4. THEORY OF EQUATIONS

Sections	No. of periods (12)	Weightage in IPE [1x2 + 1x7 =9]
1. Fundamental Concepts, Nature of roots	3	2 marks
2. Relations between coefficients and roots, Symmetric functions, Solving certain types of equations	5	2 or 7 marks
3. Transformation of equations, Missing terms.	4	7 marks
4. Reciprocal equations.	3	7 marks

"Theory of equations" deals with the study of methods of solving and the possibilities of solving polynomial equations. It also deals with the relations between the roots and coefficients of equations.

In the first section, basic concepts and theorems related to the Theory of Equations and Nature of Roots of certain polynomial equations are discussed.

In Section 2, relations between coefficients and roots of polynomial equations, symmetric functions of roots of polynomial equations and methods of solving certain types of equations are discussed.

In Section 3, Transformation of Equations, Removal of Terms with suitable transformations and solving reciprocal equations are discussed.

SYNOPSIS POINTS

1. A polynomial equation with n roots $\alpha_1,\alpha_2,....\alpha_n$ is $(x-\alpha_1)(x-\alpha_2)....(x-\alpha_n)=0$.

2.1. In an equation with real coefficients, complex roots occur in conjugate pairs.

2.2. In an equation with rational coefficients, irrational roots occur in pairs of conjugate surds.

3. If α,β,γ are the roots of $ax^3+bx^2+cx+d=0$, then
(i) $S_1=\alpha+\beta+\gamma=-b/a$ (ii) $S_2=\alpha\beta+\beta\gamma+\gamma\alpha=c/a$ (iii) $S_3=\alpha\beta\gamma=-d/a$

4. If $\alpha,\beta,\gamma,\delta$ are the roots of $ax^4+bx^3+cx^2+dx+e=0$ then
(i) $S_1=\alpha+\beta+\gamma+\delta=-b/a$ (ii) $S_2=\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=c/a$
(iii) $S_3=\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-d/a$ (iv) $S_4=\alpha\beta\gamma\delta=e/a$

5.

Progression	Roots of Cubic equation	Roots of Biquadratic equation
A.P	a–d, a, a+d	a–3d, a–d, a+d, a+3d
G.P.	a/r, a, ar	$\frac{a}{r^3},\frac{a}{r},ar,ar^3$
H.P.	$\frac{1}{a-d},\frac{1}{a},\frac{1}{a+d}$	$\frac{1}{a-3d},\frac{1}{a-d},\frac{1}{a+d},\frac{1}{a+3d}$

6.1. The equation whose roots are those of the equation f(x)=0 with signs changed is f(–x)=0
6.2. The equation whose roots are $k(\neq 0)$ times the roots of f(x)=0 is f(x/k)=0
6.3. The equation whose roots are reciprocals of the roots of f(x)=0 is f(1/x)=0
6.4. The equation whose roots are the squares of the roots of f(x)=0 is $f(\sqrt{x})=0$ expressed as a polynomial equation (i.e., radical signs should be removed)
6.5. The equation whose roots are the cubes of the roots of f(x)=0 is $f\left(\sqrt[3]{x}\right)=0$ expressed as a polynomial equation (i.e., radical signs should be removed)
6.6. The equation whose roots are exceed by h than those of f(x)=0 is f(x–h)=0
6.7. The equation whose roots are diminished by h than those of f(x)=0 is f(x+h)=0

7. **REMOVAL OF TERMS:** An assigned term of a given equation can be removed by increasing or decreasing the roots of the given equation by a suitable number h.
If $f(x)=a_0x^n+a_1x^{n-1}+....+a_n=0$ then
(i) to remove the second term in f(x)=0, diminish the roots of f(x)=0 by $h=-\frac{a_1}{na_0}$
(ii) to remove the 3rd term, diminish the roots of f(x)=0 by h such that $\frac{n(n-1)}{2}a_0h^2+(n-1)a_1h+a_2=0$

8.1. **Reciprocal equation**: An equation f(x)=0 is said to be a reciprocal equation if f(x)=0 and f(1/x)=0 are one and the same i.e., the reciprocal of every root of the equation is also its root.

8.2. A polynomial equation $f(x)=a_0x^n+a_1x^{n-1}+....+a_n=0$ is said to a reciprocal equation of Class–I if $a_k=a_{n-k}$ ($\forall$ k=0,1,2,....,n) and Class–II if $a_k=-a_{n-k}$ ($\forall$ k=0,1,2,....,n)

9.1. A reciprocal equation of class–I with even degree is called a **S**tandard **R**eciprocal **E**quation

9.2. **Procedure of solving a S.R.E**: To solve a S.R.E of order 2m, divide the equation by x^m and put $x+\frac{1}{x}=y$ and proceed accordingly.

ADDITIONAL QUESTIONS WITH SOLUTIONS

1 **If α, β and γ are the roots of $x^3-3ax+b=0$ prove that $\Sigma(\alpha-\beta)(\alpha-\gamma)=9a$.**

Sol: Given α, β, γ are the roots of $x^3-3ax+b=0$

$\Rightarrow \alpha+\beta+\gamma=0$, [Hence $\beta+\gamma=-\alpha$]; $\alpha\beta+\beta\gamma+\gamma\alpha=-3a$; $\alpha\beta\gamma=-b$

$\therefore\ \Sigma(\alpha-\beta)(\alpha-\gamma)=\Sigma[\alpha^2-\alpha(\beta+\gamma)+\beta\gamma]=\Sigma[\alpha^2-\alpha(-\alpha)+\beta\gamma]=\Sigma[\alpha^2+\alpha^2+\beta\gamma]=\Sigma[2\alpha^2+\beta\gamma]$

$=2(\alpha^2+\beta^2+\gamma^2)+(\beta\gamma+\gamma\alpha+\alpha\beta)=2[(\alpha+\beta+\gamma)^2-4(\alpha\beta+\beta\gamma+\gamma\alpha)]+(\alpha\beta+\beta\gamma+\gamma\alpha)=0-4(-3a)+(-3a)=9a$

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2 **If α, β and γ are the roots of $x^3+px^2+qx+r=0$ then find the following.** EAM Q

(i) $\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$ **(ii)** $(\beta+\gamma-3\alpha)(\gamma+\alpha-3\beta)(\alpha+\beta-3\gamma)$ **(iii)** $\Sigma\alpha^3\beta^3$

Sol: Given α, β, γ are the roots of $x^3+px^2+qx+r=0$ then $\alpha+\beta+\gamma=-p$, $\alpha\beta+\beta\gamma+\gamma\alpha=q$, $\alpha\beta\gamma=-r$

(i) $$\Sigma\frac{\beta^2+\gamma^2}{\beta\gamma}=\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{\alpha\beta^2+\alpha\gamma^2+\gamma^2\beta+\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma}{\alpha\beta\gamma}$$

$$=\frac{(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-3\alpha\beta\gamma}{\alpha\beta\gamma}=\frac{-pq+3r}{-r}=\frac{pq-3r}{r}$$

(ii) $(\alpha+\gamma-3\alpha)(\gamma+\alpha-3\beta)(\alpha+\beta-3\gamma)$

$(\alpha+\gamma-3\alpha)(\gamma+\alpha-3\beta)(\alpha+\beta-3\gamma)=(\alpha+\beta+\gamma-4\alpha)(\alpha+\beta+\gamma-4\beta)(\alpha+\beta+\gamma-4\gamma)$

$=(-p-4\alpha)(-p-4\beta)(-p-4\gamma)=-(p+4\alpha)(p+4\beta)(p+4\gamma)$

$=-(p^3+4p^2(\alpha+\beta+\gamma)+16p(\alpha\beta+\beta\gamma+\gamma\alpha)+64(\alpha\beta\gamma)=-(p^3-4p^3+16pq-64r)$

$=3p^3-16pq+64r$

(iii) $\Sigma\alpha^3\beta^3=\alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3$

We know $(\alpha\beta+\beta\gamma+\gamma\alpha)^2=\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

Hence, $q^2=\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2pr \Rightarrow \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=q^2-2pr$

$\therefore\ \alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3=(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma\Sigma\alpha^2\beta$

$=(q^2-2pr)\,q+r\,[(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-3\alpha\beta\gamma]$

$=q^3-2pqr+r(-pq+3r)=q^3-2pqr-pqr+3r^2=q^3-3pqr+3r^2$

3 **Find the condition in order that the equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ may have a pair of equal roots**

Sol: Let $\alpha,\alpha,\beta,\beta$ be the roots of $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \Rightarrow x^4 + \frac{4b}{a}x^3 + \frac{6c}{a}x^2 + \frac{4d}{a}x + \frac{e}{a} = 0$

Sum of the roots, $2(\alpha+\beta) = -\frac{4b}{a} \Rightarrow \alpha+\beta = -\frac{2b}{a}$. Let $\alpha\beta = k$

Equation with roots α,β is $x^2-(\alpha+\beta)x+\alpha\beta=0 \Rightarrow x^2 + \frac{2b}{a}x + k = 0$

$$\therefore x^4 + \frac{4b}{a}x^3 + \frac{6c}{a}x^2 + \frac{4d}{a}x + \frac{e}{a} = \left(x^2 + \frac{2b}{a}x + k\right)^2 = x^4 + \frac{4b}{a}x^3 + \left(\frac{4b^2}{a^2} + 2k\right)x^2 + \frac{4bk}{a}x + k^2$$

Comparing the coefficient of x $;\ \frac{4d}{a} = \frac{4bk}{a} \Rightarrow k = \frac{d}{b}$

Comparing the coefficient of x^2; $\frac{6c}{a} = \frac{4b^2}{a^2} + \frac{2d}{b} \Rightarrow 6abc = 4b^3 + 2a^2d \Rightarrow 3abc = 2b^3 + a^2d$

Comparing the constant terms; $k^2 = \frac{e}{a} \Rightarrow \frac{d^2}{b^2} = \frac{e}{a} \Rightarrow ad^2 = eb^2$

$\therefore$ The required conditions are $3abc = 2b^3 + a^2d$ and $ad^2 = eb^2$

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4 **Prove that the sum of any two of the roots of the equation $x^4+px^3+qx^2+rx+s = 0$ is equal to the sum of the remaining two roots of the equation iff $p^3-4pq+8r=0$.**

Sol: Let $\alpha, \beta, \gamma, \delta$ be roots of the given equation.

From the given condition, we take $\alpha+\beta = \gamma+\delta$.

Then $x^4+px^3+qx^2+rx+s = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= [x^2 - (\alpha+\beta)x+\alpha\beta][x^2 - (\gamma+\delta)x+\gamma\delta]$

$= [x^2 - (\alpha+\beta)x+\alpha\beta][x^2 - (\alpha+\beta)x+\gamma\delta]$

Put $-(\alpha+\beta)=b$, $\alpha\beta=c$, $\gamma\delta=d$ then

$x^4+px^3+qx^2+rx+s = (x^2+bx+c)(x^2+bx+d) = x^4+2bx^3+(b^2+c+d)x^2+b(c+d)x+cd$

Comparing the coefficients of like powers of x, we get

$2b = p$........(1); $b^2+c+d = q$.... (2) ; $b(c+d) = r$.....(3); $cd = s$....(4)

From (1), $b = p/2$,

Now (2)$\Rightarrow c+d = q-b^2 = q-\left(\frac{p}{2}\right)^2$ $\therefore$ (3)$\Rightarrow \frac{p}{2}\left(q-\frac{p^2}{4}\right) = r$

$$\Leftrightarrow \frac{p}{2}\left(\frac{4q-p^2}{4}\right) = r \Leftrightarrow p(4q-p^2) = 8r \Leftrightarrow 4pq-p^3 = 8r \Leftrightarrow p^3-4pq+8r = 0$$