product, Skew lines vector equations of planes	5	4 or 7 Marks
2. Vector Triple product	2	4 or 7 Marks
3. Products of Four vectors	1	4 or 7 Marks

In the previous chapter, we have learned two types of products viz., Scalar Product(.) and Vector Product (x) *of two vectors* a,b. *If we consider three vectors* a,b,c *to be operated with and then 8 products of various combinations are possible viz.,* $(a.b).c$, $a.(b.c)$, $(a.b)xc$, $a.(bxc)$, $(axb).c$, $ax(b.c)$, $(axb)xc$, $ax(bxc)$. *But, here it is evident that the products* $(a.b).c$, $a.(b.c)$, $(a.b)xc$, $ax(b.c)$ *are meaningless and* $(axb).c$, $a.(bxc)$ *are scalars where as* $(axb)xc$, $ax(bxc)$ *are vectors. Hence,* $a.(bxc)$ *or* $(axb).c$ *are called Scalar Triple Products and* $(axb)xc$, $ax(bxc)$ *are called Vector Triple Products.*

The Scalar Triple Product (S.T.P) of a,b,c *i.e.,* $a.(bxc)$ *or* $(axb).c$, *denoted by* $[a,b,c]$ *is called box product and the magnitude of its value gives the volume of the parallelopiped with* a,b,c *as its coterminous edges.*

The condition for complanarity of three vectors is determined using the box product of three vectors; i.e., a,b,c are coplanar $\Leftrightarrow [a,b,c]=0$.

A special case of two lines in space is, non-intersecting and non-parallel lines called Skew Lines. The shortest distance between two skew lines can be determined using S.T.P.

Once again, the vector equation of planes are treated here in the S.T.P mode. Here, we discuss about the vector equation of plane passing through (i) three points (ii) two points and parallel to a vector (iii) a point and parallel to two vectors. In this context, a formula to find the perpendicular distance from the origin to a plane is derived.

In section-2, Vector Triple Product is defined and it is expressed interms of dot product.

With four vectors a,b,c,d *and 2 operations . and x, we consider two useful and meaningful operations viz.,* $(axb).(cxd)$ *(Scalar Product of four vectors) and* $(axb)x(cxd)$ *(Vector Product of four vectors).*

4. PROOFS OF RESULTS ON STP

SCALAR TRIPLE PRODUCT

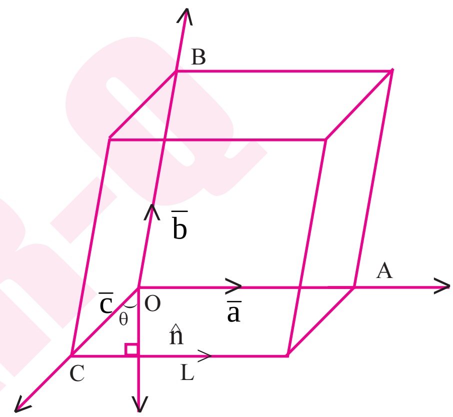
1. Def: If $\bar{a},\bar{b},\bar{c}$ are any 3 vectors then the product of the type $(\bar{a}x\bar{b}).\bar{c}$ is called scalar triple product of the vectors $\bar{a},\bar{b},\bar{c}$ and it is denoted by $[\bar{a}\,\bar{b}\,\bar{c}]$ or $(\bar{a},\bar{b},\bar{c})$ or $[\bar{a},\bar{b},\bar{c}]$.

Thus, $[\bar{a}\,\bar{b}\,\bar{c}] = (\bar{a}x\bar{b}).\bar{c}$

2. Geometrical Interpretation:

Theorem 1: **If $\bar{a},\bar{b},\bar{c}$ are 3 non-zero non coplanar vectors and V is the volume of the parallelopiped with $\bar{a},\bar{b},\bar{c}$ as coterminus edges then (i) $(\bar{a}x\bar{b}).\bar{c}$ =V if $\bar{a},\bar{b},\bar{c}$ form a right handed system of vectors (ii) $(\bar{a}x\bar{b}).\bar{c}$ =–V if $\bar{a},\bar{b},\bar{c}$ form a left handed system of vectors.**

Proof: Let $\bar{a}=\overline{OA},\bar{b}=\overline{OB},\bar{c}=\overline{OC}$ and $\hat{n}$ be the unit vector perpendicular to the plane generated by $\bar{a},\bar{b}$ such that $\bar{a},\bar{b}$, $\hat{n}$ form a right handed system of vector. Also $\bar{a},\bar{b},\bar{c}$ form a right handed system of vectors ⇒ $\bar{c}$, $\hat{n}$ lie on the same side of the plane determined by $\bar{a},\bar{b}$ such that $\theta=\langle\bar{c},\hat{n}\rangle<90^{0}$. Let L be the foot of the perpendicular from O on to the face opposite the plane of $\bar{a},\bar{b}$.



We know that the volume (V) of the parallelopiped determined by the coterminous edges $\bar{a},\bar{b},\bar{c}$ is equal to the product of area of the parallelogram with a,b as adjacent sides and $|\overline{OL}|$.

Now consider $(\bar{a}\times\bar{b}).\bar{c}=|\bar{a}\times\bar{b}||\bar{c}|\cos\theta$ $(\because \theta=\langle\bar{c},\hat{n}\rangle$ and $\hat{n}\,||\,\bar{a}\times\bar{b}\Rightarrow(\bar{a}\times\bar{b},\bar{c})=\theta)$

$$=|\bar{a}\times\bar{b}||\overline{OL}|\left(\because \text{from } \Delta OCL, \cos\theta=\frac{|\overline{OC}|}{|\overline{OL}|}\Rightarrow|\overline{OC}|\cos\theta=|\overline{OL}|\right)$$

$= V$

Here $(\bar{a}x\bar{b}).\bar{c}$ =V is positive (i.e, $\bar{a},\bar{b},\bar{c}$ form a right handed system of vectors)

If $\bar{a},\bar{b},\bar{c}$ form a left handed system of vectors then V= $(\bar{b}x\bar{a}).\bar{c}=-(\bar{a}x\bar{b}).\bar{c}$

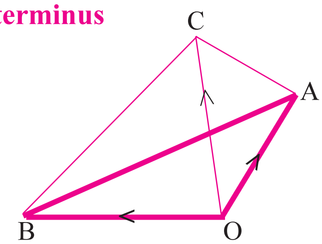
$\therefore$ $(\bar{a}x\bar{b}).\bar{c}$ =–V.

But volume is a non negative quantity hence V= $|[\bar{a}\,\bar{b}\,\bar{c}]|$

Conclusions: (i) If $(\bar{a}x\bar{b}).\bar{c}$ is positive then $\bar{a},\bar{b},\bar{c}$ form a right handed system of vectors.

(ii) If $(\bar{a}x\bar{b}).\bar{c}$ is negative then $\bar{a},\bar{b},\bar{c}$ form a left handed system of vectors.

Theorem 2: **The volume of the tetrahedron with $\bar{a},\bar{b},\bar{c}$ as co-terminus edges is $\frac{1}{6}|[\bar{a}\,\bar{b}\,\bar{c}]|$**



Proof: Let $\overline{OA}=\bar{a},\overline{OB}=\bar{b},\overline{OC}=\bar{c}$ be the co-terminus edges of a tetrahedron OABC such that $[\bar{a}\ \bar{b}\ \bar{c}]$ is a right handed vector traid.

The volume V of the tetrahedron OABC is

$$V=\frac{1}{3}(\text{Area of }\Delta OAB)\text{ x (length of the perpendicular from C to the plane OAB)}$$

$$\text{Area of }\Delta OAB=\frac{1}{2}(\bar{a}\times\bar{b})$$

Length of the perpendicular from C to the plane OAB=Projection of C in the direction of

$$\bar{a}\times\bar{b}=\frac{(\bar{a}\times\bar{b}).\bar{c}}{|\bar{a}\times\bar{b}|}=\frac{[\bar{a}\bar{b}\bar{c}]}{|\bar{a}\times\bar{b}|}$$

$$\therefore V=\frac{1}{3}\left(\frac{1}{2}|\bar{a}\times\bar{b}|\frac{[\bar{a}\bar{b}\bar{c}]}{|\bar{a}\times\bar{b}|}\right)=\frac{1}{6}|[\bar{a}\bar{b}\bar{c}]|$$

3. Properties of Scalar Triple Product:

3.1.Theorem 3: Three nonzero noncollinear vectors $\bar{a},\bar{b},\bar{c}$ are coplanar iff $[\bar{a}\ \bar{b}\ \bar{c}]=0$

Proof: Suppose that $\bar{a},\bar{b},\bar{c}$ are coplanar.

Given that $\bar{a},\bar{b}$ are noncollinear $\Rightarrow$ $\bar{c}$ lies in the plane generated by $\bar{a},\bar{b}$ and hence $\bar{c}$ is perpendicular to $\bar{a}x\bar{b}$. Thus $(\bar{a}x\bar{b}).\bar{c}=0\Rightarrow[\bar{a}\ \bar{b}\ \bar{c}]=0$.

Conversely, suppose that $[\bar{a}\ \bar{b}\ \bar{c}]=0$.

Given that $\bar{a},\bar{b}$ are nonzero and noncollinear vectors $\Rightarrow$ $\bar{a}x\bar{b}\neq\bar{0}$.

But $[\bar{a}\ \bar{b}\ \bar{c}]=0\Rightarrow(\bar{a}x\bar{b}).\bar{c}=0\Rightarrow\bar{a}x\bar{b}$ is perpendicular to $\bar{c}$ ($\because\ \bar{a}x\bar{b}\neq\bar{0},\bar{c}\neq\bar{0}$)

But $\bar{a}x\bar{b}$ is a vector perpendicular to the plane generated by $\bar{a},\bar{b}$ hence it follows that $\bar{c}$ lies in the plane generated by $\bar{a},\bar{b}$. Thus the vectors $\bar{a},\bar{b},\bar{c}$ are coplanar.

Note: If $[\bar{a}\,\bar{b}\,\bar{c}]=0$ then $\bar{a},\bar{b},\bar{c}$ are coplanar (or) any two of $\bar{a},\bar{b},\bar{c}$ are parallel (or) any one of $\bar{a},\bar{b},\bar{c}$ is a zero vector.

3.2. Theorem 4: If $\bar{a},\bar{b},\bar{c}$ are vectors then $(\bar{a}x\bar{b}).\bar{c}=(\bar{b}x\bar{c}).\bar{a}=(\bar{c}x\bar{a}).\bar{b}$.

Proof: If $\bar{a}=\bar{0}$ or $\bar{b}=\bar{0}$ or $\bar{c}=\bar{0}$ then it is clear that $(\bar{a}x\bar{b}).\bar{c}=(\bar{b}x\bar{c}).\bar{a}=(\bar{c}x\bar{a}).\bar{b}=0$

Let $a\neq\bar{0},b\neq\bar{0},\bar{c}\neq\bar{0}$ and suppose that $\bar{a},\bar{b},\bar{c}$ are three coplanar vectors.

Then the planes generated by $\bar{a}\ \&\ \bar{b};\bar{b}\ \&\ \bar{c};\bar{c}\ \&\ \bar{a}$ coincide.

Let the plane containing the vectors $\bar{a},\bar{b},\bar{c}$ be denoted by π

Then $\bar{a}x\bar{b},\bar{b}x\bar{c},\bar{c}x\bar{a}$ are perpendicular to the plane π and hence $(\bar{a}x\bar{b}).\bar{c}=0$

$\therefore\ (\bar{a}x\bar{b}).\bar{c}=(\bar{b}x\bar{c}).\bar{a}=(\bar{c}x\bar{a}).\bar{b}=0$

Suppose $\bar{a},\bar{b},\bar{c}$ are three non coplanar vectors.

Let V be the volume of the parallelopiped with $\bar{a},\bar{b},\bar{c}$ as coterminous edges.

Suppose $\bar{a},\bar{b},\bar{c}$ form a right handed system.

Then $(\bar{a}x\bar{b}).\bar{c}=V$

Since $\bar{a},\bar{b},\bar{c}$ form a right handed system, $\bar{b},\bar{c},\bar{a}$ and $\bar{c},\bar{a},\bar{b}$ also form right handed systems.

$\therefore\ (\bar{b}x\bar{c}).\bar{a}=V,(\bar{c}x\bar{a}).\bar{b}=V$

$\therefore\ (\bar{a}x\bar{b}).\bar{c}=(\bar{b}x\bar{c}).\bar{a}=(\bar{c}x\bar{a}).\bar{b}=V$

Similarly if $\bar{a},\bar{b},\bar{c}$ form left handed system then we can s.t $(\bar{a}x\bar{b}).\bar{c}=(\bar{b}x\bar{c}).\bar{a}=(\bar{c}x\bar{a}).\bar{b}=-V$

3.3. Corollary: If $\bar{a},\bar{b},\bar{c}$ are vectors, then $(\bar{a}x\bar{b}).\bar{c}=\bar{a}.(\bar{b}x\bar{c})$ (that is '.' and 'x' are interchangeable)

Proof: $(\bar{a}x\bar{b}).\bar{c}=(\bar{b}x\bar{c}).\bar{a}=\bar{a}.(\bar{b}x\bar{c})$ [Since, dot product is commutative]

3.4. Theorem 5: If $\bar{a},\bar{b},\bar{c}$ are three vectors, l,m,n are three real numbers then $[l\bar{a},m\bar{b},n\bar{c}]=lmn[\bar{a},\bar{b},\bar{c}]$

Proof: $[l\bar{a},m\bar{b},n\bar{c}]=(l\bar{a}\,x\,m\bar{b}).n\bar{c}=lm(\bar{a}x\bar{b}).n\bar{c}$

$=lmn\{(\bar{a}x\bar{b}).\bar{c}\}=mn[\bar{a},\bar{b},\bar{c}]$

3.5. Theorem 6: If $\bar{a},\bar{b},\bar{c},\bar{d}$ are any 4 vectors then $[\bar{a}+\bar{b},\bar{c},\bar{d}]=[\bar{a},\bar{c},\bar{d}]+[\bar{b},\bar{c},\bar{d}]$

Proof: $[\bar{a}+\bar{b},\bar{c},\bar{d}]=((\bar{a}+\bar{b})x\bar{c}).\bar{d}=(\bar{a}x\bar{c}+\bar{b}x\bar{c}).\bar{d}$

$=(\bar{a}x\bar{c}).\bar{d}+(\bar{b}x\bar{c}).\bar{d}$

$=[\bar{a},\bar{c},\bar{d}]+[\bar{b},\bar{c},\bar{d}]$

3.6. Scalar triple product on the orthonormal traid $(\bar{i},\bar{j},\bar{k})$

$(\bar{i}x\bar{j}).\bar{k}=\bar{k}.\bar{k}=1$

Hence $[\bar{i}\ \bar{j}\ \bar{k}]=1=[\bar{j}\ \bar{k}\ \bar{i}]=[\bar{k}\ \bar{i}\ \bar{j}]=-[\bar{j}\ \bar{i}\ \bar{k}]=-[\bar{i}\ \bar{k}\ \bar{j}]=-[\bar{k}\ \bar{j}\ \bar{i}]$

3.7. Theorem 7: If $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k},\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k},\bar{c}=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}$ **then** $[\bar{a}\,\bar{b}\,\bar{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

Proof: $\bar{b}\,x\,\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\bar{i}(b_2c_3-b_3c_2)-\bar{j}(b_1c_3-b_3c_1)+\bar{k}(b_1c_2-b_2c_1)$

$\bar{a}.(\bar{b}x\bar{c})=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

3.8. Result: If $\bar{a}=a_1\bar{p}+a_2\bar{q}+a_3\bar{r},\bar{b}=b_1\bar{p}+b_2\bar{q}+b_3\bar{r},\bar{c}=c_1\bar{p}+c_2\bar{q}+c_3\bar{r}$ where $\bar{p},\bar{q},\bar{r}$ form a right handed system of noncoplanar vectors, then $[\bar{a}\ \bar{b}\ \bar{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}[\bar{p}\ \bar{q}\ \bar{r}]$

3.9. Theorem 8: If $\bar{a},\bar{b},\bar{c}$ are three vectors, then (i) $\bar{a}x(\bar{b}+\bar{c})=(\bar{a}x\bar{b})+(\bar{a}x\bar{c})$

(ii) $(\bar{b}+\bar{c})x\bar{a}=(\bar{b}x\bar{a})+(\bar{c}x\bar{a})$

Proof: (i) If $\bar{r}$ is any vector, then $\bar{r}.\{\bar{a}x(\bar{b}+\bar{c})\}=(\bar{r}x\bar{a}).(\bar{b}+\bar{c})$ $(\because \bar{a}.(\bar{b}x\bar{c})=(\bar{a}x\bar{b}).\bar{c})$

$=(\bar{r}x\bar{a}).\bar{b}+(\bar{r}x\bar{a}).\bar{c}=\bar{r}.(\bar{a}x\bar{b})+\bar{r}.(\bar{a}x\bar{c})$

$\Rightarrow \bar{r}.\{\bar{a}x(\bar{b}+\bar{c})\}-\bar{r}.(\bar{a}x\bar{b})-\bar{r}(\bar{a}x\bar{c})=0$

$\Rightarrow \bar{r}.\{\bar{a}x(\bar{b}+\bar{c})-\bar{a}x\bar{b}-\bar{a}x\bar{c}\}=0 \Rightarrow \bar{a}x(\bar{b}+\bar{c})-\bar{a}x\bar{b}-\bar{a}x\bar{c}=0$

$\Rightarrow \bar{a}x(\bar{b}+\bar{c})=(\bar{a}x\bar{b})+(\bar{a}x\bar{c})$

(ii) $(\bar{b}+\bar{c})x\bar{a}=-\bar{a}x(\bar{b}+\bar{c})=-[(\bar{a}x\bar{b})+(\bar{a}x\bar{c})]$

$=-\bar{a}x\bar{b}-\bar{a}x\bar{c}=(\bar{b}x\bar{a})+(\bar{c}x\bar{a})$

5. PROOFS OF RESULTS ON VE OF PLANES IN STP

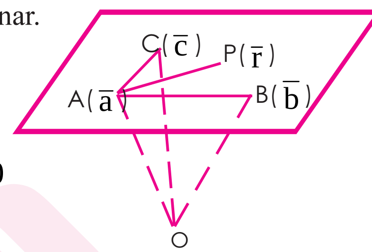
VECTOR EQUATIONS OF PLANES (INTERMS OF STP)

Theorem 1: **The vector equation of the plane passing through the points A($\bar{a}$),B($\bar{b}$),C($\bar{c}$) is** $[\bar{r}\ \bar{a}\ \bar{b}]+[\bar{r}\ \bar{b}\ \bar{c}]+[\bar{r}\ \bar{c}\ \bar{a}]=[\bar{a}\ \bar{b}\ \bar{c}]$ **(or)** $\bar{r}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})=[\bar{a}\ \bar{b}\ \bar{c}]$.

Proof: Let P be any point on the plane of ABC and O be the origin of reference then

$\overline{OA}=\bar{a},\overline{OB}=\bar{b},\overline{OC}=\bar{c}$ and $\overline{OP}=\bar{r}$.

The condition to be satisfied by P is that P,A,B,C are coplanar.



$\Rightarrow\ \overline{AP},\overline{AB},\overline{AC}$ are coplanar. $\Rightarrow[\overline{AP},\overline{AB},\overline{AC}]=0$

$\Rightarrow[\overline{OP}-\overline{OA},\overline{OB}-\overline{OA},\overline{OC}-\overline{OA}]=0$

$\Rightarrow[\bar{r}-\bar{a},\bar{b}-\bar{a},\bar{c}-\bar{a}]=0\Rightarrow(\bar{r}-\bar{a}).(\bar{b}-\bar{a})x(\bar{c}-\bar{a})=0$

$\Rightarrow(\bar{r}-\bar{a}).(\bar{b}x\bar{c}-\bar{b}x\bar{a}-\bar{a}x\bar{c}+\bar{a}x\bar{a})=0$

$\Rightarrow(\bar{r}-\bar{a}).(\bar{b}x\bar{c}+\bar{a}x\bar{b}+\bar{c}x\bar{a}+\bar{0})=0$

$\Rightarrow(\bar{r}-\bar{a}).(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})=0$

$\Rightarrow\bar{r}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})-\bar{a}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})=0$

$\Rightarrow\bar{r}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})-[(\bar{a}.\bar{a}x\bar{b})+\bar{a}.(\bar{b}x\bar{c})+\bar{a}.(\bar{c}x\bar{a})]=0$

$\Rightarrow\bar{r}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})-([\bar{a}\ \bar{a}\ \bar{b}]+[\bar{a}\ \bar{b}\ \bar{c}]+[\bar{a}\ \bar{c}\ \bar{a}])=0$

$\Rightarrow\bar{r}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})-(0+[\bar{a}\ \bar{b}\ \bar{c}]+0)=0$

$\Rightarrow\bar{r}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})-[\bar{a}\ \bar{b}\ \bar{c}]=0$

$\Rightarrow\bar{r}.(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})=[\bar{a}\ \bar{b}\ \bar{c}]$ ....(1)

$\Rightarrow[\bar{r}\ \bar{a}\ \bar{b}]+[\bar{r}\ \bar{b}\ \bar{c}]+[\bar{r}\ \bar{c}\ \bar{a}]=[\bar{a}\ \bar{b}\ \bar{c}]$

Corollary: The perpendicular distance from the origin to the plane passing through the points

A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) is $\dfrac{|[\bar{a}\ \bar{b}\ \bar{c}]|}{|(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})|}$

Proof: Let us reduce (1) into the equation of the plane in the normal form.

Dividing (1) by $|\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a}|$, we get

$$r.\frac{(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})}{|(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})|}=\frac{[\bar{a}\ \bar{b}\ \bar{c}]}{|(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})|}$$

, which is in the normal form $\bar{r}.\hat{n}=p$

$\therefore$ the perpendicular distance from the origin to the plane is $\dfrac{|[\bar{a}\ \bar{b}\ \bar{c}]|}{|(\bar{a}x\bar{b}+\bar{b}x\bar{c}+\bar{c}x\bar{a})|}$

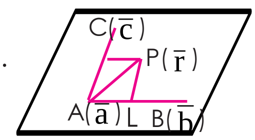
Note: The cartesian equation of the plane passing through the points A(a_1,a_2,a_3), B(b_1,b_2,b_3), C(c_1,c_2,c_3) is given by $[\overline{AP},\overline{AB},\overline{AC}]=0\Rightarrow\begin{vmatrix}x-a_1 & y-a_2 & z-a_3\\ b_1-a_1 & b_2-a_2 & b_3-a_3\\ c_1-a_1 & c_2-a_2 & c_3-a_3\end{vmatrix}=0$

Theorem 2: The vector equation of the plane passing through $A(\bar{a})$ & parallel to $\bar{b}, \bar{c}$ is $[\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$

Proof: Let P be an point on the plane passing through $A(\bar{a})$ and parallel to $\bar{b}, \bar{c}$

Let O be the origin of reference such that $\overline{OP} = \bar{r}, \overline{OA} = \bar{a}$.

The point P on the plane is such that $\overline{AP}, \bar{b}, \bar{c}$ are on parallel planes.



$\Rightarrow [\overline{AP}, \bar{b}, \bar{c}] = 0 \Rightarrow [\overline{OP} - \overline{OA}, \bar{b}, \bar{c}] = 0$

$\Rightarrow [\bar{r} - \bar{a}, \bar{b}, \bar{c}] = 0 \Rightarrow [\bar{r}\ \bar{b}\ \bar{c}] - [\bar{a}\ \bar{b}\ \bar{c}] = 0$

$\Rightarrow [\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$

Note: The Cartesian equation of the plane passing through the point $A(a_1,a_2,a_3)$ and parallel to the vectors $\bar{b} = (b_1,b_2,b_3)$, $\bar{c} = (c_1,c_2,c_3)$ is given by $[\overline{AP}, \bar{b}, \bar{c}] = 0 \Rightarrow \begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

Remark: The vector equation of the plane containing the line $\bar{r} = \bar{a} + t\bar{b}, t \in R$ and per pendicular to the plane $\bar{r}.\bar{c} = q$ is $[\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$

Proof: For the plane $\bar{r}.\bar{c} = q$, the vector $\bar{c}$ is a normal. Since the plane contains the line $\bar{r} = \bar{a} + t\bar{b}$, it passes through the point $\bar{a}$ and is parallel to the vectors $\bar{b}$ and $\bar{c}$.

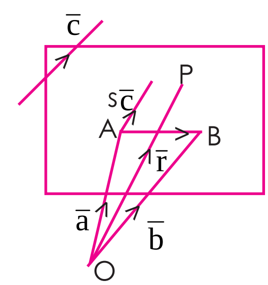
$\therefore$ The vector equation of the plane is $[\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$

Theorem 3: The vector equation of the plane passing through $A(\bar{a})$, $B(\bar{b})$ and parallel to $\bar{c}$ is $[\bar{r}\ \bar{b}\ \bar{c}] + [\bar{r}\ \bar{c}\ \bar{a}] = [\bar{a}\ \bar{b}\ \bar{c}]$ (or) $\bar{r}.(\bar{b}x\bar{c} + \bar{c}x\bar{a}) = [\bar{a}\ \bar{b}\ \bar{c}]$

Proof: Let P be a point on the plane passing through $A(\bar{a})$, $B(\bar{b})$ and parallel to $\bar{c}$

Let O be the origin of reference such that $\overline{OP} = \bar{r}, \overline{OA} = \bar{a}, \overline{OB} = \bar{b}$.

The point P on the plane is such that $\overline{AP}$ and $\overline{AB}$ are parallel to $\bar{c}$



$\Rightarrow [\overline{AP}\ \overline{AB}\ \bar{c}] = 0 \Rightarrow [\overline{OP} - \overline{OA}, \overline{OB} - \overline{OA}, \bar{c}] = 0$

$\Rightarrow [\bar{r} - \bar{a}, \bar{b} - \bar{a}, \bar{c}] = 0 \Rightarrow (\bar{r} - \bar{a}).((\bar{b} - \bar{a})x\bar{c}) = 0$

$\Rightarrow (\bar{r} - \bar{a}).(\bar{b}x\bar{c} - \bar{a}x\bar{c}) = 0$

$\Rightarrow \bar{r}.(\bar{b}x\bar{c} - \bar{a}x\bar{c}) - \bar{a}.(\bar{b}x\bar{c} - \bar{a}x\bar{c}) = 0$

$\Rightarrow \bar{r}.(\bar{b}x\bar{c} + \bar{c}x\bar{a}) - (\bar{a}.(\bar{b}x\bar{c}) + \bar{a}(\bar{a}x\bar{c})) = 0$

$\Rightarrow \bar{r}.(\bar{b}x\bar{c} + \bar{c}x\bar{a}) - ([\bar{a}\ \bar{b}\ \bar{c}] + [\bar{a}\ \bar{a}\ \bar{c}]) = 0$

$\Rightarrow \bar{r}.(\bar{b}x\bar{c} + \bar{c}x\bar{a}) - ([\bar{a}\ \bar{b}\ \bar{c}] + 0) = 0$

$\Rightarrow \bar{r}.(\bar{b}x\bar{c} + \bar{c}x\bar{a}) - [\bar{a}\ \bar{b}\ \bar{c}] = 0$

$\Rightarrow \bar{r}.(\bar{b}x\bar{c} + \bar{c}x\bar{a}) = [\bar{a}\ \bar{b}\ \bar{c}] \Rightarrow [\bar{r}\ \bar{b}\ \bar{c}] + [\bar{r}\ \bar{c}\ \bar{a}] = [\bar{a}\ \bar{b}\ \bar{c}]$

Note : The Cartesian equation of the plane passing through the points $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$ and parallel to the vector $\bar{c} = (c_1,c_2,c_3)$ is given by $[\overline{AP}, \overline{AB}, \bar{c}] = 0 \Rightarrow \begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

6. PROOFS OF RESULTS ON VECTOR TRIPLE PRODUCT

1. Def: If $\bar{a},\bar{b},\bar{c}$ are any 3 vectors then the vectors of the type $(\bar{a}x\bar{b})x\bar{c}$ and $\bar{a}x(\bar{b}x\bar{c})$ are called Vector Triple products of $\bar{a},\bar{b},\bar{c}$.

Note 1: $(\bar{a}x\bar{b})x\bar{c}$ is a vector perpendicular to both $\bar{a}x\bar{b}$ and $\bar{c}$.

Note 2: In general, $(\bar{a}x\bar{b})x\bar{c} \neq \bar{a}x(\bar{b}x\bar{c})$

Note 3: $(\bar{a}x\bar{b})x\bar{c} = \bar{0} \Rightarrow$ (i) $\bar{a}x\bar{b} \parallel \bar{c}$ i.e., $\bar{c}$ is perpendicular to plane containing $\bar{a},\bar{b}$

(ii) $\bar{a},\bar{b},\bar{c}$ are mutually perpendicular

(iii) $\bar{a} \parallel \bar{b}$ when $\bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}, \bar{c} \neq \bar{0}$

(iv) either of $\bar{a},\bar{b},\bar{c}$ is a zero vector.

2. Theorem 1: If $\bar{a},\bar{b},\bar{c}$ are 3 vectors then prove that $(\bar{a}x\bar{b})x\bar{c} = (\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a}$.

Proof: **Case 1**: If $\bar{a}=\bar{0}$ or $\bar{b}=\bar{0}$ or $\bar{c}=\bar{0}$ then it is clear that $(\bar{a}x\bar{b})x\bar{c} = \bar{0} = (\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a}$.

Case 2: Let $\bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}, c \neq \bar{0}$

Sub-case1: Suppose $\bar{a},\bar{b}$ are parallel then $\bar{b} = \lambda\bar{a}$, $\lambda \in R$ and $\bar{a}x\bar{b} = \bar{0} \Rightarrow (\bar{a}x\bar{b})x\bar{c} = \bar{0}$

Now, $(\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a} = (\bar{c}.\bar{a})(\lambda\bar{a}) - [\bar{c}.(\lambda\bar{a})]\bar{a} = \lambda(\bar{c}.\bar{a})\bar{a} - \lambda(\bar{c}.\bar{a})\bar{a} = \bar{0}$

$\therefore (\bar{a}x\bar{b})x\bar{c} = \bar{0} = (\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a}$

Sub-case 2: Suppose $\bar{a},\bar{b}$ are not parallel and $\bar{c}$ is not parallel to $\bar{a} \times \bar{b}$

Without loss of generality, we take $\bar{a} = a_1\bar{i}$, $\bar{b} = b_1\bar{i} + b_2\bar{j}$, $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

Now $\bar{a}x\bar{b} = a_1\bar{i}x(b_1\bar{i} + b_2\bar{j}) = (a_1\bar{i}xb_1\bar{i}) + (a_1\bar{i}xb_2\bar{j})$

$= (a_1b_1)(\bar{i}x\bar{i}) + (a_1b_2)(\bar{i}x\bar{j}) = (a_1b_1)\bar{0} + (a_1b_2)\bar{k} = a_1b_2\bar{k}$

$$\therefore (\bar{a}x\bar{b})x\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & a_1b_2 \\ c_1 & c_2 & c_3 \end{vmatrix} = \bar{i}(0 - c_2(a_1b_2) - \bar{j}(0 - c_1(a_1b_2) + \bar{k}(0) = a_1b_2c_1\bar{j} - a_1b_2c_2\bar{i}$$

Now, $(\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a} = [a_1\bar{i}.(c_1\bar{i} + c_2\bar{j} + c_3\bar{k})](b_1\bar{i} + b_2\bar{j}) - [(b_1\bar{i} + b_2\bar{j}).(c_1\bar{i} + c_2\bar{j} + c_3\bar{k})]a_1\bar{i}$

$= a_1c_1(b_1\bar{i} + b_2\bar{j}) - (b_1c_1 + b_2c_2)a_1\bar{i} = a_1c_1b_1\bar{i} + a_1c_1b_2\bar{j} - b_1c_1a_1\bar{i} - b_2c_2a_1\bar{i} = a_1b_2c_1\bar{j} - a_1b_2c_2\bar{i}$

$\therefore (\bar{a}x\bar{b})x\bar{c} = (\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a}$

Corollary: If $\bar{a},\bar{b},\bar{c}$ are any 3 vectors then prove that $\bar{a}x(\bar{b}x\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$.

Proof: $\bar{a}x(\bar{b}x\bar{c}) = -(\bar{b}x\bar{c})x\bar{a} = -\{(\bar{a}.\bar{b})\bar{c} - (\bar{a}.\bar{c})\bar{b}\}$ (From the above theorem)

$= -(\bar{a}.\bar{b})\bar{c} + (\bar{a}.\bar{c})\bar{b} = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$.

Note: From the above theorem and corollary, it follows that

$(\bar{a}x\bar{b})x\bar{c} = (\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a}$, which is a vector in the plane of $\bar{a},\bar{b}$

$\bar{a}x(\bar{b}x\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$, which is a vector in the plane of $\bar{b},\bar{c}$

3.Theorem 2: Prove that $(\bar{a}x\bar{b})x\bar{c} = \bar{a}x(\bar{b}x\bar{c}) \Rightarrow \bar{a},\bar{c}$ **are collinear or** $(\bar{c}x\bar{a})x\bar{b} = \bar{0}$ **.**

Proof: Let $(\bar{a}x\bar{b})x\bar{c} = \bar{a}x(\bar{b}x\bar{c}) \Leftrightarrow (\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a} = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$.

$\Leftrightarrow (\bar{b}.\bar{c})\bar{a} = (\bar{a}.\bar{b})\bar{c} \Rightarrow \bar{a},\bar{c}$ are collinear ($\because$ $\bar{a} = t\bar{c}$, $t \in R$)

$\Rightarrow (\bar{b}.\bar{c})\bar{a} - (\bar{a}.\bar{b})\bar{c} = \bar{0} \Leftrightarrow (\bar{c}x\bar{a})x\bar{b} = \bar{0}$

4. Show that $(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a}) = [\bar{a}\bar{b}\bar{c}]\bar{c}$**, using this prove that** $[\bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}\ \ \bar{a}\times\bar{b}] = [\bar{a}\bar{b}\bar{c}]^2$ **&write a conclusion.**

Proof: $(\bar{b}x\bar{c})x(\bar{c}x\bar{a}) = \{(\bar{b}x\bar{c}).\bar{a}\}\bar{c} - \{(\bar{b}x\bar{c}).\bar{c}\}\bar{a}$ ($\because \bar{a}x(\bar{b}x\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$)

$= [\bar{b}\ \bar{c}\ \bar{a}]\bar{c} - [\bar{b}\ \bar{c}\ \bar{c}]\bar{a}$

$= [\bar{a}\ \bar{b}\ \bar{c}]\bar{c} - 0 = [\bar{a}\ \bar{b}\ \bar{c}]\bar{c}$ ....(1)

Now, $[\bar{b}x\bar{c}\ \ \bar{c}x\bar{a}\ \ \bar{a}x\bar{b}] = [\bar{a}x\bar{b}\ \ \bar{b}x\bar{c}\ \ \bar{c}x\bar{a}]$

$= (\bar{a}x\bar{b}).\{(\bar{b}x\bar{c})x(\bar{c}x\bar{a})\}$ $[\because [abc] = \bar{a}.(\bar{b}x\bar{c})]$

$= (\bar{a}x\bar{b}).[\bar{a}\ \bar{b}\ \bar{c}]\bar{c}$ [from (1)]

$= ((\bar{a}x\bar{b}).\bar{c})[\bar{a}\bar{b}\bar{c}]$

$= [\bar{a}\ \bar{b}\ \bar{c}][\bar{a}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]^2$

Conclusion : If $\bar{a},\bar{b},\bar{c}$ are non coplanar then $[\bar{a}\ \bar{b}\ \bar{c}] \neq 0 \Rightarrow [\bar{a}\ \bar{b}\ \bar{c}]^2 \neq 0$

$\therefore$ if $\bar{a},\bar{b},\bar{c}$ are non coplanar then $(\bar{a}x\bar{b}),(\bar{b}x\bar{c}),(\bar{c}x\bar{a})$ are also non coplanar.

7. PROOFS OF RESULTS ON PRODUCT OF FOUR VECTORS

1. Scalar product of 4 vectors: If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are any 4 vectors then the scalar of the type $(\bar{a}x\bar{b}).(\bar{c}x\bar{d})$ is called a scalar product of the 4 vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$.

2. Theorem: If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are any four vectors then P.T $(\bar{a}x\bar{b}).(\bar{c}x\bar{d}) = (\bar{a}.\bar{c})(\bar{b}.\bar{d}) - (\bar{a}.\bar{d})(\bar{b}.\bar{c}) = \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix}$

Proof: $(\bar{a}x\bar{b}).(\bar{c}x\bar{d})$ can be treated as the scalar triple product of $\bar{a}, \bar{b}$ and $\bar{c}x\bar{d}$

$$\therefore (\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = \bar{a}.\{\bar{b}\times(\bar{c}\times\bar{d})\} [\because (\bar{a}\times\bar{b}).\bar{c} = \bar{a}.(\bar{b}\times\bar{c})] = \bar{a}.\{(\bar{b}.\bar{d})\bar{c} - (\bar{b}.\bar{c})\bar{d}\}$$

$$= (\bar{a}.\bar{c})(\bar{b}.\bar{d}) - (\bar{a}.\bar{d})(\bar{b}.\bar{c}). \ (\because (\bar{b}.\bar{d}) \text{ and } (\bar{b}.\bar{c}) \text{ are scalars})$$

$$= \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix}$$

Note: In the above result, instead of $\bar{c}$ and $\bar{d}$ if $\bar{a}$ and $\bar{b}$ are taken, then we have

$$(\bar{a}x\bar{b})x(\bar{a}x\bar{b}) = |\bar{a}x\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - |\bar{a}.\bar{b}|^2.$$

3. Vector product of 4 vectors: If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are any 4 vectors then the vector of the type $(\bar{a}x\bar{b})x(\bar{c}x\bar{d})$ is called a vector product of the 4 vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$.

4. Theorem: If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are any 4 vectors then P.T $(\bar{a}x\bar{b})x(\bar{c}x\bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a} = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}.$.

Proof: Let $\bar{c}x\bar{d} = \bar{m}$

$$\therefore (\bar{a}x\bar{b})x(\bar{c}x\bar{d}) = (\bar{a}x\bar{b})x\bar{m} = (\bar{a}.\bar{m})\bar{b} - (\bar{b}.\bar{m})\bar{a},,$$

$$= (\bar{a}.\bar{c}x\bar{d})\bar{b} - (\bar{b}.\bar{c}x\bar{d})\bar{a}$$

$$\therefore (\bar{a}x\bar{b})x(\bar{c}x\bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a} \quad ....(1)$$

Again $(\bar{a}x\bar{b})x(\bar{c}x\bar{d}) = -(\bar{c}x\bar{d})x(\bar{a}x\bar{b}) = -\{[\bar{c}\ \bar{a}\ \bar{b}]\bar{d} - [\bar{d}\ \bar{a}\ \bar{b}]\bar{c}\}$ [as from (1)]

$$= [\bar{d}\ \bar{a}\ \bar{b}]\bar{c} - [\bar{c}\ \bar{a}\ \bar{b}]\bar{d}$$

$$\therefore (\bar{a}x\bar{b})x(\bar{c}x\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{b} \quad ....(2)$$

From (1) and (2), $(\bar{a}x\bar{b})x(\bar{c}x\bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a} = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$.

ADDITIONAL QUESTIONS ON VECTOR PRODUCT

1. **$\bar{a},\bar{b},\bar{c}$ and $\bar{d}$ are the position vectors of four coplanar points such that $(\bar{a}-\bar{d}).(\bar{b}-\bar{c})=(\bar{b}-\bar{d}).(\bar{c}-\bar{a})=0$. Show that the point $\bar{d}$ represents the orthocentre of the traingle with $\bar{a},\bar{b}$ and $\bar{c}$ vertices.**

Sol: Let P.V's of A,B,C,D are $\overline{OA}=\bar{a},\ \overline{OB}=\bar{b},\ \overline{OC}=\bar{c},\ \overline{OD}=\bar{d}$

(i) Given $(\bar{a}-\bar{d}).(\bar{b}-\bar{c})=0$

$\Rightarrow\left(\overline{OA}-\overline{OD}\right).\left(\overline{OB}-\overline{OC}\right)=0\Rightarrow\overline{DA}.\overline{CB}=0\Rightarrow\overline{DA}\perp\overline{CB}$ .....(1)

(ii) Given $(\bar{b}-\bar{d}).(\bar{c}-\bar{a})=0$

$\Rightarrow\left(\overline{OB}-\overline{OD}\right).\left(\overline{OC}-\overline{OA}\right)=0\Rightarrow\overline{DB}.\overline{AC}=0\Rightarrow\overline{DB}\perp\overline{AC}$ .....(2)

From (1) & (2) we conclude that D($\bar{d}$) is the orthocentre of ΔABC.

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2. **For any two vectors $\bar{a}$ and $\bar{b}$, show that**

(i) $|\bar{a}.\bar{b}|\le|\bar{a}||\bar{b}|$ (Cauchy-Schwartz inequality)

(ii) $|\bar{a}+\bar{b}|\le|\bar{a}|+|\bar{b}|$ (Triangle inequality)

Sol: If $\bar{a}=\bar{0}$ or $\bar{b}=\bar{0}$ then the given inequalities hold trivally.

So, assume that $|\bar{a}|\ne 0\ne|\bar{b}|$. Then $\dfrac{|\bar{a}.\bar{b}|}{|\bar{a}|.|\bar{b}|}=|\cos\theta|\le 1$

Hence $|\bar{a}.\bar{b}|\le|\bar{a}||\bar{b}|$

(ii) Consider $|\bar{a}+\bar{b}|^2=(\bar{a}+\bar{b})^2=(\bar{a}+\bar{b}).(\bar{a}+\bar{b})=\bar{a}.\bar{a}+\bar{a}.\bar{b}+\bar{b}.\bar{a}+\bar{b}.\bar{b}$

$=|\bar{a}|^2+2(\bar{a}.\bar{b})+|\bar{b}|^2$, (Scalar product is commutative)

$\le|\bar{a}|^2+2|\bar{a}.\bar{b}|+|\bar{b}|^2$ $(\because x\le|x|,\forall x\in R)$

$\le|\bar{a}|^2+2|\bar{a}||\bar{b}|+|\bar{b}|^2$ (from (i)) $=(|\bar{a}|+|\bar{b}|)^2$

Hence $|\bar{a}+\bar{b}|\le|\bar{a}|+|\bar{b}|$

3. G is the centroid of ΔABC and a,b,c are the lengths of the sides BC,CA and AB respectively. Prove that $a^2+b^2+c^2=3(OA^2+OB^2+OC^2)-9(OG)^2$ where 'O' is any pont.

Sol: Let the P.Vs of vertices A,B,C of ΔABC are $\overline{OA}, \overline{OB}, \overline{OC}$.

Also $|\overline{BC}|=a, |\overline{CA}|=b, |\overline{AB}|=c$;

G is the centroid of $\Delta ABC \Rightarrow \overline{OG}=\dfrac{\overline{OA}+\overline{OB}+\overline{OC}}{3}$

$\Rightarrow 3\overline{OG}=\overline{OA}+\overline{OB}+\overline{OC}$

$\Rightarrow 9OG^2=(\overline{OA}+\overline{OB}+\overline{OC})^2$

R.H.S= $3(OA^2+OB^2+OC^2)-9(OG)^2=3(OA^2+OB^2+OC^2)-(\overline{OA}+\overline{OB}+\overline{OC})^2$

$=3OA^2+3OB^2+3OC^2-OA^2-OB^2-OC^2-2\overline{OA}.\overline{OB}-2\overline{OB}.\overline{OC}-2\overline{OC}.\overline{OA}$

$=2OA^2+2OB^2+2OC^2-2\overline{OA}.\overline{OB}-2\overline{OB}.\overline{OC}-2\overline{OC}.\overline{OA}$

$=[OC^2+OB^2-2\overline{OB}.\overline{OC}]+[OA^2+OC^2-2\overline{OA}.\overline{OC}]+[OA^2+OB^2-2\overline{OA}.\overline{OB}]$

$=(\overline{OC}-\overline{OB})^2+(\overline{OA}-\overline{OC})^2+(\overline{OB}-\overline{OA})^2=\overline{BC}^2+\overline{CA}^2+\overline{AB}^2=a^2+b^2+c^2=\text{L.H.S}$

4. Let $\bar{a} = 2\bar{i} + 3\bar{j} + \bar{k}$, $\bar{b} = 4\bar{i} + \bar{j}$ and $\bar{c} = \bar{i} - 3\bar{j} - 7\bar{k}$. Find the vector $\bar{r}$ such that $\bar{r}.\bar{a} = 9$, $\bar{r}.\bar{b} = 7$ and $\bar{r}.\bar{c} = 6$

Semi Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Now $\bar{r}.\bar{a} = 9 \Rightarrow (x\bar{i} + y\bar{j} + z\bar{k}).(2\bar{i} + 3\bar{j} + \bar{k}) = 9 \Rightarrow 2x + 3y + z = 9$ ....(1)

$\bar{r}.\bar{b} = 7 \Rightarrow (x\bar{i} + y\bar{j} + z\bar{k}).(4\bar{i} + \bar{j}) = 7 \Rightarrow 4x + y = 7$ ....(2)

$\bar{r}.\bar{c} = 6 \Rightarrow (x\bar{i} + y\bar{j} + z\bar{k}).(\bar{i} - 3\bar{j} - 7\bar{k}) = 6 \Rightarrow x - 3y - 7z = 6$ ....(3)

Solving these equations we get x=1, y=3, z= –2

$\therefore \bar{r} = \bar{i} + 3\bar{j} - 2\bar{k}$

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5. Find unit vector orthogonal to the vector $3\bar{i} + 2\bar{j} + 6\bar{k}$ and coplanar with the vectors $2\bar{i} + \bar{j} + \bar{k}$ and $\bar{i} - \bar{j} + \bar{k}$

Semi Sol: Let $\bar{a} = 2\bar{i} + \bar{j} + \bar{k}$, $\bar{b} = \bar{i} - \bar{j} + \bar{k}$, $\bar{c} = 3\bar{i} + 2\bar{j} + 6\bar{k}$

Let $\bar{r}$ be a vector coplnarar with $\bar{a}, \bar{b}$ and orthogonal to $\bar{c}$.

Then $\bar{r} = x\bar{a} + y\bar{b}$ where x,y are scalars. Also $\bar{r}.\bar{c} = \bar{0}$ and $|\bar{r}| = 1$ ....(1)

Now, $\bar{r} = x\bar{a} + y\bar{b} = x(2\bar{i} + \bar{j} + \bar{k})\bar{i} + y(\bar{i} - \bar{j} + \bar{k})$

$\Rightarrow \bar{r} = (2x + y)\bar{i} + (x - y)\bar{j} + (x + y)\bar{k}$.......(2)

From (1),(2) $\bar{r}.\bar{c} = 0 \Rightarrow [(2x + y)\bar{i} + (x - y)\bar{j} + (x + y)\bar{k}].(3\bar{i} + 2\bar{j} + 6\bar{k})$

$= 3(2x + y) + 2(x - y) + 6(x + y) = 0$

$\Rightarrow 14x + 7y = 0 \Rightarrow y = -2x$

Also from (1), $|\bar{r}| = 1 \Rightarrow (2x + y)^2 + (x - y)^2 + (x + y)^2 = 1$

$\Rightarrow (4x^2 + y^2 + \cancel{4xy}) + (x^2 + y^2 - 2xy) + (x^2 + y^2 + 2xy) = 1 \Rightarrow 9x^2 + x^2 = 1 \Rightarrow 10x^2 = 1$

$\Rightarrow x = \pm\frac{1}{\sqrt{10}}$ Hence y= –2x = $\pm\frac{2}{\sqrt{10}}$

$\therefore$ From (2) we get $\bar{r} = \pm\frac{1}{\sqrt{10}}(3\bar{j} - \bar{k})$

6. If $\bar{a}, \bar{b}, \bar{c}$ are 3 vectors then prove that $(\bar{a}\times\bar{b})\times\bar{c} = (\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a}$

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Sol: We take $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}, \bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}, \bar{c}=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \bar{i}(a_2b_3 - b_2a_3) - \bar{j}(a_1b_3 - b_1a_3) + \bar{k}(a_1b_2 - b_1a_2)$$

$$= \bar{i}(a_2b_3 - a_3b_2) + \bar{j}(a_3b_1 - a_1b_3) + \bar{k}(a_1b_2 - a_2b_1)$$

$$(\bar{a}\times\bar{b})\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \bar{i}[(a_3b_1 - a_1b_3)c_3 - c_2(a_1b_2 - a_2b_1)]$$

$$-\bar{j}[(a_2b_3 - a_3b_2)c_3 - c_1(a_1b_2 - a_2b_1)]$$

$$+\bar{k}[(a_2b_3 - a_3b_2)c_2 - c_1(a_3b_1 - a_1b_3)]$$

$$= \bar{i}[a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2]$$

$$-\bar{j}[a_2b_3c_3 - a_3b_2c_3 - a_1b_2c_1 + a_2b_1c_1]$$

$$+\bar{k}[a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1] \text{...........(1)}$$

$$(\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a}$$

$$= (c_1a_1 + c_2a_2 + c_3a_3)(b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) - (c_1b_1 + c_2b_2 + c_3b_3)(a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$$

$$= \bar{i}[(c_1a_1 + c_2a_2 + c_3a_3)b_1 - (c_1b_1 + c_2b_2 + c_3b_3)a_1]$$

$$-\bar{j}[-(c_1a_1 + c_2a_2 + c_3a_3)b_2 + (c_1b_1 + c_2b_2 + c_3b_3)a_2]$$

$$+\bar{k}[(c_1a_1 + c_2a_2 + c_3a_3)b_3 - (c_1b_1 + c_2b_2 + c_3b_3)a_3]$$

$$= \bar{i}(\cancel{a_1b_1c_1} + a_2b_1c_2 + a_3b_1c_3 \cancel{-a_1b_1c_1} - a_1b_2c_2 - a_1b_3c_3)$$

$$-\bar{j}(-a_1b_2c_1 \cancel{-a_2b_2c_2} - a_3b_2c_3 + a_2b_1c_1 \cancel{+a_2b_2c_2} + a_2b_3c_3)$$

$$+\bar{k}(a_1b_3c_1 + a_2b_3c_2 \cancel{+a_3b_3c_3} - a_3b_1c_1 - a_3b_2c_2 \cancel{-a_3b_3c_3})$$

$$= \bar{i}(a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2)$$

$$-\bar{j}(a_2b_3c_3 - a_3b_2c_3 - a_1b_2c_1 + a_2b_1c_1)$$

$$+\bar{k}(a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1) \text{...........(2)}$$

From (1) & (2)

$(\bar{a}\times\bar{b})\times\bar{c}=(\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a}$

7. If $\bar{a} = \bar{i} - 2\bar{j} + \bar{k}, \bar{b} = 2\bar{i} + \bar{j} + \bar{k}, \bar{c} = \bar{i} + 2\bar{j} - \bar{k}$ then find $\bar{a}\times(\bar{b}\times\bar{c})$ and $|(\bar{a}\times\bar{b})\times\bar{c}|$.

Sol: Given $\bar{a} = \bar{i} - 2\bar{j} + \bar{k}, \bar{b} = 2\bar{i} + \bar{j} + \bar{k}, \bar{c} = \bar{i} + 2\bar{j} - \bar{k}$

1) To find $\bar{a}\times(\bar{b}\times\bar{c})$, first we have to find $\bar{b}\times\bar{c}$ (term in the bracket)

$$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \bar{i}(-1-2) - \bar{j}(-2-1) + \bar{k}(4-1) = -3\bar{i} + 3\bar{j} + 3\bar{k}$$

$$\therefore \bar{a}\times(\bar{b}\times\bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix} = \bar{i}(-6-3) - \bar{j}(3+3) + \bar{k}(3-6)$$

$$= -9\bar{i} - 6\bar{j} - 3\bar{k}$$

2) To find $(\bar{a}\times\bar{b})\times\bar{c}$, we have to find $\bar{a}\times\bar{b}$ (term in the bracket)

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \bar{i}(-2-1) - \bar{j}(1-2) + \bar{k}(1+4) = -3\bar{i} + \bar{j} + 5\bar{k}$$

$$\therefore (\bar{a}\times\bar{b})\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -3 & 1 & 5 \\ 1 & 2 & -1 \end{vmatrix} = \bar{i}(-1-10) - \bar{j}(3-5) + \bar{k}(-6-1)$$

$$= -11\bar{i} + 2\bar{j} - 7\bar{k}$$

$$\therefore |(\bar{a}\times\bar{b})\times\bar{c}| = |-11\bar{i} + 2\bar{j} - 7\bar{k}|$$

$$= \sqrt{(-11)^2 + (2)^2 + (-7)^2} = \sqrt{121+4+49} = \sqrt{174}$$

8. If $\bar{a} = \bar{i} - 2\bar{j} - 3\bar{k}$, $\bar{b} = 2\bar{i} + \bar{j} - \bar{k}, \bar{c} = \bar{i} + 3\bar{j} - 2\bar{k}$ then, show that $\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$

A.

$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{vmatrix} = \bar{i}(-2+3) - \bar{j}(4+1) + \bar{k}(6-1) = \bar{i} + 3\bar{j} + 5\bar{k}$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix} = \bar{i}(-10+9) - \bar{j}(5+3) + \bar{k}(3+2) = -\bar{i} - 8\bar{j} + 5\bar{k}$$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & -3 \\ 2 & 1 & 5 \end{vmatrix} = \bar{i}(2+3) - \bar{j}(-1+6) + \bar{k}(1+4) = 5\bar{i} - 5\bar{j} + 5\bar{k}$$

$$(\bar{a} \times \bar{b}) \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5 & -5 & 5 \\ 1 & 3 & -2 \end{vmatrix} = \bar{i}(10-5) - \bar{j}(-10-5) + \bar{k}(15+5) = -5\bar{i} + 15\bar{j} + 20\bar{k}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$$

9. If $\bar{a},\bar{b}$ and $\bar{c}$ represent the vertices A, B and C respectively of Δ ABC, then prove that $\left|(\bar{a}\times\bar{b})+(\bar{b}\times\bar{c})+(\bar{c}\times\bar{a})\right|$ is twice the area of Δ ABC.

Sol: P.V's of A,B,C are $\overline{OA}=\bar{a},\overline{OB}=\bar{b}$, and $\overline{OC}=\bar{c}$. Then

$\overline{AB}=\overline{OB}-\overline{OA}=\bar{b}-\bar{a}$ and $\overline{AC}=\overline{OC}-\overline{OA}=\bar{c}-\bar{a}$

Area of Δ ABC = $\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|$

$\Rightarrow 2(\text{Area}\,\Delta\,ABC)=\left|\overline{AB}\times\overline{AC}\right|=\left|(\overline{OB}-\overline{OA})\times(\overline{OC}-\overline{OA})\right|=\left|(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})\right|$

$=\left|\bar{b}\times\bar{c}-\bar{b}\times\bar{a}-\bar{a}\times\bar{c}+\bar{a}\times\bar{a}\right|=\left|\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a}+\bar{0}\right|=\left|(\bar{a}\times\bar{b})+(\bar{b}\times\bar{c})+(\bar{c}\times\bar{a})\right|$

Hence proved the result.

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10. If $\bar{a}=\bar{i}+\bar{j}+\bar{k},\bar{c}=\bar{j}-\bar{k}$ then find vector $\bar{b}$ such that $\bar{a}\times\bar{b}=\bar{c}$ and $\bar{a}.\bar{b}=3$

Sol: Given $\bar{a}=\bar{i}+\bar{j}+\bar{k},\bar{c}=\bar{j}-\bar{k}$ and $\bar{a}.\bar{b}=3$

Let $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$

$\bar{a}.\bar{b}=3\Rightarrow(\bar{i}+\bar{j}+\bar{k}).(b_1\bar{i}+b_2\bar{j}+b_3\bar{k})=3\Rightarrow b_1+b_2+b_3=3$..........(1)

Also $\bar{a}\times\bar{b}=\bar{c}\Rightarrow\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\1&1&1\\b_1&b_2&b_3\end{vmatrix}=(\bar{j}-\bar{k})$

$\Rightarrow\bar{i}(b_3-b_2)-\bar{j}(b_3-b_1)+\bar{k}(b_2-b_1)=\bar{j}-\bar{k}$

$\Rightarrow\bar{i}(b_3-b_2)+\bar{j}(b_1-b_3)-\bar{k}(b_1-b_2)=\bar{j}-\bar{k}$

Equating the coefficients of $\bar{i},\bar{j},\bar{k}$ respectively we have

$b_3-b_2=0;\ b_1-b_3=1\ ;(b_1-b_2)=1$

$\Rightarrow b_3=b_2\ ;b_1-b_3=1\ ;\ b_1-b_2=1\Rightarrow b_1=1+b_2$.............(2)

Also from (1), $b_1+b_2+b_3=3\Rightarrow(1+b_2)+b_2+b_2=3\Rightarrow 3b_2=2\Rightarrow b_2=\frac{2}{3}=b_3$

Hence, $b_1=1+b_2=1+\frac{2}{3}=\frac{5}{3}$

$\therefore$ Required vector $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}=\frac{5}{3}\bar{i}+\frac{2}{3}\bar{j}+\frac{2}{3}\bar{k}$

11. Let $\bar{e}_1$ and $\bar{e}_2$ be unit vectors including angle θ. If $\frac{1}{2}|\bar{e}_1-\bar{e}_2|=\sin\lambda\theta$ then find λ.

Sol: Given $\bar{e}_1, \bar{e}_2$ are unit vectors and $(\bar{e}_1,\bar{e}_2)=\theta$ Also, $\frac{1}{2}|\bar{e}_1-\bar{e}_2|=\sin\lambda\theta \Rightarrow |\bar{e}_1-\bar{e}_2|=2\sin\lambda\theta$

$\Rightarrow |\bar{e}_1-\bar{e}_2|^2=(2\sin\lambda\theta)^2 \Rightarrow (|\bar{e}_1|^2+|\bar{e}_2|^2-2\bar{e}_1.\bar{e}_2)=4\sin^2(\lambda\theta)$

$\Rightarrow (1+1-2|\bar{e}_1||\bar{e}_2|\cos\theta)=4\sin^2\lambda\theta$

$\Rightarrow (2-2(1)(1)\cos\theta)=4\sin^2\lambda\theta \Rightarrow 2(1-\cos\theta)=4\sin^2\lambda\theta$

$\Rightarrow \cancel{2}\left(\cancel{2}\sin^2\frac{\theta}{2}\right)=\cancel{4}\sin^2\lambda\theta \Rightarrow \sin^2\frac{\theta}{2}=\sin^2\lambda\theta \qquad \therefore \lambda=\frac{1}{2}$

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12. If $\bar{a},\bar{b},\bar{c}$ are three vectors of equal magnitudes and each of them is inclined at an angle of 60° to the others . If $|\bar{a}+\bar{b}+\bar{c}|=\sqrt{6}$ then find $|\bar{a}|$

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Sol: Given that $\bar{a},\bar{b},\bar{c}$ have equal magntiude $\Rightarrow |\bar{a}|=|\bar{b}|=|\bar{c}|=\lambda$(say)

Angle between the pair of vectors is 60° $\Rightarrow (\bar{a},\bar{b})=(\bar{b},\bar{c})=(\bar{c},\bar{a})=60^\circ$

Now $|\bar{a}+\bar{b}+\bar{c}|^2=|\bar{a}|^2+|\bar{b}|^2+|\bar{c}|^2+2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a})$

$=\lambda^2+\lambda^2+\lambda^2+2|\bar{a}||\bar{b}|\cos60^\circ+2|\bar{b}||\bar{c}|\cos60^\circ+2|\bar{c}||\bar{a}|\cos60^\circ$

$=3\lambda^2+\cancel{2}\lambda^2\left(\frac{1}{\cancel{2}}\right)+\cancel{2}\lambda^2\left(\frac{1}{\cancel{2}}\right)+\cancel{2}\lambda^2\left(\frac{1}{\cancel{2}}\right)=6\lambda^2 \qquad \therefore |\bar{a}+\bar{b}+\bar{c}|=\sqrt{6}\lambda$

Given $|\bar{a}+\bar{b}+\bar{c}|=\sqrt{6} \therefore \lambda=1 \qquad \therefore |\bar{a}|=1$

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13. Let $\bar{a},\bar{b}$ be two non-collinear unit vectors. If $\bar{\alpha}=\bar{a}-(\bar{a}.\bar{b})\bar{b}$ and $\bar{\beta}=\bar{a}\times\bar{b}$ then show that $|\bar{\beta}|=|\bar{\alpha}|$.

Sol: Given that $\bar{a},\bar{b}$ are unit vectors. $\Rightarrow |\bar{a}|=1, |\bar{b}|=1$

$\therefore \bar{a}.\bar{b}=|\bar{a}||\bar{b}|\cos\theta=1.1.\cos\theta=\cos\theta$

Now, $|\bar{\beta}|^2=|\bar{a}\times\bar{b}|^2=|\bar{a}|^2|\bar{b}|^2-(\bar{a}.\bar{b})^2=1^2\times1^2-(\bar{a}.\bar{b})^2=1-\cos^2\theta=\sin^2\theta$ ....(1)

Also $|\bar{\alpha}|^2=|\bar{a}-(\bar{a}.\bar{b})\bar{b}|^2=|\bar{a}|^2+(\bar{a}.\bar{b})^2|\bar{b}|^2-2(\bar{a}.\bar{b})^2$

$=1+\cos^2\theta-2\cos^2\theta=1-\cos^2\theta=\sin^2\theta$ ....(2)

From (1) & (2), $|\bar{\beta}|=|\bar{\alpha}|$

14. For any two vectors $\bar{a}, \bar{b}$ show that $(1+|\bar{a}|^2)(1+|\bar{b}|^2) = |1-\bar{a}.\bar{b}|^2 + |\bar{a}+\bar{b}+\bar{a}\times\bar{b}|^2$

Sol: Let $(\bar{a},\bar{b}) = \theta$

R.H.S= $|1-\bar{a}.\bar{b}|^2 + |\bar{a}+\bar{b}+(\bar{a}\times\bar{b})|^2$

$= [1+(\bar{a}.\bar{b})^2 - 2(\bar{a}.\bar{b})] + [|\bar{a}|^2 + |\bar{b}|^2 + |\bar{a}\times\bar{b}|^2 + 2(\bar{a}.\bar{b}) + 2[\bar{b}.(\bar{a}\times\bar{b})] + 2[\bar{a}.(\bar{a}\times\bar{b})]]$

$= 1+|\bar{a}|^2|\bar{b}|^2\cos^2\theta - \cancel{2(\bar{a}.\bar{b})} + |\bar{a}|^2 + |\bar{b}|^2 + |\bar{a}\times\bar{b}|^2 + \cancel{2(\bar{a}.\bar{b})} + 0 + 0 \ [\because \bar{b}.(\bar{a}\times\bar{b}) = 0 \text{ and } \bar{a}.(\bar{a}\times\bar{b}) = 0]$

$= 1+|\bar{a}|^2|\bar{b}|^2(1-\sin^2\theta) + |\bar{a}|^2 + |\bar{b}|^2 + |\bar{a}\times\bar{b}|^2$

$= 1+|\bar{a}|^2|\bar{b}|^2 - |\bar{a}|^2|\bar{b}|^2\sin^2\theta + |\bar{a}|^2 + |\bar{b}|^2 + |\bar{a}\times\bar{b}|^2$

$= 1+|\bar{a}|^2|\bar{b}|^2 - \cancel{|\bar{a}|^2|\bar{b}|^2\sin^2\theta} + |\bar{a}|^2 + |\bar{b}|^2 + \cancel{|\bar{a}|^2|\bar{b}|^2\sin^2\theta} \ [\because |\bar{a}\times\bar{b}| = |\bar{a}|\times|\bar{b}|\sin\theta]$

$= 1+|\bar{a}|^2|\bar{b}|^2 + |\bar{a}|^2 + |\bar{b}|^2 = (1+|\bar{a}|^2)(1+|\bar{b}|^2) = \text{L.H.S}$

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15. If A=(1,a,a²), B= (1,b,b²) and C= (1,c,c²) are non-coplanar vectors and $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ then show that abc+1=0

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Sol: Given that $A=(1,a,a^2)$, $B=(1,b,b^2)$ and $C=(1,c,c^2)$ are non-coplanar vectors.

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0 \text{..........(1)}$$

Given that $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0 \qquad \Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\Rightarrow \Delta + (abc).\Delta = 0 \Rightarrow \Delta(1+abc) = 0 \Rightarrow 1+abc = 0 \quad [\because \Delta \neq 0]$

16. **If $\bar{a},\bar{b},\bar{c}$ are non-zero vectors, then prove that $|(\bar{a}\times\bar{b}.\bar{c})| = |\bar{a}||\bar{b}||\bar{c}| \Leftrightarrow \bar{a}.\bar{b} = \bar{b}.\bar{c} = \bar{c}.\bar{a} = 0$**

Sol: $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})\bar{n}$ where $\bar{n}$ is a unit vector perpendicular to $\bar{a}$ and $\bar{b}$

Now $(\bar{a}\times\bar{b}).\bar{c} = |\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})(\bar{n}.\bar{c}) = |\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})|\bar{n}||\bar{c}|\cos(\bar{n},\bar{c})$

$= |\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})\cos(\bar{n},\bar{c})|\bar{c}|(1) = |\bar{a}||\bar{b}||\bar{c}|\sin(\bar{a},\bar{b})\cos(\bar{n},\bar{c}) \qquad [\because |\bar{n}| = 1]$

Given that $|(\bar{a}\times\bar{b}.\bar{c})| = |\bar{a}||\bar{b}||\bar{c}| \Rightarrow |\bar{a}||\bar{b}||\bar{c}|\sin(\bar{a},\bar{b})\cos(\bar{n},\bar{c}) = |\bar{a}||\bar{b}||\bar{c}| \Rightarrow \sin(\bar{a},\bar{b})\cos(\bar{n},\bar{c}) = 1$

$\Rightarrow \sin(\bar{a},\bar{b}) = 1$ and $\cos(\bar{n},\bar{c}) = 1 \Rightarrow \bar{a}$ and $\bar{b}$ are perpendicular and $\bar{n}$ is parallel to $\bar{c}$

$\Rightarrow \bar{a}$ and $\bar{b}$ are perpendicular and $\bar{c}$ is perpendicular to both $\bar{a}$ and $\bar{b}$

$\therefore \bar{a}$ is perpendicular to $\bar{b}$, $\bar{b}$ is perpendicular to $\bar{c}$ and $\bar{c}$ is perpendicular to $\bar{a}$

$\Leftrightarrow \bar{a}.\bar{b} = \bar{b}.\bar{c} = \bar{c}.\bar{a} = 0$

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17. **Let $\bar{a} = \bar{i} - \bar{k}$, $\bar{b} = x\bar{i} + \bar{j} + (1-x)\bar{k}$ and $\bar{c} = y\bar{i} + x\bar{j} + (1+x-y)\bar{k}$, prove that the scalar triple product $[\bar{a}\ \bar{b}\ \bar{c}]$ is independent of both x and y**

Sol: Given $\bar{a} = \bar{i} - \bar{k}$, $\bar{b} = x\bar{i} + \bar{j} + (1-x)\bar{k}$ and $\bar{c} = y\bar{i} + x\bar{j} + (1+x-y)\bar{k}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & -1 \\ x & 1 & 1-x \end{vmatrix} = [\bar{i}(0+1) - \bar{j}(1-x+x) + \bar{k}(1-0)] = \bar{i} - \bar{j} + \bar{k}$$

Now $[\bar{a}\ \bar{b}\ \bar{c}] = (\bar{a}\times\bar{b}).\bar{c} = (\bar{i} - \bar{j} + \bar{k})[y\bar{i} + x\bar{j} + (1+x-y)\bar{k}]$

$= 1(y)+(-1)(x)+(1)(1+x-y) = \cancel{y} - \cancel{x} + 1 + \cancel{x} - \cancel{y} = 1 =$ a constant

Hence $[\bar{a}\ \bar{b}\ \bar{c}]$ is independent of both x and y.

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18. **For any four vectors $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$, prove that**

(i) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\bar{c}\bar{d}]\bar{b} - [\bar{b}\bar{c}\bar{d}]\bar{a}$ and (ii) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\bar{b}\bar{d}]\bar{c} - [\bar{a}\bar{b}\bar{c}]\bar{d}$

Sol: (i) Let $\bar{c}\times\bar{d} = \bar{p}$

$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = (\bar{a}\times\bar{b})\times\bar{p} = (\bar{a}.\bar{p})\bar{b} - (\bar{b}.\bar{p})\bar{a} = (\bar{a}.(\bar{c}\times\bar{d}))\bar{b} - (\bar{b}.(\bar{c}\times\bar{d}))\bar{a}$

$= [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$

(ii) Let $\bar{a}\times\bar{b} = \bar{q}$

$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = \bar{q}\times(\bar{c}\times\bar{d}) = (\bar{q}.\bar{d})\bar{c} - (\bar{q}.\bar{c})\bar{d} = ((\bar{a}\times\bar{b}).\bar{d}.)\bar{c} - (\bar{a}\times\bar{b}).\bar{c}.)\bar{d}$

$= [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

19. Let $\bar{b} = 2\bar{i} + \bar{j} - \bar{k}, \bar{c} = \bar{i} + 3\bar{k}$. If $\bar{a}$ is a unit vector then find the maximum value of $[\bar{a}\ \bar{b}\ \bar{c}]$

Sol: Let $\bar{a} = x\bar{i} + y\bar{j} + z\bar{k}$ such that $x^2 + y^2 + z^2 = 1$ Also $\bar{b} = 2\bar{i} + \bar{j} - \bar{k}, \bar{c} = \bar{i} + 3\bar{k}$.

$$\therefore [\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} x & y & z \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix} = x(3-0) - y(6+1) + z(0-1) = 3x - 7y - z$$

But $x^2 + y^2 + z^2 = 1$, $\therefore$ the maximum value of $3x-7y-z$ is

$$\sqrt{(3)^2 + (-7)^2 + (-1)^2} = \sqrt{9+49+1} = \sqrt{59}$$

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20. Let $\bar{a} = \bar{i} - \bar{j}, \bar{b} = \bar{i} - \bar{k}, \ \bar{c} = \bar{k} - \bar{i}$. Find unit vector $\bar{d}$ such that $\bar{a}.\bar{d} = 0 = [\bar{b}\ \bar{c}\ \bar{d}]$

Sol: Given that $\bar{a} = \bar{i} - \bar{j}, \bar{b} = \bar{i} - \bar{k}, \ \bar{c} = \bar{k} - \bar{i}$

Let $\bar{d} = x\bar{i} + y\bar{j} + z\bar{k}$ be a unit vector $\Rightarrow |\bar{d}| = 1 \Rightarrow x^2 + y^2 + z^2 = 1$

Since $\bar{a}.\bar{d} = 0 \Rightarrow (\bar{i} - \bar{j}).(x\bar{i} + y\bar{j} + z\bar{k}) = 0 \Rightarrow x - y = 0 \Rightarrow x = y$.........(1)

and $[\bar{b}\ \bar{c}\ \bar{d}] = 0 \Rightarrow \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ x & y & z \end{vmatrix} = 0 \Rightarrow 0(0-0) - 1(-z-x) - 1(-y-0) = 0 \Rightarrow x + y + z = 0$ ....(2)

From (1) & (2) $\Rightarrow x+x+z=0 \Rightarrow z= -2x$

Now $x : y : z = x : x : -2x = 1 : 1 : -2$

Put $x=\lambda$ then $y=\lambda$ and $z= -2\lambda$. Then $\bar{d} = \lambda(\bar{i} + \bar{j} - 2\bar{k}), \lambda \in R$

Unit vector along $\bar{d}$ is $\hat{d} = \pm\dfrac{\bar{d}}{|\bar{d}|} = \pm\dfrac{\lambda(\bar{i} + \bar{j} - 2\bar{k})}{\lambda(\sqrt{1+1+4})} = \pm\left(\dfrac{\bar{i} + \bar{j} - 2\bar{k}}{\sqrt{6}}\right)$